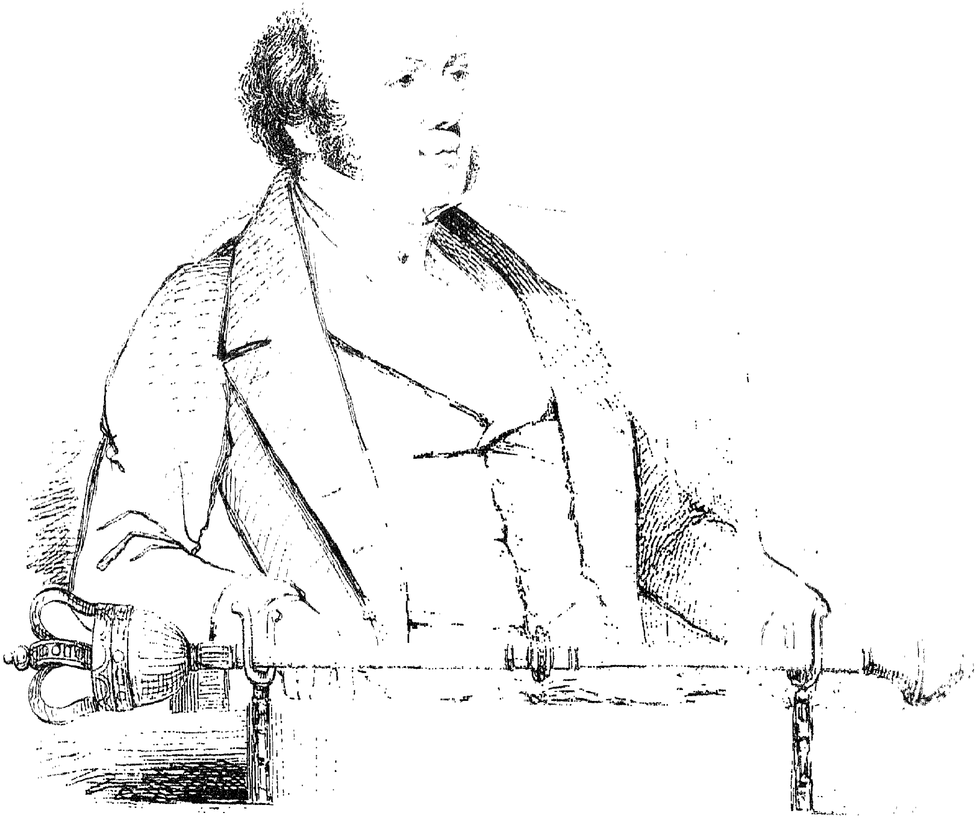
THE MATHEMATICAL PAPERS OF SIR WILLIAM ROWAN HAMILTON

CAMBRIDGE
UNIVERSITY PRESS
LONDON: BENTLEY HOUSE
NEW YORK, TORONTO, BOMBAY
CALCUTTA, MADRAS: MACMILLAN
TOKYO: MARUZEN COMPANY LTD

All rights reserved



William Rowan Hamilton.

Cunningham Memoir No. XIV

THE
MATHEMATICAL PAPERS
OF
SIR WILLIAM ROWAN HAMILTON

VOL. II

DYNAMICS

EDITED FOR THE
ROYAL IRISH ACADEMY

BY
A. W. CONWAY, D.Sc., Sc.D., LL.D., F.R.S., P.R.I.A.
Professor of Mathematical Physics,
University College, Dublin
AND
A. J. McCONNELL, Sc.D., M.R.I.A.
Fellow and University Professor of
Natural Philosophy, Trinity
College, Dublin

CAMBRIDGE
AT THE UNIVERSITY PRESS
1940

PREFACE

The second volume of the Mathematical Papers of Sir William Rowan Hamilton is mainly devoted to dynamics but it was thought desirable to include in it Hamilton's Calculus of Principal Relations, which is a natural and obvious extension of his method of the principal function. The greater part of this volume is now appearing in print for the first time, the only papers previously published being his two famous essays on dynamics and a number of minor papers. A detailed description of the contents will be found in the Introduction.

The frontispiece is reproduced from an etching in the Dublin University Magazine, Vol. XIX (1842), facing page 94.

The Royal Irish Academy wishes to acknowledge with gratitude generous financial assistance towards the publication of this volume from the Board of Trinity College, Dublin, the Senate of the National University of Ireland and the Council of the Royal Society of London.

The Editors desire to thank all those who have helped by advice and interest in the production of the volume, and in particular to record their appreciation of the care which the Cambridge University Press has shown in the printing.

A. W. C.
A. J. McC.

DUBLIN,
August, 1938

CONTENTS

William Rowan Hamilton . . . . *frontispiece*

PAGE

Preface v

Introduction ix

PART I. DYNAMICS OF MATERIAL BODIES

I. Problem of Three Bodies by my Characteristic Function. (1833) 1
[Manuscript]

II. On a General Method in Dynamics; by which the Study of the Motions of all free Systems of attracting or repelling Points is reduced to the Search and Differentiation of one central Relation, or Characteristic Function. (1834) 103
[*Phil. Trans. Roy. Soc.* Part II for 1834, pp. 247–308.]

III. Second Essay on a General Method in Dynamics. (1835) . 162
[*Phil. Trans. Roy. Soc.* Part I for 1835, pp. 95–144.]

IV. On the Application to Dynamics of a General Mathematical Method previously applied to Optics. (1834) 212
[*Brit. Assoc. Report*, 1834, pp. 513–518.]

V. On nearly Circular Orbits. (1836) 217
[Manuscript.]

VI. Theory of the Moon. (1837) 238
[Manuscript.]

VII. Correspondence with J. W. Lubbock. (1837) 249
[Manuscript.]

VIII. On the Composition of Forces. (1841) 284
[*Proc. Roy. Irish Acad.* Vol. II (1840–44), pp. 166–168.]

IX. On Theorems of Central Forces. (1846) 286
[*Proc. Roy. Irish Acad.* Vol. III (1845–47), pp. 308, 309.]

X. The Hodograph, or a new method of expressing in symbolical language the Newtonian Law of Attraction. (1846) . . 287
[*Proc. Roy. Irish Acad.* Vol. III (1845–47), pp. 344–353.]

XI. On Theorems of Hodographic and Anthodographic Isochronism. (1847) 293
[*Proc. Roy. Irish Acad.* Vol. III (1845–47), pp. 417, 465.]

PART II. CALCULUS OF PRINCIPAL RELATIONS

PAGE

XII. Calculus of Principal Relations. (1836) 297
[Manuscript.]

XIII. Calculus of Principal Relations. (1836) 332
[Manuscript.]

XIV. Calculus of Principal Relations. A new series of investigations. (1836) 358
[Manuscript.]

XV. Integration of Partial Differential Equations by the Calculus of Variations. (1836) 391
[Manuscript.]

XVI. Calculus of Principal Relations. (1836) 408
[*Brit. Assoc. Report*, 1836, pp. 41–44.]

PART III. DYNAMICS OF LIGHT

XVII. On the Propagation of Light in Crystals. (1835–38) . . 413
[Manuscript.]

XVIII. On the Propagation of Light *in vacuo*. (1838) 446
[*Brit. Assoc. Report*, 1838, pp. 2–6.]

On the Propagation of Light in Crystals. (1838) . . . 450
[*Brit. Assoc. Report*, 1838, p. 6.]

XIX. Researches respecting Vibration connected with the Theory of Light. (1839) 451
[Manuscript.]

XX. Propagation of Motion in Elastic Medium—Discrete Molecules. (1839) 527
[Manuscript.]

XXI. Researches on the Dynamics of Light. (1839) 576
[*Proc. Roy. Irish Acad.* Vol. I (1836–40), pp. 267–270.]

XXII. Researches respecting Vibration connected with the Theory of Light. (1839) 578
[*Proc. Roy. Irish Acad.* Vol. I (1836–40), pp. 341–349.]

XXIII. Correspondence. (1835–39) 583
[Manuscript.]

Editors' Appendix 609

Index 643

INTRODUCTION

This is the second volume of the collected mathematical works of Sir William R. Hamilton and is devoted to his researches on Dynamics and topics closely related to them. As in the first volume, it is thought desirable that in this Introduction we should give a connected account of Hamilton's work over the period in question, as it will help to put the papers appearing in this volume in their proper perspective and show their close connection with one another.

Following the decision of the Editors of the first volume to divide the material into volumes by subject rather than by date, we include here all Hamilton's contributions to dynamics apart from those that arose as applications of quaternions; nevertheless most of the papers of the volume belong to a definite period of Hamilton's life, from 1833 to 1839. The contents of this Introduction may be stated as follows:

1. Arrangement and editing.
2. The application of the characteristic and principal functions to dynamics.
3. The calculus of principal relations.
4. The dynamics of light.
5. Abstracts.

1. *Arrangement and editing.*

The present volume falls naturally into three main divisions. In the first part we have included Hamilton's two famous Essays in which he defines and obtains the chief properties of his characteristic and principal functions together with several manuscripts, hitherto unpublished, where he applies his general method to specific problems. We have also included a few miscellaneous papers on dynamics having no direct relation to his principal function.

In the second part, most of which consists of manuscript, Hamilton is dealing with a new calculus, called by him the Calculus of Principal Relations, which is really an extension of the method of his principal function to obtain integrals of total and even partial differential equations.

The last part is taken up with Hamilton's researches in the dynamics of vibration and its application to the theory of light. As will be seen this part also consists of unpublished manuscript with the exception of one or two brief abstracts of some of his results in this field.

This second volume differs considerably in character from the first. The latter was very homogeneous in its matter, being Hamilton's contribution to optics, and at least three-quarters of it had already appeared in print. The present volume, although dealing mainly with dynamical problems, is much more varied in character and less than one-quarter has been published before. As the Introduction of the first volume has emphasised, Hamilton arrived at his conclusions by heavy and laborious preliminary work and by the working of many examples he often reached the general results gradually through the particular. When he finally published his results, they appeared in very compact and condensed form, which made his published papers somewhat difficult reading and gave very little indication of the labour by which he arrived at the end of his journey. Most of the manuscripts printed here for the first time are in a very unfinished condition, but they show clearly Hamilton's method of working as well as his great power of manipulation. The Editors have no hesitation in printing these manuscripts in full, though Hamilton himself was obviously not entirely satisfied with them and published only brief abstracts of their contents.

We have tried to make this volume as uniform as possible with the first and have adopted the same plan of editing. Footnotes of the Editors are inserted in square brackets where it was considered necessary and longer notes on difficulties that arise in the text are printed in an Editors' Appendix at the end.

2. *The application of the characteristic and principal functions to dynamics.*

Much of Hamilton's researches on optics could be at once carried over into dynamics and he at once began in characteristic manner to study particular problems before producing anything for publication. Problems in dynamics at this date were largely of an astronomical character, so that it is not surprising that his first attempt was made on the celebrated problem of three bodies which had already been treated by Laplace, Lagrange and others. None of the previous attempts consisted of much more than obtaining the classical integrals and applying the known methods of approximation to special cases. It cannot be said that Hamilton's work, now published for the first time, did much more than verify the results already known, although his method of attack in utilising his characteristic function was quite different. Restricting himself to the special case of the Sun, Jupiter and Saturn, the orbits of which were supposed to lie in the same plane, he obtained approximations to the motion. Of course, in his optical work, no reference to time was necessary and the question of finding the approximation to the time of Saturn required new developments. Under the supposition that the time followed the same law in its approximation as the eccentricity or the major axis, his first attempts seemed unsatisfactory. All the particular cases which, as was his habit, he tried showed even that there was a definite error somewhere. Returning to the simplest case of two bodies, the Kepler or Newtonian orbit, he discovered the result which he required to complete his theory, viz. that by differentiating the action function with respect to the *vis viva* constant he obtained the time. Calculations more extensive at once enabled him to prove that this result was general for what would now be called conservative systems and he at once started on his First Essay "On a General Method in Dynamics."

There is no necessity to give a résumé here of this famous Essay and it is sufficient to observe that before he had completed it he had discovered a more manageable function, the principal function.

This function he tested by means of a particular case of the three-body problem, Sun, Earth and Moon, and obtained approximations of already known type. His "Second Essay" contained not only the development of his principal function but also here appeared for the first time his canonical equations of dynamics. It is curious to remark that Hamilton hardly made any use of these equations and was more interested in reducing all work to the consideration of one function and of carrying out his approximations by means of it.

3. *The calculus of principal relations.*

Hamilton immediately saw that the general method he had expounded in dynamical theory admitted of a considerable extension. If S is the principal function of a dynamical system whose kinetic energy is T and force function U, then

$$dS = (T - U)\,dt,$$

and $S = S\,(x_1, x_2, \ldots x_n, t, a_1, a_2, \ldots a_n, t_0)$ is a principal relation that enables us to solve the above total differential equation and the equations associated with it by the calculus of variations, namely, the equations of motion of the system. So in the same way if we take any total differential equation

$$f(x, x', x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n') = 0,$$

where dashes denote derivatives with respect to a variable u, and if we associate with it the equations given by the calculus of variations, namely

$$\frac{\dfrac{\partial f}{\partial x} - \left(\dfrac{\partial f}{\partial x'}\right)'}{\dfrac{\partial f}{\partial x'}} = \frac{\dfrac{\partial f}{\partial x_1} - \left(\dfrac{\partial f}{\partial x_1'}\right)'}{\dfrac{\partial f}{\partial x_1'}} = \ldots = \frac{\dfrac{\partial f}{\partial x_n} - \left(\dfrac{\partial f}{\partial x_n'}\right)'}{\dfrac{\partial f}{\partial x_n'}},$$

we are able to deduce a principal or central relation

$$F\,(x, x_1, x_2, \ldots x_n, a, a_1, a_2, \ldots a_n) = 0,$$

the a's being initial values of the x's, the knowledge of which will furnish the solution of the total differential equation and the auxiliary equations. Exactly as in the dynamical case, it is found that the function F must satisfy two partial differential equations of the first order, and if we can find the appropriate solution of these equations, our problem is solved. Hamilton has been criticised in that he did not indicate how the particular solution required of his partial differential equation was to be distinguished from other solutions, and also that he did not show how this integral was to be obtained. In the first manuscript on the Calculus of Principal Relations, Hamilton gave a means of identifying this required integral and described an interesting method of successive approximation by which the principal relation or function can be rigorously obtained without any integrations whatever. We have given an account of this method in the important dynamical case in one of the Appendix notes.

The next step was to extend his results, and he applied the same methods to a system of total differential equations of any order,

$$f_1\,(x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n', x_1'', \ldots x_n'', \ldots x_1^{(i)}, \ldots x_n^{(i)}) = 0,$$

$$f_k\,(x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n', x_1'', \ldots x_n'', \ldots x_1^{(i)}, \ldots x_n^{(i)}) = 0.$$

He associated with these the auxiliary equations of the calculus of variations, and indicated that we ought to have a principal relation of the form

$$F\,(x_1, x_2, \ldots x_n, x_1', \ldots x_n', \ldots x_1^{(i-1)}, \ldots x_n^{(i-1)}, a_1, a_2, \ldots a_n, a_1', \ldots a_n', \ldots a_1^{(i-1)}, \ldots a_n^{(i-1)}) = 0.$$

Hamilton did not develop his ideas much further in this direction and left his investigations in a very unfinished condition. A great deal of his work on this calculus deals with methods of approximation to his principal relation or function, and obviously his work has a close connection with the investigations that were afterwards conducted in the methods known generally as Rayleigh's Principle, which also deals with successive approximations to the solutions of variational equations. Hamilton himself gave his methods an interesting application to dynamical astronomy in a series of letters to J. W. Lubbock, which are reproduced on pp. 249–283 of this volume.

Applying a similar type of argument to partial differential equations, Hamilton succeeded in obtaining a very interesting method of solving partial differential equations of the first and second orders. His method is analogous to Cauchy's theory of characteristics for equations of the first order, but he approached the problem from a different point of view and one which suggested a similar method for equations of the second order. To elucidate Hamilton's manuscript on the subject, which is fragmentary in character, we have added a note in the Appendix giving a short account of his method and its relation to Cauchy's theory.

4. *The dynamics of light.*

On taking up the dynamical theory of optics he attacked the problem of the propagation of light in a crystalline medium. He obtained much the same results as Cauchy but anticipated him in later researches. He pushed his investigations very much further than any of his contemporaries but after much numerical calculation concluded that the accuracy of the measurements available was not sufficient to decide the much debated question of the direction of the light vector. His strict adherence to the classical dynamical principles and his use of discrete attracting particles form a contrast to the ideas of his great colleague MacCullagh, whose idea of a continuous medium and the resulting equations form a direct forerunner of the Faraday-Maxwell aether. Concurrently he developed his "Skotodynamics" or dynamics of darkness. That is, he studied the exact manner in which an elasto-optical disturbance travelled into the undisturbed part of the medium. This brought him into various new paths in mathematics which ultimately led to a great paper entitled "Fluctuating Functions" (to be included in Vol. III of his works) and incidentally led to his discovery of the difference between group-velocity and phase-velocity and of many of the properties of the functions afterwards called Bessel Functions. In fact, he was the first to give a complete asymptotic expansion of these functions.

5. *Abstracts.*

The following abstracts of the papers in this volume have been made as concise as possible. The dates given in the case of published papers are those of communication.

PART I. Dynamics of Material Bodies

1. *Problem of Three Bodies by my Characteristic Function.* (1833.)

This paper is an attempt by Hamilton to solve the problem of three bodies by the aid of the characteristic function. It contains a number of curious and interesting results, among others the well-known form of the characteristic function for two bodies, $V = \int_{-\chi}^{+\chi} \pm \sqrt{\frac{1}{r+r_0+\chi} - \frac{1}{4a}}\, d\chi$. He also discovers in this paper the important formula $\partial V/\partial h = t$. Towards the end of the paper is a proof that his two partial differential equations for V have a common solution in this particular case.

II. *On a General Method in Dynamics.* (1834.)

This is the first of Hamilton's two famous Essays on dynamics, in which he develops the whole of theoretical dynamics by the aid of one central function. In this Essay he defines his characteristic function by analogy with his researches in optics, and develops its chief properties for a general system of points in any system of coordinates. The remainder of the paper is devoted to methods of approximation with a view to applying them to perturbations of astronomical bodies. At the end he introduces another function, the principal function, which he develops in the Second Essay.

III. *Second Essay on a General Method in Dynamics.* (1835.)

Hamilton develops the properties of the principal function in much the same way as in the previous Essay, but he here establishes for the first time his well-known equations of motion. He applies his method to a case of planetary motion, using a system of canonical elements.

IV. *On the Application to Dynamics of a General Mathematical Method previously applied to Optics.* (1834.)

This is a Report to the British Association on the methods developed in his two Essays.

V. *On nearly Circular Orbits.* (1836.)

The principal function in the case of a binary system is expressed in a form suitable for approximation when the orbit is nearly circular and similar results are also given for the three-body problem.

VI. *Theory of the Moon.* (1837.)

The method of the principal function is applied to the theory of the Moon and approximations are obtained which he compares with results already obtained by Plana and Lubbock.

VII. *Correspondence with J. W. Lubbock.* (1837.)

This long correspondence between Hamilton and Lubbock arose from an enquiry on Lubbock's part as to whether Hamilton had considered the application of his general methods to lunar theory as the methods in use were so laborious and unreliable. Hamilton replied in a series of letters giving an interesting account of his methods differing somewhat in point of view from those in his two Essays. These methods are in fact his applications of the calculus of principal relations to this particular problem, and some of his results are already contained in the previous paper VI.

VIII. *On the Composition of Forces.* (1841.)

This contains Hamilton's statical proof of the parallelogram of forces.

IX. *On Theorems of Central Forces.* (1846.)

General formulae are given for the description of conics and sphero-conics under central forces.

X. *The Hodograph.* (1846.)

The hodograph of a particle is here defined and it is applied to the case of central orbits. Various interesting theorems are deduced concerning the relation between the hodograph and the law of force.

XI. *On Theorems of Hodographic and Anthodographic Isochronism.* (1847.)

In these two notes, the anthode of a particle is defined and two theorems of isochronism, one hodographic and the other anthodic, are stated.

PART II. CALCULUS OF PRINCIPAL RELATIONS

XII. *Calculus of Principal Relations.* (1836).

This is the incomplete manuscript of a text-book which Hamilton had intended to write on the calculus of principal relations. It consists only of an introduction and scrappy portions of Chapter I. It does little more than state the basic problem and merely indicates the method of solution.

XIII. *Calculus of Principal Relations.* (1836).

This is really material for his proposed treatise expressed in rough and incomplete form. In addition to stating the problem, he gives an interesting method of obtaining his principal function from the partial differential equation by successive approximation. The manuscript ends with applications to some examples.

XIV. *Calculus of Principal Relations: A new series of investigations.* (1836.)

This is a continuation of XIII in which he states the fundamental problem in a more general form and also sketches how the method may be extended to deal with a system of total differential equations of any order.

XV. *Integration of Partial Differential Equations by the Calculus of Variations.* (1836.)

Hamilton here shows how the calculus of principal relations can be used to solve partial differential equations of the first and second orders.

XVI. *Calculus of Principal Relations.* (1836.)

A British Association Report in which Hamilton gives a sketch of the method of principal relations.

PART III. DYNAMICS OF LIGHT

XVII. *On the Propagation of Light in Crystals.* (1835–38.)

A dynamical attempt to reproduce the phenomena of the propagation of light through crystals, and an attempt at comparison with existing physical measurements.

XVIII. *On the Propagation of Light* in vacuo. (1838.)

A brief discussion of the law of force between "particles of the ether." Hamilton decides against the inverse fourth power law and suggests a formula of Poisson as more suitable.

On the Propagation of Light in Crystals. (1838.)

This is an abstract of XVII, which appeared as a British Association Report.

XIX. *Researches respecting Vibration connected with the Theory of Light.* (1839.)

A series of equidistant particles lie in a line, the end particles being fixed, and each particle is acted upon by its two adjacent particles. The problem of small transverse vibrations is worked out in great detail. Many interesting results are obtained, such as, an asymptotic value of what were afterwards called Bessel functions, the reciprocal theorem in dynamics, and the distinction between phase- and group-velocity. The term fluctuating function is first mentioned here.

XX. *Propagation of Motion in Elastic Medium—Discrete Molecules.* (1839.)

This paper deals with the same problems as XIX and extends them to two and three dimensions. The method of attack is quite different and consists of very interesting operational methods, what is now known as Heaviside's operator being extensively used. The asymptotic values of various integrals are obtained and a detailed account of the wave front is given.

XXI, XXII. *Researches on the Dynamics of Light.* (1839.)

These papers give an account of some of the problems dealt with in XIX and XX.

XXIII. *Correspondence.* (1835–39.)

Correspondence with Professor Powell and with Sir John Herschel on the dynamics of light.

A. W. CONWAY
A. J. McCONNELL

Part I

DYNAMICS OF MATERIAL BODIES

I.

PROBLEM OF THREE BODIES BY MY CHARACTERISTIC FUNCTION

[1833.]

[Note Book 29.]

[It has been considered advisable to add a table of contents to this paper. It is a trial of the Characteristic Function in an attack on a classical problem. It led Hamilton to a new result $\frac{\partial V}{\partial h}=t$ and to various subsidiary investigations.

CONTENTS

PAGE

[1.] [2.] [3.] General statement of problem. Sun, Jupiter, Saturn in centrobaric coordinates. 2

[4.] [5.] Introduction of the Characteristic Function V and its variation δV. 3

[6.]–[9.] Approximate equations solved. 5

[10.] [11.] Perturbations of Jupiter obtained and compared with results of Laplace. . . 8

[12.] Perturbations of Saturn. 12

[13.] Approximate determination of the eight elements a, e, ϵ, ϖ, a', e', ϵ', ϖ' in terms of initial coordinates and initial velocities. 14

[14.] [15.] Expressions for *vis viva* in form $h=H_{,}+mm'H_{,,}$. 17

[16.] [17.] Expressions for perturbations $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$. 20

[18.] [19.] [20.] Putting $r=(r_{,})+\Delta r_{,}+m'(r_{,,})+m'\Delta r_{,,}$, $\Delta H_{,,}$, $\Delta V_{,,}$ and δV are calculated. . 24

[21.] Perturbations of nearly circular orbits. 30

[22.] Expressions for variation of constants a, e, ϵ, ... in terms of δr, δr_0, $\delta\theta$, $\delta\theta_0$, ... 32

[23.] The disturbing function W. 34

[24.] [25.] [26.] Difficulties arising from the assumption $t=t_{,}+mm't_{,,}$. 37

[27.] [28.] Restatement of problem with new definitions of a, e, etc. 43

[29.] The characteristic function of elliptic motion obtained in the form

$$\int_{-\chi}^{\chi} \pm \sqrt{\frac{1}{r+r_0+\chi}-\frac{1}{4a}}\,.d\chi.$$

The equation $\frac{\delta V}{\delta h}=t$. 46

[30.] Expression for disturbing function and comparison with Laplace. General proof of $\frac{\delta V}{\delta h}=t$. 49

[31.]–[38.] Various investigations regarding the disturbing function and heliocentric coordinates. Comparison with Lagrange. 53

[39.] Independence of perturbations on disturbed mass when squares and products of masses are neglected. 78

[40.] Proof that the partial differential equations

$$\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2=2F(r),\quad \left(\frac{\delta V}{\delta x_0}\right)^2+\left(\frac{\delta V}{\delta y_0}\right)^2+\left(\frac{\delta V}{\delta z_0}\right)^2=2F(r_0)$$

have a common solution. The expansion of V for various laws of force. . 79

[41.] **[42.]** **[43.]** Various transformations of the partial differential equations.] . 95

[*General statement of problem. Sun, Jupiter, Saturn in centrobaric coordinates.*]

(Dec. 16th, 1833.)

[1.] Let Sun, Jupiter, Saturn have their masses 1, m, m'; let their common centre of gravity be fixed at the origin of coordinates; let them be considered as moving in the plane of the ecliptic which we shall take for the plane of xy; let, in this plane, the rectangular coordinates of Jupiter be x, y, and those of Saturn x', y', so that those of the Sun must be $-mx-m'x'$, $-my-m'y'$; then, t being the time, the equations of motion of Jupiter will be

$$(1)\ldots\quad 0=\frac{d}{dt}\frac{dx}{dt}+(x+mx+m'x')\{(x+mx+m'x')^2+(y+my+m'y')^2\}^{-\frac{3}{2}}$$
$$+m'(x-x')\{(x-x')^2+(y-y')^2\}^{-\frac{3}{2}};$$

$$(2)\ldots\quad 0=\frac{d}{dt}\frac{dy}{dt}+(y+my+m'y')\{(x+mx+m'x')^2+(y+my+m'y')^2\}^{-\frac{3}{2}}$$
$$+m'(y-y')\{(x-x')^2+(y-y')^2\}^{-\frac{3}{2}};$$

and those of Saturn will be

$$(3)\ldots\quad 0=\frac{d}{dt}\frac{dx'}{dt}+(x'+m'x'+mx)\{(x'+m'x'+mx)^2+(y'+m'y'+my)^2\}^{-\frac{3}{2}}$$
$$+m(x'-x)\{(x'-x)^2+(y'-y)^2\}^{-\frac{3}{2}};$$

$$(4)\ldots\quad 0=\frac{d}{dt}\frac{dy'}{dt}+(y'+m'y'+my)\{(x'+m'x'+mx)^2+(y'+m'y'+my)^2\}^{-\frac{3}{2}}$$
$$+m(y'-y)\{(x'-x)^2+(y'-y)^2\}^{-\frac{3}{2}};$$

while the Sun's coordinates are linear functions of those of the 2 planets, obtained, as above, from the condition that the centre of gravity of the system is fixed at the origin of coordinates.

[2.] The integration of the 4 foregoing equations would give x, y, x', y', the four varying coordinates of Jupiter and Saturn in the plane in which both are here supposed to revolve, as functions of the time t; of the planetary masses m, m' (Sun's mass being unity); and of 8 arbitrary constants, which we may suppose, if we choose, to be the four initial coordinates x_0, y_0, x'_0, y'_0, corresponding to the origin of time, and their first differential coefficients $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dx'_0}{dt}$, $\frac{dy'_0}{dt}$. And reciprocally, the problem of the 3 bodies would be solved (neglecting inclinations) if we could find these four functions; that is, if we could determine the dependence of the 4 coordinates x, y, x', y' on the eleven quantities t, m, m', x_0, y_0, x'_0, y'_0, $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dx'_0}{dt}$, $\frac{dy'_0}{dt}$.

[3.] The equations (1) (2) (3) (4) give, as one of their 1st integrals, the equation of living force; which may be obtained by multiplying

$$(1)\text{ by } m(1+m)\frac{dx}{dt}+mm'\frac{dx'}{dt}; \qquad (3)\text{ by } m'(1+m')\frac{dx'}{dt}+mm'\frac{dx}{dt};\ \&\text{c.};$$

and may be thus written,

$$(5)\ldots\quad \frac{m}{2}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}+\frac{m'}{2}\left\{\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2\right\}+\frac{1}{2}\left\{\left(m\frac{dx}{dt}+m'\frac{dx'}{dt}\right)^2+\left(m\frac{dy}{dt}+m'\frac{dy'}{dt}\right)^2\right.$$
$$-m\{(x+mx+m'x')^2+(y+my+m'y')^2\}^{-\frac{1}{2}}$$
$$-m'\{(x'+m'x'+mx)^2+(y'+m'y'+my)^2\}^{-\frac{1}{2}}-mm'\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}=h,$$

h being independent of the time. If therefore we had determined the four functions x, y, x', y' and substituted them in this last equation (5), we should obtain a relation between the eleven quantities h, m, m', x_0, y_0, x'_0, y'_0, $\frac{dx_0}{dt}$, $\frac{dx'_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dy'_0}{dt}$ not involving the time t; which may indeed be obtained at once from (5) by changing the final to the initial quantities. And combining this relation with the 4 which determine the forms of the four functions x, y, x', y', we could, by elimination, determine or at least conceive determined the four initial components of velocities $\frac{dx_0}{dt}, \ldots \frac{dy'_0}{dt}$ as a function of the eleven quantities $h, m, m', x_0, y_0, x'_0, y'_0, x, y, x', y'$. The time t would also be determined as a function of these last eleven.

[*Introduction of the Characteristic Function V and its variation δV.*]

[4.] Now my fundamental dynamical theorem* is that the 4 initial components of velocities $\frac{dx_0}{dt}, \ldots \frac{dy'_0}{dt}$ and also the 4 final components may be expressed by the partial differential coefficients of the first order of another function V of the same eleven quantities; and this function, which I call the characteristic function, is $=\int_0^t \Sigma mv^2\,dt$, that is, the integral of the product, the living force of the system × the element of the time; so that we have

$$(6)\ldots\quad V=\int_0^t\left[m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}+m'\left\{\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)\right.\right.$$
$$\left.+\left(m\frac{dx}{dt}+m'\frac{dx'}{dt}\right)^2+\left(m\frac{dy}{dt}+m'\frac{dy'}{dt}\right)^2\right]dt.$$

It is, as I have said, to be regarded as a function of the 4 initial and 4 final coordinates, the planetary masses, and the constant h in the equation of living force; and its variation δV, corresponding to any infinitesimal variations of the initial and final coordinates of all the bodies of the system, the Sun included, is equal to the sum of all those variations multiplied respectively by the components of initial momentum, taken negatively, and by the components of final momentum. That is, we have,

---

* [Cf. Vol. I, pp. 330–332.]

$$(7)\dots\quad \delta V = m\left(\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y\right) + m'\left(\frac{dx'}{dt}\,\delta x' + \frac{dy'}{dt}\,\delta y'\right)$$

$$+\left(m\,\frac{dx}{dt} + m'\,\frac{dx'}{dt}\right)(m\,\delta x + m'\delta x') + \left(m\,\frac{dy}{dt} + m'\,\frac{dy'}{dt}\right)(m\,\delta y + m'\,\delta y')$$

$$-m\left(\frac{dx_0}{dt}\,\delta x_0 + \frac{dy_0}{dt}\,\delta y_0\right) - m'\left(\frac{dx_0'}{dt}\,\delta x_0' + \frac{dy_0'}{dt}\,\delta y_0'\right)$$

$$-\left(m\,\frac{dx_0}{dt} + m'\,\frac{dx_0'}{dt}\right)(m\,\delta x_0 + m'\,\delta x_0') - \left(m\,\frac{dy_0}{dt} + m'\,\frac{dy_0'}{dt}\right)(m\,\delta y_0 + m'\,\delta y_0');$$

and therefore

$$(8)\dots\quad \frac{1}{m}\,\frac{\delta V}{\delta x} = \frac{dx}{dt}\,(1+m) + m'\,\frac{dx'}{dt}; \qquad (9)\dots\quad \frac{1}{m'}\,\frac{\delta V}{\delta x'} = \frac{dx'}{dt}\,(1+m') + m\,\frac{dx}{dt};$$

$$(10)\dots\quad \frac{1}{m}\,\frac{\delta V}{\delta y} = \frac{dy}{dt}\,(1+m) + m'\,\frac{dy'}{dt}; \qquad (11)\dots\quad \frac{1}{m'}\,\frac{\delta V}{\delta y'} =$$

$$(12)\dots\quad -\frac{1}{m}\,\frac{\delta V}{\delta x_0} = \frac{dx_0}{dt}\,(1+m) + m'\,\frac{dx_0'}{dt}; \qquad (13)\dots\quad -\frac{1}{m'}\,\frac{\delta V}{\delta x_0'} =$$

$$(14)\dots\quad -\frac{1}{m}\,\frac{\delta V}{\delta y_0} = \frac{dy_0}{dt}\,(1+m) + m'\,\frac{dy_0'}{dt}; \qquad (15)\dots\quad \frac{1}{m'}\,\frac{\delta V}{\delta y_0'} =$$

Hence also

$$(16)\dots\quad (1+m+m')\,\frac{dx}{dt} = \frac{1+m'}{m}\,\frac{\delta V}{\delta x} - \frac{\delta V}{\delta x'}; \qquad (17)\dots\quad (1+m+m')\,\frac{dx'}{dt} = \frac{1+m}{m'}\,\frac{\delta V}{\delta x'} - \frac{\delta V}{\delta x}$$

$$(18)\dots\quad (1+m+m')\,\frac{dy}{dt} = \frac{1+m'}{m}\,\frac{\delta V}{\delta y} - \frac{\delta V}{\delta y'}; \qquad (19)\dots$$

$$(20)\dots\quad (\qquad\qquad)\,\frac{dx_0}{dt} = -\frac{1+m'}{m}\,\frac{\delta V}{\delta x_0} + \frac{\delta V}{\delta x_0'}; \qquad (21)\dots$$

$$(22)\dots\quad (\qquad\qquad)\,\frac{dy_0}{dt} = -\frac{1+m'}{m}\,\frac{\delta V}{\delta y_0} + \frac{\delta V}{\delta y_0'}; \qquad (23)\dots$$

and these 4 last equations contain the expressions sought, for the four initial components velocity by means of the partial differential coefficients of the 1st order of the Characteri Function V, taken with respect to the four initial coordinates.

[5.] The equations (16)...(23) may be put under the following forms:

$$(24)\dots\quad \frac{dx}{dt} = \frac{1}{m}\,\frac{\delta V}{\delta x} - \frac{1}{1+m+m'}\left(\frac{\delta V}{\delta x} + \frac{\delta V}{\delta x'}\right); \qquad (25)\dots\quad \frac{dx'}{dt} = \frac{1}{m'}\,\frac{\delta V}{\delta x'} - \frac{1}{1+m+m'}\left(\frac{\delta V}{\delta x} + \frac{\delta V}{\delta x'}\right)$$

$$(26)\dots\quad \frac{dy}{dt} = \frac{1}{m}\,\frac{\delta V}{\delta y} - \frac{1}{1+m+m'}\left(\frac{\delta V}{\delta y} + \frac{\delta V}{\delta y'}\right); \qquad (27)\dots$$

$$(28)\dots\quad \frac{dx_0}{dt} = -\frac{1}{m}\,\frac{\delta V}{\delta x_0} + \frac{1}{1+m+m'}\left(\frac{\delta V}{\delta x_0} + \frac{\delta V}{\delta x_0'}\right); \qquad (29)\dots$$

$$(30)\dots\quad \frac{dy_0}{dt} = -\frac{1}{m}\,\frac{\delta V}{\delta y_0} + \frac{1}{1+m+m'}\left(\frac{\delta V}{\delta y_0} + \frac{\delta V}{\delta y_0'}\right); \qquad (31)\dots$$

and they give

$$(32)\ldots \qquad m\frac{dx}{dt}+m'\frac{dx'}{dt}=\frac{1}{1+m+m'}\left(\frac{\delta V}{\delta x}+\frac{\delta V}{\delta x'}\right),$$

$$(33)\ldots \qquad m\frac{dy}{dt}+m'\frac{dy'}{dt}=\frac{1}{1+m+m'}\left(\frac{\delta V}{\delta y}+\frac{\delta V}{\delta y'}\right);$$

$$(34)\ldots \qquad m\frac{dx_0}{dt}+m'\frac{dx_0'}{dt}=-\frac{1}{1+m+m'}\left(\frac{\delta V}{\delta x_0}+\frac{\delta V}{\delta x_0'}\right);$$

$$(35)\ldots \qquad m\frac{dy_0}{dt}+m'\frac{dy_0'}{dt}=-\frac{1}{1+m+m'}\left(\frac{\delta V}{\delta y_0}+\frac{\delta V}{\delta y_0'}\right).$$

Hence the living force of the system, at the time t, may be thus expressed:

$$(36)\ldots \quad \frac{dV}{dt}=\Sigma mv^2=\frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2\right\}+\frac{1}{m'}\left\{\left(\frac{\delta V}{\delta x'}\right)^2+\left(\frac{\delta V}{\delta y'}\right)^2\right\}$$
$$-\frac{1}{1+m+m'}\left\{\left(\frac{\delta V}{\delta x}+\frac{\delta V}{\delta x'}\right)^2+\left(\frac{\delta V}{\delta y}+\frac{\delta V}{\delta y'}\right)^2\right\};$$

an expression which results either from (6), or from the formula

$$(37)\ldots \qquad \frac{dV}{dt}=\frac{\delta V}{\delta x}\frac{dx}{dt}+\frac{\delta V}{\delta y}\frac{dy}{dt}+\frac{\delta V}{\delta x'}\frac{dx'}{dt}+\frac{\delta V}{\delta y'}\frac{dy'}{dt},$$

combined with (24)...(27). The equation of living force, (5), gives* ∵ the 2 following partial differential equations which the function V must satisfy:

$$(38)\ldots \quad \frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2\right\}+\frac{1}{m'}\left\{\left(\frac{\delta V}{\delta x'}\right)^2+\left(\frac{\delta V}{\delta y'}\right)^2\right\}-\frac{1}{1+m+m'}\left\{\left(\frac{\delta V}{\delta x}+\frac{\delta V}{\delta x'}\right)^2+\left(\frac{\delta V}{\delta y}+\frac{\delta V}{\delta y'}\right)\right.$$
$$=2m\{(x+mx+m'x')^2+(y+my+m'y')^2\}^{-\frac{1}{2}}$$
$$+2m'\{(x'+m'x'+mx)^2+(y'+m'y'+my)^2\}^{-\frac{1}{2}}$$
$$+2mm'\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}+2h;$$

$$(39)\ldots \quad \frac{1}{m}\left\{\left(\frac{\delta V}{\delta x_0}\right)^2+\left(\frac{\delta V}{\delta y_0}\right)^2\right\}+\frac{1}{m'}\left\{\left(\frac{\delta V}{\delta x_0'}\right)^2+\left(\frac{\delta V}{\delta y_0'}\right)^2\right\}-\frac{1}{1+m+m'}\left\{\left(\frac{\delta V}{\delta x_0}+\frac{\delta V}{\delta x_0'}\right)^2+\left(\frac{\delta V}{\delta y_0}+\frac{\delta V}{\delta y_0'}\right.\right.$$
$$=2m\{(x_0+mx_0+m'x_0')^2+(y_0+my_0+m'y_0')^2\}^{-\frac{1}{2}}$$
$$+2m'\{(x_0'+m'x_0'+mx_0)^2+(y_0'+m'y_0'+my_0)^2\}^{-\frac{1}{2}}$$
$$+2mm'\{(x_0-x_0')^2+(y_0-y_0')^2\}^{-\frac{1}{2}}+2h.$$

[*Approximate equations solved.*]

[**6.**] So far all is rigorous and nothing has yet been neglected for the sake of approximation. If, however, we were now to neglect m' in (1) (2) and m in (3) (4), we should be able to integrate the fundamental differential equations; and we should get approximate values for the functions x, y, x', y', which we may call $x_{,}, y_{,}, x'_{,}, y'_{,}$. But to approximate to the mutual perturbations we shall only change x, y, x', y' to the following expressions $x_{,}+\xi m'$, $y_{,}+\eta m'$, $x'_{,}+\xi' m$, $y'_{,}+\eta' m$

---

* [∵ = therefore. See footnote, Vol. I, p. 370.]

and the differential equations (1) &c. will resolve themselves into the following, which are rigorous,

(40)... $0=\frac{d}{dt}\frac{dx_{,}}{dt}+\frac{x_{,}(1+m)^{-2}}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}}$; (41)... $0=\frac{d}{dt}\frac{dx'_{,}}{dt}+\frac{x'_{,}(1+m')^{-2}}{(x'^2_{,}+y'^2_{,})^{\frac{3}{2}}}$;

(42)... $0=\frac{d}{dt}\frac{dy_{,}}{dt}+\frac{y_{,}(1+m)^{-2}}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}}$; (43)... $0=\frac{d}{dt}\frac{dy'_{,}}{dt}+\frac{y'_{,}(1+m')^{-2}}{(x'^2_{,}+y'^2_{,})^{\frac{3}{2}}}$;

and the following, which are approximate,

(44)... $$0=\frac{d}{dt}\frac{d\xi}{dt}+\frac{x'_{,}+\xi}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}}+\frac{x_{,}-x'_{,}}{\{(x_{,}-x'_{,})^2+(y_{,}-y'_{,})^2\}^{\frac{3}{2}}}-\frac{3x_{,}(x_{,}x'_{,}+y_{,}y'_{,}+x_{,}\xi+y_{,}\eta)}{(x_{,}^2+y_{,}^2)^{\frac{5}{2}}};$$

(45)... $$0=\frac{d}{dt}\frac{d\eta}{dt}+\frac{y'_{,}+\eta}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}}+\frac{y_{,}-y'_{,}}{\{(x_{,}-x'_{,})^2+(y_{,}-y'_{,})^2\}^{\frac{3}{2}}}\quad\begin{matrix}3y_{,}\qquad\qquad\qquad)\\(x_{,}^2+y_{,}^2)^{\frac{5}{2}}\end{matrix};$$

(46)... $$0=\frac{d}{dt}\frac{d\xi'}{dt}+\frac{x_{,}+\xi'}{(x'^2_{,}+y'^2_{,})^{\frac{3}{2}}}+\frac{x'_{,}-x_{,}}{\{(x_{,}-x'_{,})^2+(y_{,}-y'_{,})^2\}^{\frac{3}{2}}}-\frac{3x'_{,}(x_{,}x'_{,}+y_{,}y'_{,}+x'_{,}\xi'+y'_{,}\eta')}{(x'^2_{,}+y'^2_{,})^{\frac{5}{2}}};$$

(47)... $$0=\frac{d}{dt}\frac{d\eta'}{dt}+\frac{y_{,}+\eta'}{(x'^2_{,}+y'^2_{,})^{\frac{3}{2}}}+\frac{y'_{,}-y_{,}}{\{(x_{,}-x'_{,})^2+(y_{,}-y'_{,})^2\}^{\frac{3}{2}}}\quad\begin{matrix}3y'_{,}(\qquad\qquad\qquad)\\(x'^2_{,}+y'^2_{,})^{\frac{5}{2}}\end{matrix}$$

[7.] The equations (40)...(43) can be completely integrated by the theory of elliptic motion. They have for approximate integrals, (corresponding to the case of nearly circular orbits,)

(48)... $$x_{,}=a\cos(nt+\epsilon)+\frac{ae}{2}\cos(2nt+2\epsilon-\varpi)-\frac{3ae}{2}\cos\varpi;$$

(49)... $$y_{,}=a\sin(nt+\epsilon)+\frac{ae}{2}\sin(2nt+2\epsilon-\varpi)-\frac{3ae}{2}\sin\varpi;$$

(50)... $$x'_{,}=a'\cos(n't+\epsilon')+\frac{a'e'}{2}\cos(2n't+2\epsilon'-\varpi')-\frac{3a'e'}{2}\cos\varpi';$$

(51)... $$y'_{,}=a'\sin(n't+\epsilon')+\frac{a'e'}{2}\sin(2n't+2\epsilon'-\varpi')-\frac{3a'e'}{2}\sin\varpi';$$

which are derived from the following:

(52)... $r_{,}=a-ae\cos(nt+\epsilon-\varpi)$; (53)... $\theta_{,}=nt+\epsilon+2e\sin(nt+\epsilon-\varpi)$;

(54)... $r'_{,}=a'-a'e'\cos(n't+\epsilon'-\varpi')$; (55)... $\theta'_{,}=n't+\epsilon'+2e'\sin(n't+\epsilon'-\varpi')$;

(56)... $x_{,}=r_{,}\cos\theta_{,}$; (57)... $y_{,}=r_{,}\sin\theta_{,}$;

(58)... $x'_{,}=r'_{,}\cos\theta'_{,}$; (59)... $y'_{,}=r'_{,}\sin\theta'_{,}$;

and in which

(60)... $n=\frac{a^{-\frac{3}{2}}}{1+m}$; (61)... $n'=\frac{a'^{-\frac{3}{2}}}{1+m'}$.

In fact the equations (48) (49) (60) give, when we neglect e^2, but not any power of m,

(62)... $(x_{,}^2+y_{,}^2)^{\frac{1}{2}}=\ a-ae\cos(nt+\epsilon-\varpi)$;

(63)... $(x'^2_{,}+y'^2_{,})^{\frac{1}{2}}=\ a'-a'e'\cos(n't+\epsilon'-\varpi')$;

(64)... $$\frac{d}{dt}\left(\frac{dx_{,}}{dt}\right)=-an^2\{\cos(nt+\epsilon)+2e\cos(2nt+2\epsilon-\varpi)\}=-\frac{x_{,}(1+m)^{-2}}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}};$$

(65)... $$\frac{d}{dt}\left(\frac{dy_{,}}{dt}\right)=-an^2\{\sin(nt+\epsilon)+2e\sin(2nt+2\epsilon-\varpi)\}=-\frac{y_{,}(1+m)^{-2}}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}};$$

observing that

$$\frac{a^2x_{,}}{(x_{,}^2+y_{,}^2)^{\frac{3}{2}}}=\{1+3e\cos(nt+\epsilon-\varpi)\}\left\{\cos(nt+\epsilon)+\frac{e}{2}\cos(2nt+2\epsilon-\varpi)-\frac{3e}{2}\cos\varpi\right\}$$
$$=\cos(nt+\epsilon)+\frac{e}{2}\cos(2nt+2\epsilon-\varpi)-\frac{3e}{2}\cos\varpi+\frac{3e}{2}\{\cos(2nt+2\epsilon-\varpi)+\cos\varpi\};\ \&c.$$

[**8**.] In integrating (44)...(47) we shall neglect the masses and the excentricities, and thus we shall have

$$(66)\ldots\quad 0=\frac{d}{dt}\frac{d\xi}{dt}+n^2\xi+n^2a'\cos(n't+\epsilon')+\frac{a\cos(nt+\epsilon)-a'\cos(n't+\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}$$
$$3n^2\cos(nt+\epsilon)\{a'\cos(nt-n't+\epsilon-\epsilon')+\xi\cos(nt+\epsilon)+\eta\sin(nt+\epsilon)\};$$

$$(67)\ldots\quad 0=\frac{d}{dt}\frac{d\eta}{dt}+n^2\eta+n^2a'\sin(n't+\epsilon')+\frac{a\sin(nt+\epsilon)-a'\sin(n't+\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}$$
$$-3n^2\sin(nt+\epsilon)\{a'\cos(nt-n't+\epsilon-\epsilon')+\xi\cos(nt+\epsilon)+\eta\sin(nt+\epsilon)\};$$

$$(68)\ldots\quad 0=\frac{d}{dt}\frac{d\xi'}{dt}+n'^2\xi'+n'^2a\cos(nt+\epsilon)+\frac{a'\cos(n't+\epsilon')-a\cos(nt+\epsilon)}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}$$
$$-3n'^2\cos(n't+\epsilon')\{a\cos(nt-n't+\epsilon-\epsilon')+\xi'\cos(n't+\epsilon')+\eta'\sin(n't+\epsilon')\};$$

$$(69)\ldots\quad 0=\frac{d}{dt}\frac{d\eta'}{dt}+n'^2\eta'+n'^2a\sin(nt+\epsilon)+\frac{a'\sin(n't+\epsilon')-a\sin(nt+\epsilon)}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}$$
$$-3n'^2\sin(n't+\epsilon')\{a\cos(nt-n't+\epsilon-\epsilon')+\xi'\cos(n't+\epsilon')+\eta'\sin(n't+\epsilon')\}.$$

[**9**.] Multiplying (66) by $\cos(nt+\epsilon)$ and (67) by $\sin(nt+\epsilon)$ and adding the products, we find

$$(70)\ldots\quad 0=\cos(nt+\epsilon)\frac{d}{dt}\frac{d\xi}{dt}+\sin(nt+\epsilon)\frac{d}{dt}\frac{d\eta}{dt}$$
$$-2n^2\{\xi\cos(nt+\epsilon)+\eta\sin(nt+\epsilon)+a'\cos(nt-n't+\epsilon-\epsilon')\}$$
$$+\{a-a'\cos(nt-n't+\epsilon-\epsilon')\}\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{-\frac{3}{2}};$$

and by an analogous combination,

$$(71)\ldots\quad 0=\sin(nt+\epsilon)\frac{d}{dt}\frac{d\xi}{dt}-\cos(nt+\epsilon)\frac{d}{dt}\frac{d\eta}{dt}+n^2\{\xi\sin(nt+\epsilon)-\eta\cos(nt+\epsilon)\}$$
$$+n^2a'\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

Put, for abbreviation,

$$(72)\ldots\quad \cos(nt+\epsilon)\,.\,\xi+\sin(nt+\epsilon)\,.\,\eta=r_{,,};\qquad (73)\ldots\quad -\xi\sin(nt+\epsilon)+\eta\cos(nt+\epsilon)=a\theta_{,,};$$

then,

$$(74)\ldots\quad \xi=r_{,,}\cos(nt+\epsilon)-a\theta_{,,}\sin(nt+\epsilon);\qquad (75)\ldots\quad \eta=r_{,,}\sin(nt+\epsilon)+a\theta_{,,}\cos(nt+\epsilon);$$

$$(76)\ldots\quad \frac{d}{dt}\frac{d\xi}{dt}=\cos(nt+\epsilon)\frac{d}{dt}\frac{dr_{,,}}{dt}-a\sin(nt+\epsilon)\frac{d}{dt}\frac{d\theta_{,,}}{dt}$$
$$-2n\sin(nt+\epsilon)\frac{dr_{,,}}{dt}-2an\cos(nt+\epsilon)\frac{d\theta_{,,}}{dt}-n^2\xi;$$

$$(77)\ldots\quad \frac{d}{dt}\frac{d\eta}{dt}=\sin(nt+\epsilon)\frac{d}{dt}\frac{dr_{,,}}{dt}+a\cos(nt+\epsilon)\frac{d}{dt}\frac{d\theta_{,,}}{dt}$$
$$+2n\cos(nt+\epsilon)\frac{dr_{,,}}{dt}-2an\sin(nt+\epsilon)\frac{d\theta_{,,}}{dt}-n^2\eta;$$

& $\therefore$

$$(78)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}-3n^2r_{\prime\prime}-2na\frac{d\theta_{\prime\prime}}{dt}-2n^2a'\cos(nt-n't+\epsilon-\epsilon') \\ +\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}};$$

$$(79)\ldots\quad 0=-a\frac{d}{dt}\frac{d\theta_{\prime\prime}}{dt}-2n\frac{dr_{\prime\prime}}{dt} \\ +n^2a'\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}$$

This last equation can be immediately integrated and gives

$$(80)\ldots\quad -a\frac{d\theta_{\prime\prime}}{dt}=2nr_{\prime\prime}+\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(nt-n't+\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\} \\ -\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}.$$

if we suppose, which we are at liberty to do, that ξ, η, $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, & $\therefore$ $r_{\prime\prime}$, $\theta_{\prime\prime}$, $\frac{dr_{\prime\prime}}{dt}$, $\frac{d\theta_{\prime\prime}}{dt}$, vanish with t. On this last supposition we shall have by integrating (80)

$$(81)\ldots\quad a\theta_{\prime\prime}=-2n\int_0^t r_{\prime\prime}dt+\frac{n^2a't}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}-\frac{n^2a'}{(n-n')^2} \\ \times\left(-\sin(\epsilon-\epsilon')+\sin(nt-n't+\epsilon-\epsilon')\right)+\frac{n^2a^2a'^{-1}}{n-n'}\int_0^t\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}dt.$$

We have also, by substituting (80) in (78),

$$(82)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\frac{2n^2n'a'}{n-n'}\cos(nt-n't+\epsilon-\epsilon') \\ -\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\} \\ -\frac{2n^3a^2a'^{-1}}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}.$$

We are $\therefore$ to integrate this equation and to determine the constants by the condition that $r_{\prime\prime}$ and $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t.

[*Perturbations of Jupiter obtained and compared with results of Laplace.*]

[10.] Since $a'>a$, we have, nearly,

$$(83)\ldots\quad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=1+\frac{a}{a'}\cos\omega+\frac{1}{2}\frac{a^2}{a'^2}(-1+3\cos\omega^2) \\ =1+\frac{a}{a'}\cos\omega+\frac{1}{4}\frac{a^2}{a'^2}(1+3\cos2\omega);$$

or, exactly, in the notation of the *Méc. Cél.**

$$(84)\ldots \qquad \left(1-\frac{2a}{a'}\cos\omega+\frac{w}{a'^2}\right) \quad = \tfrac{1}{2}b^{(0)}_{\frac{1}{2}}+b^{(1)}_{\frac{1}{2}}\cos\omega+b^{(2)}_{\frac{1}{2}}\cos 2\omega+\&c.;$$

$$\left[\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{3}{2}}=1+3\frac{a}{a'}\cos\omega+\frac{3}{4}\frac{a^2}{a'^2}(3+5\cos 2\omega)+\&c. \\ =\tfrac{1}{2}b^{(0)}_{\frac{3}{2}}+b^{(1)}_{\frac{3}{2}}\cos\omega+b^{(2)}_{\frac{3}{2}}\cos 2\omega+\&c.;\right]$$

∵ if we put (82) under the form

$$(85)\ldots \quad \frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\tfrac{1}{2}R_0+R_1\cos(nt-n't+\epsilon-\epsilon')+R_2\cos 2(nt-n't+\epsilon-\epsilon')+\&c.=0,$$

we have

$$(86)\ldots \quad \tfrac{1}{2}R_0=-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1-2\frac{a}{a'}\cos\overline{\epsilon-\epsilon'}+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}\right\}-\frac{n^3a^2a'^{-1}}{n-n'}b^{(0)}_{\frac{1}{2}}-\frac{1}{2a'}\frac{db^{(0)}_{\frac{1}{2}}}{da};$$

observing that

$$\frac{a-a'\cos\omega}{(a^2-2aa'\cos\omega+a'^2)^{\frac{3}{2}}}=-\frac{d}{da}(a^2-2aa'\cos\omega+a'^2)^{-\frac{1}{2}}$$

$$=-\frac{1}{a'}\frac{d}{da}\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=-\frac{1}{2a'}\frac{db^{(0)}_{\frac{1}{2}}}{da}-\frac{1}{a'}\frac{db^{(1)}_{\frac{1}{2}}}{da}\cos\omega-\frac{1}{a'}\frac{db^{(2)}_{\frac{1}{2}}}{da}\cos 2\omega-\&c.;$$

and similarly

$$(87)\ldots \qquad R_1=\frac{2n^2n'a'}{n-n'}-\frac{2n^3a^2}{(n-n')a'}b^{(1)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(1)}_{\frac{1}{2}}}{da};$$

$$(88)\ldots \qquad R_2=\qquad -\frac{2n^3a^2}{(n-n')a'}b^{(2)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(2)}_{\frac{1}{2}}}{da};$$

$$(89)\ldots \qquad R_3=\qquad -\frac{2n^3a^2}{(n-n')a'}b^{(3)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(3)}_{\frac{1}{2}}}{da};\ \&c.$$

The integral of (85) is

$$(90)\ldots \quad r_{\prime\prime}=-\frac{R_0}{2n^2}+\frac{R_1\cos(nt-n't+\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{R_2\cos 2(nt-n't+\epsilon-\epsilon')}{4(n-n')^2-n^2} \\ +\frac{R_3\cos 3(nt-n't+\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.+R_{\prime}\cos(nt+\epsilon)+R_{\prime\prime}\sin(nt+\epsilon);$$

in which the arbitrary constants $R_{\prime}$ & $R_{\prime\prime}$ may be determined by the conditions before proposed to be employed, namely that $r_{\prime\prime}$ & $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t; and thus we shall have

$$(91)\ldots \quad R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon=\frac{R_0}{2n^2}-\frac{R_1\cos(\epsilon-\epsilon')}{(n-n')^2-n^2}-\frac{R_2\cos 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}-\frac{R_3\cos 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}-\&c.;$$

$$(92)\ldots \quad R_{\prime}\sin\epsilon-R_{\prime\prime}\cos\epsilon=-\frac{n-n'}{n}\left\{\frac{R_1\sin(\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{2R_2\sin 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}+\frac{3R_3\sin 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.\right\}.$$

We have ∵ determined $r_{\prime\prime}$, which is, very nearly, the perturbation of the radius vector of Jupiter, divided by the mass of Saturn or the coefficient of this latter disturbing mass, in the

---

* [Tome I, Livre 2, No. 49.]

& ∵

$$(78)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}-3n^2r_{\prime\prime}-2na\frac{d\theta_{\prime\prime}}{dt}-2n^2a'\cos(nt-n't+\epsilon-\epsilon')$$
$$+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}};$$

$$(79)\ldots\quad 0=-a\frac{d}{dt}\frac{d\theta_{\prime\prime}}{dt}-2n\frac{dr_{\prime\prime}}{dt}$$
$$+n^2a'\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

This last equation can be immediately integrated and gives

$$(80)\ldots\quad -a\frac{d\theta_{\prime\prime}}{dt}=2nr_{\prime\prime}+\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(nt-n't+\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}$$
$$-\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\};$$

if we suppose, which we are at liberty to do, that ξ, η, $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, & ∵ $r_{\prime\prime}$, $\theta_{\prime\prime}$, $\frac{dr_{\prime\prime}}{dt}$, $\frac{d\theta_{\prime\prime}}{dt}$, vanish with t. On this last supposition we shall have by integrating (80)

$$(81)\ldots\quad a\theta_{\prime\prime}=-2n\int_0^t r_{\prime\prime}\,dt+\frac{n^2a't}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}-\frac{n^2a'}{(n-n')^2}$$
$$\times\left(-\sin(\epsilon-\epsilon')+\sin(nt-n't+\epsilon-\epsilon')\right)+\frac{n^2a^2a'^{-1}}{n-n'}\int_0^t\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}dt.$$

We have also, by substituting (80) in (78),

$$(82)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\frac{2n^2n'a'}{n-n'}\cos(nt-n't+\epsilon-\epsilon')$$
$$-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}$$
$$-\frac{2n^3a^2a'^{-1}}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}.$$

We are ∵ to integrate this equation and to determine the constants by the condition that $r_{\prime\prime}$ and $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t.

[*Perturbations of Jupiter obtained and compared with results of Laplace.*]

[**10.**] Since $a'>a$, we have, nearly,

$$(83)\ldots\quad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=1+\frac{a}{a'}\cos\omega+\frac{1}{2}\frac{a^2}{a'^2}(-1+3\cos\omega^2)$$
$$=1+\frac{a}{a'}\cos\omega+\frac{1}{4}\frac{a^2}{a'^2}(1+3\cos2\omega):$$

or, exactly, in the notation of the *Méc. Cél.*[*]

$$(84)\ldots \qquad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=\tfrac{1}{2}b_{\frac{1}{2}}^{(0)}+b_{\frac{1}{2}}^{(1)}\cos\omega+b_{\frac{1}{2}}^{(2)}\cos 2\omega+\&c.;$$

$$\left[\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{3}{2}}=1+3\frac{a}{a'}\cos\omega+\frac{3}{4}\frac{a^2}{a'^2}(3+5\cos 2\omega)+\&c.\right.$$
$$\left.=\tfrac{1}{2}b_{\frac{3}{2}}^{(0)}+b_{\frac{3}{2}}^{(1)}\cos\omega+b_{\frac{3}{2}}^{(2)}\cos 2\omega+\&c.;\right]$$

$\because$ if we put (82) under the form

$$(85)\ldots \quad \frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\tfrac{1}{2}R_0+R_1\cos(nt-n't+\epsilon-\epsilon')+R_2\cos 2(nt-n't+\epsilon-\epsilon')+\&c.=0,$$

we have

$$(86)\ldots \quad \tfrac{1}{2}R_0=-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1-2\frac{a}{a'}\cos\overline{\epsilon-\epsilon'}+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}\right\}-\frac{n^3a^2a'^{-1}}{n-n'}b_{\frac{1}{2}}^{(0)}-\frac{1}{2a'}\frac{db_{\frac{1}{2}}^{(0)}}{da};$$

observing that

$$\frac{a-a'\cos\omega}{(a^2-2aa'\cos\omega+a'^2)^{\frac{3}{2}}}=-\frac{a}{da}(a^2-2aa'\cos\omega+a'^2)^{-\frac{1}{2}}$$
$$=-\frac{1}{a'}\frac{d}{da}\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=-\frac{1}{2a'}\frac{db_{\frac{1}{2}}^{(0)}}{da}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(1)}}{da}\cos\omega-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(2)}}{da}\cos 2\omega-\&c.;$$

and similarly

$$(87)\ldots \qquad R_1=\frac{2n^2n'a'}{n-n'}-\frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(1)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(1)}}{da};$$

$$(88)\ldots \qquad R_2=\qquad -\frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(2)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(2)}}{da};$$

$$(89)\ldots \qquad R_3=\qquad \frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(3)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(3)}}{da};\ \ \&c.\Big|$$

The integral of (85) is

$$(90)\ldots \quad r_{\prime\prime}=-\frac{R_0}{2n^2}+\frac{R_1\cos(nt-n't+\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{R_2\cos 2(nt-n't+\epsilon-\epsilon')}{4(n-n')^2-n^2}$$
$$+\frac{R_3\cos 3(nt-n't+\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.+R_{\prime}\cos(nt+\epsilon)+R_{\prime\prime}\sin(nt+\epsilon);$$

in which the arbitrary constants $R_{\prime}$ & $R_{\prime\prime}$ may be determined by the conditions before proposed to be employed, namely that $r_{\prime\prime}$ & $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t; and thus we shall have

$$(91)\ldots \quad R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon=\frac{R_0}{2n^2}-\frac{R_1\cos(\epsilon-\epsilon')}{(n-n')^2-n^2}-\frac{R_2\cos 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}-\frac{R_3\cos 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}-\&c.;$$

$$(92)\ldots \quad R_{\prime}\sin\epsilon-R_{\prime\prime}\cos\epsilon=-\frac{n-n'}{n}\left\{\frac{R_1\sin(\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{2R_2\sin 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}+\frac{3R_3\sin 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.\right\}.$$

We have $\because$ determined $r_{\prime\prime}$, which is, very nearly, the perturbation of the radius vector of Jupiter, divided by the mass of Saturn or the coefficient of this latter disturbing mass, in the

---

[*] [Tome I, Livre 2, No. 49.]

& ∵

$$(78)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}-3n^2r_{\prime\prime}-2na\frac{d\theta_{\prime\prime}}{dt}-2n^2a'\cos(nt-n't+\epsilon-\epsilon')$$
$$+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}};$$

$$(79)\ldots\quad 0=-a\frac{d}{dt}\frac{d\theta_{\prime\prime}}{dt}-2n\frac{dr_{\prime\prime}}{dt}$$
$$+n^2a'\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

This last equation can be immediately integrated and gives

$$(80)\ldots\quad -a\frac{d\theta_{\prime\prime}}{dt}=2nr_{\prime\prime}+\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(nt-n't+\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right.$$
$$\left.-\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\};$$

if we suppose, which we are at liberty to do, that ξ, η, $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, & ∵ $r_{\prime\prime}$, $\theta_{\prime\prime}$, $\frac{dr_{\prime\prime}}{dt}$, $\frac{d\theta_{\prime\prime}}{dt}$, vanish with t. On this last supposition we shall have by integrating (80)

$$(81)\ldots\quad a\theta_{\prime\prime}=-2n\int_0^t r_{\prime\prime}\,dt+\frac{n^2a't}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}-\frac{n^2a'}{(n-n')^2}$$
$$\times\left(-\sin(\epsilon-\epsilon')+\sin(nt-n't+\epsilon-\epsilon')\right)+\frac{n^2a^2a'^{-1}}{n-n'}\int_0^t\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}dt.$$

We have also, by substituting (80) in (78),

$$(82)\ldots\quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\frac{2n^2n'a'}{n-n'}\cos(nt-n't+\epsilon-\epsilon')$$
$$-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}$$
$$-\frac{2n^3a^2a'^{-1}}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}.$$

We are ∵ to integrate this equation and to determine the constants by the condition that $r_{\prime\prime}$ and $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t.

[*Perturbations of Jupiter obtained and compared with results of Laplace.*]

[**10.**] Since $a'>a$, we have, nearly,

$$(83)\ldots\quad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=1+\frac{a}{a'}\cos\omega+\frac{1}{2}\frac{a^2}{a'^2}(-1+3\cos\omega^2)$$
$$=1+\frac{a}{a'}\cos\omega+\frac{1}{4}\frac{a^2}{a'^2}(1+3\cos2\omega);$$

or, exactly, in the notation of the *Méc. Cél.**

$$(84)\ldots \qquad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=\tfrac{1}{2}b^{(0)}_{\frac{1}{2}}+b^{(1)}_{\frac{1}{2}}\cos\omega+b^{(2)}_{\frac{1}{2}}\cos 2\omega+\&c.;$$

$$\left[\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{3}{2}}=1+3\frac{a}{a'}\cos\omega+\frac{3}{4}\frac{a^2}{a'^2}(3+5\cos 2\omega)+\&c.\right.$$
$$\left.=\tfrac{1}{2}b^{(0)}_{\frac{3}{2}}+b^{(1)}_{\frac{3}{2}}\cos\omega+b^{(2)}_{\frac{3}{2}}\cos 2\omega+\&c.;\right]$$

∵ if we put (82) under the form

$$(85)\ldots \qquad \frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\tfrac{1}{2}R_0+R_1\cos(nt-n't+\epsilon-\epsilon')+R_2\cos 2(nt-n't+\epsilon-\epsilon')+\&c.=0,$$

we have

$$(86)\ldots \qquad \tfrac{1}{2}R_0=-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1-2\frac{a}{a'}\cos\overline{\epsilon-\epsilon'}+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}\right\}-\frac{n^3a^2a'^{-1}}{n-n'}b^{(0)}_{\frac{1}{2}}-\frac{1}{2a'}\frac{db^{(0)}_{\frac{1}{2}}}{da};$$

observing that

$$\frac{a-a'\cos\omega}{(a^2-2aa'\cos\omega+a'^2)^{\frac{3}{2}}}=-\frac{a}{da}(a^2-2aa'\cos\omega+a'^2)^{-\frac{1}{2}}$$
$$=-\frac{1}{a'}\frac{d}{da}\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=-\frac{1}{2a'}\frac{db^{(0)}_{\frac{1}{2}}}{da}-\frac{1}{a'}\frac{db^{(1)}_{\frac{1}{2}}}{da}\cos\omega-\frac{1}{a'}\frac{db^{(2)}_{\frac{1}{2}}}{da}\cos 2\omega-\&c.;$$

and similarly

$$(87)\ldots \qquad R_1=\frac{2n^2n'a'}{n-n'}-\frac{2n^3a^2}{(n-n')a'}b^{(1)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(1)}_{\frac{1}{2}}}{da};$$

$$(88)\ldots \qquad R_2=\qquad -\frac{2n^3a^2}{(n-n')a'}b^{(2)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(2)}_{\frac{1}{2}}}{da}.$$

$$(89)\ldots \qquad R_3=\qquad -\frac{2n^3a^2}{(n-n')a'}b^{(3)}_{\frac{1}{2}}-\frac{1}{a'}\frac{db^{(3)}_{\frac{1}{2}}}{da};\quad \&c. \Big|$$

The integral of (85) is

$$(90)\ldots \qquad r_{\prime\prime}=-\frac{R_0}{2n^2}+\frac{R_1\cos(nt-n't+\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{R_2\cos 2(nt-n't+\epsilon-\epsilon')}{4(n-n')^2-n^2}$$
$$+\frac{R_3\cos 3(nt-n't+\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.+R_{\prime}\cos(nt+\epsilon)+R_{\prime\prime}\sin(nt+\epsilon);$$

in which the arbitrary constants $R_{\prime}$ & $R_{\prime\prime}$ may be determined by the conditions before proposed to be employed, namely that $r_{\prime\prime}$ & $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t; and thus we shall have

$$(91)\ldots \qquad R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon=\frac{R_0}{2n^2}-\frac{R_1\cos(\epsilon-\epsilon')}{(n-n')^2-n^2}-\frac{R_2\cos 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}-\frac{R_3\cos 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}-\&c.;$$

$$(92)\ldots \qquad R_{\prime}\sin\epsilon-R_{\prime\prime}\cos\epsilon=-\frac{n-n'}{n}\left\{\frac{R_1\sin(\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{2R_2\sin 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}+\frac{3R_3\sin 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&c.\right\}.$$

We have ∵ determined $r_{\prime\prime}$, which is, very nearly, the perturbation of the radius vector of Jupiter, divided by the mass of Saturn or the coefficient of this latter disturbing mass, in the

& ∴

$$(78)\ldots \quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}-3n^2r_{\prime\prime}-2na\frac{d\theta_{\prime\prime}}{dt}-2n^2a'\cos(nt-n't+\epsilon-\epsilon') + \frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}};$$

$$(79)\ldots \quad 0=-a\frac{d}{dt}\frac{d\theta_{\prime\prime}}{dt}-2n\frac{dr_{\prime\prime}}{dt} + n^2a'\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

This last equation can be immediately integrated and gives

$$(80)\ldots \quad -a\frac{d\theta_{\prime\prime}}{dt}=2nr_{\prime\prime}+\frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(nt-n't+\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\} - \frac{n^2a}{n-n'}\left\{\frac{a'}{a}\cos(\epsilon-\epsilon')-\left(1+\frac{a'^2}{a^2}-\frac{2a'}{a}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\};$$

if we suppose, which we are at liberty to do, that ξ, η, $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, & ∴ $r_{\prime\prime}$, $\theta_{\prime\prime}$, $\frac{dr_{\prime\prime}}{dt}$, $\frac{d\theta_{\prime\prime}}{dt}$, vanish with t. On this last supposition we shall have by integrating (80)

$$(81)\ldots \quad a\theta_{\prime\prime}=-2n\int_0^t r_{\prime\prime}\,dt+\frac{n^2a't}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}}\right\}-\frac{n^2a'}{(n-n')^2} \left(-\sin(\epsilon-\epsilon')+\sin(nt-n't+\epsilon-\epsilon')\right)+\frac{n^2a^2a'^{-1}}{n-n'}\int_0^t\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}dt.$$

We have also, by substituting (80) in (78),

$$(82)\ldots \quad 0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\frac{2n^2n'a'}{n-n'}\cos(nt-n't+\epsilon-\epsilon') -\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)\right. -\frac{2n^3a^2a'^{-1}}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}}+\frac{a-a'\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}}.$$

We are ∴ to integrate this equation and to determine the constants by the condition that $r_{\prime\prime}$ and $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t.

[*Perturbations of Jupiter obtained and compared with results of Laplace.*]

[10.] Since $a'>a$, we have, nearly,

$$(83)\ldots \quad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=1+\frac{a}{a'}\cos\omega+\frac{1}{2}\frac{a^2}{a'^2}(-1+3\cos\omega^2)$$

$$=1+\frac{a}{a'}\cos\omega+\frac{1}{4}\frac{a^2}{a'^2}(1+3\cos 2\omega);$$

or, exactly, in the notation of the *Méc. Cél.**

$$(84)\ldots \qquad \left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=\tfrac{1}{2}b_{\frac{1}{2}}^{(0)}+b_{\frac{1}{2}}^{(1)}\cos\omega+b_{\frac{1}{2}}^{(2)}\cos 2\omega+\&\text{c.};$$

$$\left[\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{3}{2}}=1+3\frac{a}{a'}\cos\omega+\frac{3}{4}\frac{a^2}{a'^2}(3+5\cos 2\omega)+\&\text{c}\right.$$
$$\left.=\tfrac{1}{2}b_{\frac{3}{2}}^{(0)}+b_{\frac{3}{2}}^{(1)}\cos\omega+b_{\frac{3}{2}}^{(2)}\cos 2\omega+\&\text{c.};\right]$$

∵ if we put (82) under the form

$$(85)\ldots \quad \frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+\tfrac{1}{2}R_0+R_1\cos(nt-n't+\epsilon-\epsilon')+R_2\cos 2(nt-n't+\epsilon-\epsilon')+\&\text{c.}=0,$$

we have

$$(86)\ldots \quad \tfrac{1}{2}R_0=-\frac{2n^3a'}{n-n'}\left\{\cos(\epsilon-\epsilon')-\frac{a^2}{a'^2}\left(1-2\frac{a}{a'}\cos\overline{\epsilon-\epsilon'}+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}\right\}-\frac{n^3a^2a'^{-1}}{n-n'}b_{\frac{1}{2}}^{(0)}-\frac{1}{2a'}\frac{db_{\frac{1}{2}}^{(0)}}{da};$$

observing that

$$\frac{a-a'\cos\omega}{(a^2-2aa'\cos\omega+a'^2)^{\frac{3}{2}}}=-\frac{d}{da}(a^2-2aa'\cos\omega+a'^2)^{-\frac{1}{2}}$$
$$=-\frac{1}{a'}\frac{d}{da}\left(1-\frac{2a}{a'}\cos\omega+\frac{a^2}{a'^2}\right)^{-\frac{1}{2}}=-\frac{1}{2a'}\frac{db_{\frac{1}{2}}^{(0)}}{da}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(1)}}{da}\cos\omega-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(2)}}{da}\cos 2\omega-\&\text{c.};$$

and similarly

$$(87)\ldots \qquad R_1=\frac{2n^2n'a'}{n-n'}-\frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(1)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(1)}}{da};$$

$$(88)\ldots \qquad R_2=\qquad\frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(2)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(2)}}{da};$$

$$(89)\ldots \qquad R_3=\qquad\frac{2n^3a^2}{(n-n')a'}b_{\frac{1}{2}}^{(3)}-\frac{1}{a'}\frac{db_{\frac{1}{2}}^{(3)}}{da};\ \ \&\text{c.}$$

The integral of (85) is

$$(90)\ldots \quad r_{\prime\prime}=-\frac{R_0}{2n^2}+\frac{R_1\cos(nt-n't+\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{R_2\cos 2(nt-n't+\epsilon-\epsilon')}{4(n-n')^2-n^2}$$
$$+\frac{R_3\cos 3(nt-n't+\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&\text{c.}+R_{\prime}\cos(nt+\epsilon)+R_{\prime\prime}\sin(nt+\epsilon);$$

in which the arbitrary constants $R_{\prime}$ & $R_{\prime\prime}$ may be determined by the conditions before proposed to be employed, namely that $r_{\prime\prime}$ & $\frac{dr_{\prime\prime}}{dt}$ shall vanish with t; and thus we shall have

$$(91)\ldots \quad R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon=\frac{R_0}{2n^2}-\frac{R_1\cos(\epsilon-\epsilon')}{(n-n')^2-n^2}-\frac{R_2\cos 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}-\frac{R_3\cos 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}-\&\text{c.};$$

$$(92)\ldots \quad R_{\prime}\sin\epsilon-R_{\prime\prime}\cos\epsilon=-\frac{n-n'}{n}\left\{\frac{R_1\sin(\epsilon-\epsilon')}{(n-n')^2-n^2}+\frac{2R_2\sin 2(\epsilon-\epsilon')}{4(n-n')^2-n^2}+\frac{3R_3\sin 3(\epsilon-\epsilon')}{9(n-n')^2-n^2}+\&\text{c.}\right\}.$$

We have ∵ determined $r_{\prime\prime}$, which is, very nearly, the perturbation of the radius vector of Jupiter, divided by the mass of Saturn or the coefficient of this latter disturbing mass, in the expression of the former radius vector. It must be observed that the radius vector here is measured from the centre of gravity of the system. The distance from the Sun is

---

* [Tome I, Livre 2, No. 49.]

$$(93)\ldots\quad \sqrt{\{(x_{,}+m'\xi+mx_{,}+m'x'_{,})^2+(y_{,}+m'\eta+my_{,}+m'y'_{,})^2\}}$$
$$=(1+m)\sqrt{(x_{,}^2+y_{,}^2)}+m'r_{\prime\prime}+m'a'\cos(nt-n't+\epsilon-\epsilon')$$

[so that in order to have $r_{\prime\prime}$ in (90) refer to the perturbation of the distance from Jupiter to Sun] we must add to R_1 the term $a'n'(n'-2n)$, & consequently change in (87) the term $\frac{2n^2n'a'}{n-n'}$ to $\frac{a'n'}{n-n'}\{2n^2+(n-n')(n'-2n)\}$, that is, to $\frac{a'n'^2(3n-n')}{n-n'}$; and then it will agree with the result in the *Méc. Cél.* In fact, we find then for the coefficient corresponding to my R_i, when $i>0$, the expression $\frac{dA^{(i)}}{da}+\frac{2nA^{(i)}}{(n-n')a}$, in which $A^{(i)}=-\frac{1}{a'}b^{(i)}_{\frac{1}{2}}$ if $i>1$ and $A^{(1)}=\frac{a}{a'^2}-\frac{1}{a'}b^{(1)}_{\frac{1}{2}}$; and

$$\left(\frac{d}{da}+\frac{2na^{-1}}{n-n'}\right)\frac{a}{a'^2}=\left(1+\frac{2n}{n-n'}\right)a'n'^2=\frac{a'n'^2(3n-n')}{n-n'}\ \text{as above.}$$

[11.] Substituting in (81) the value (90) of $r_{\prime\prime}$, we find

$$(94)\ldots\quad a\theta_{\prime\prime}=\int_0^t\{\psi(t)-\psi(0)\}\,dt;$$

$$(95)\ldots\quad \psi(t)=-2nr_{\prime\prime}+\frac{n^2a'}{n-n'}\left\{-\cos(nt-n't+\epsilon-\epsilon')+\frac{1}{2}\frac{a^2}{a'^2}\sum_{(i)-\infty}^{\infty}b^{(i)}_{\frac{1}{2}}\cos i(nt-n't+\epsilon-\epsilon')\right\};$$

that is,

$$(96)\ldots\quad \psi(t)=-2nR_{\prime}\cos(nt+\epsilon)-2nR_{\prime\prime}\sin(nt+\epsilon)+\tfrac{1}{2}\sum_{(i)-\infty}^{\infty}\Psi_i\cos i(nt-n't+\epsilon-\epsilon');$$

in which

$$(97)\ldots\quad \Psi_0=\frac{2R_0}{n}+\frac{n^2a^2b^{(0)}_{\frac{1}{2}}}{a'(n-n')};$$

$$(98)\ldots\quad \Psi_1=-\frac{n^2a'}{n-n'}-\frac{2nR_1}{(n-n')^2-n^2}+\frac{n^2a^2b^{(1)}_{\frac{1}{2}}}{a'(n-n')};$$

$$(99)\ldots\quad \Psi_2=\frac{2nR_2}{4(n-n')^2-n^2}+\frac{n^2a^2b^{(2)}_{\frac{1}{2}}}{a'(n-n')},$$

and generally, when $i>1$,

$$(100)\ldots\quad \Psi_i=-\frac{2nR_i}{i^2(n-n')^2-n^2}+\frac{n^2a^2b^{(i)}_{\frac{1}{2}}}{a'(n-n')};\qquad (\Psi_{-i}=\Psi_i,\ b^{(-i)}_{\frac{1}{2}}=b^{(i)}_{\frac{1}{2}}).$$

Hence, by (94),

$$(101)\ldots\quad a\theta_{\prime\prime}=-2nR_{\prime}\int_0^t\cos(nt+\epsilon)\,dt-2nR_{\prime\prime}\int_0^t\sin(nt+\epsilon)\,dt$$
$$+\sum_{(i)1}^{\infty}\Psi_i\left(\int_0^t\cos i(n\quad n't+\epsilon-\epsilon')\,dt-t\cos i(\epsilon-\epsilon')\right)+2nt(R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon);$$

that is,

$$(102)\ldots\quad \theta_{\prime\prime}=-\frac{2R_{\prime}}{a}\Big(\sin(nt+\epsilon)-\sin\epsilon\Big)-\frac{2R_{\prime\prime}}{a}\Big(\cos\epsilon-\cos(nt+\epsilon)\Big)$$
$$+\frac{t}{a}\left\{2n(R_{\prime}\cos\epsilon+R_{\prime\prime}\sin\epsilon)-\sum_{(i)1}^{\infty}\Psi_i\cos i(\epsilon-\epsilon')\right\}+\sum_{(i)1}^{\infty}\frac{\Psi_i}{i}\frac{\sin i(nt-n't+\epsilon-\epsilon')-\sin i(\epsilon-\epsilon')}{a(n-n')}.$$

This result is concerned with the perturbation of longitude of Jupiter referred to the centre of gravity of the system. For this perturbation

$$(103)\ldots\quad = m'\frac{\delta}{\delta m'}\tan^{-1}\frac{y}{x} = m'\,\frac{x_{,}\dfrac{\delta y}{\delta m'} - y_{,}\dfrac{\delta x}{\delta m'}}{x_{,}^2 + y_{,}^2} = \frac{m'}{a}\{\eta\cos(nt+\epsilon) - \xi\sin(nt+\epsilon)\} = m'\theta_{\prime\prime};$$

and if we refer the longitude to the Sun, then its perturbation

$$(104)\ldots\quad = m'\frac{\delta}{\delta m'}\tan^{-1}\frac{y+my+m'y'}{x+mx+m'x'}$$

$$= \frac{m'}{x_{,}^2+y_{,}^2}\{x_{,}(y'+\eta) - y_{,}(x'+\xi)\} = \frac{m'}{a}\{(\eta+y')\cos(nt+\epsilon) - (\xi+x')\sin(nt+\epsilon)\}$$

$$= m'\theta_{\prime\prime} - \frac{m'a'}{a}\sin(nt-n't+\epsilon-\epsilon').$$

Hence, in the perturbation of Jupiter's heliocentric longitude, the coefficient of

$$-\frac{m'\sin(nt-n't+\epsilon-\epsilon')}{aa'(n-n')^2\{(n-n')^2-n^2\}}$$

is $(105)\ldots\quad a'\{(n-n')^2a' - \Psi_1(n-n')\}\{(n-n')^2-n^2\}$

$$= 2nR_1(n-n')a' - n^2a^2b^{(1)}_{\frac{1}{2}}\{(n-n')^2-n^2\} + a'^2\{(n-n')^4-n^4\}$$

$$= a'^2\{(n-n')^4 - n^4 + 4n^3n'\} - n^2a^2b^{(1)}_{\frac{1}{2}}\{(n-n')^2+3n^2\} - 2n(n-n')\frac{db^{(1)}_{\frac{1}{2}}}{da}$$

$$= a'^2n'^2(n'^2-4nn'+6n^2) - n^2a^2b^{(1)}_{\frac{1}{2}}(4n^2-2nn'+n'^2) - 2n(n-n')\frac{db^{(1)}_{\frac{1}{2}}}{da}.$$

The *Méc. Cél.* gives for the same coefficient

$$n^2a^2a'A^{(1)}(n'^2-2nn') + 2n^3a^2a'\left\{a(n-n')\frac{dA^{(1)}}{da} + 2nA^{(1)}\right\}$$

$$= n^2a^2a'(4n^2-2nn'+n'^2)A^{(1)} + 2n(n-n')a'\frac{dA^{(1)}}{da},$$

in which

$$A^{(1)} = \frac{a}{a'^2} - \frac{1}{a'}b^{(1)}_{\frac{1}{2}};\quad\text{and}\quad 4n^2-2nn'+n'^2+2n(n-n') = n'^2-4nn'+6n^2;$$

∵ the coefficients agree. When $i > 1$, my coefficient of $\dfrac{m'\sin i(nt-n't+\epsilon-\epsilon')}{aa'(n-n')^2\{i^2(n-n')^2-n^2\}}$ is

$$(106)\ldots\quad n^2a^2b^{(i)}_{\frac{1}{2}}\{i^2(n-n')^2-n^2\} - 2na'R_i(n-n')$$

$$= n^2a^2b^{(i)}_{\frac{1}{2}}\{i^2(n-n')^2+3n^2\} + 2n(n-n')\frac{db^{(i)}_{\frac{1}{2}}}{da};$$

while the corresponding coefficient in the *Méc. Cél.* is

$$-a^2n^2a'A^{(i)}\{i^2(n-n')^2-n^2\} - 2a^2n^3a'\left\{a(n-n')\frac{dA^{(i)}}{da} + 2nA^{(i)}\right\}$$

$$= -a^2n^2a'A^{(i)}\{i^2(n-n')^2+3n^2\} - 2n(n-n')a'\frac{dA^{(i)}}{da},$$

in which $A^{(i)} = -\dfrac{1}{a'}\dfrac{db^{(i)}_{\frac{1}{2}}}{da}$; the results therefore agree. My coefficients are therefore the same as those of Laplace for the cosines and sines of the multiples of the elongation in the perturbations produced by Saturn of the heliocentric radius vector and longitude of Jupiter; but I have a different constant term in the perturbation of the radius vector and a term proportional to the time in the perturbation of longitude. I have also, in both perturbations, terms depending on the mean longitude of Jupiter and independent of that of Saturn. These differences arise from my different manner of determining the arbitrary constants introduced by integration in the expressions for the two perturbations.*

[*Perturbations of Saturn.*]

[12.] To investigate now the perturbations of Saturn, that is, to integrate the equations (68) (69), multiplying the former by $\cos(n't+\epsilon')$ and the latter by $\sin(n't+\epsilon')$ and adding the products, we get

$$(107)\ldots\quad 0=\cos(n't+\epsilon')\frac{d}{dt}\frac{d\xi'}{dt}+\sin(n't+\epsilon')\frac{d}{dt}\frac{d\eta'}{dt}-2n'^2\{\xi'\cos(n't+\epsilon')+\eta'\sin(n't+\epsilon')$$
$$+a\cos(nt-n't+\epsilon-\epsilon')\}+\{a'-a\cos(nt-n't+\epsilon-\epsilon')\}\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{-\frac{3}{2}};$$

& similarly, by another combination,

$$(108)\ldots\quad 0=\sin(n't+\epsilon')\frac{d}{dt}\frac{d\xi'}{dt}-\cos(n't+\epsilon')\frac{d}{dt}\frac{d\eta'}{dt}+n'^2\{\xi'\sin(n't+\epsilon')-\eta'\cos(n't+\epsilon')\}$$
$$-n'^2a\sin(nt-n't+\epsilon-\epsilon')\left\{1-\left(1+\frac{a^2}{a'^2}-2\frac{a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

If then we put

$$(109)\ldots\qquad \xi'\cos(n't+\epsilon')+\eta'\sin(n't+\epsilon')=r'_{\prime\prime};$$

$$(110)\ldots\qquad -\xi'\sin(n't+\epsilon')+\eta'\cos(n't+\epsilon')=a'\theta'_{\prime\prime};$$

and ∵

$$(111)\ldots\qquad \xi'=r'_{\prime\prime}\cos(n't+\epsilon')-a'\theta'_{\prime\prime}\sin(n't+\epsilon');$$

$$(112)\ldots\qquad \eta'=r'_{\prime\prime}\sin(n't+\epsilon')+a'\theta'_{\prime\prime}\cos(n't+\epsilon');$$

we shall have, by differentiation,

$$(113)\ldots\qquad \frac{d\xi'}{dt}=\frac{dr'_{\prime\prime}}{dt}\cos(\quad)-n'r'_{\prime\prime}\sin(\quad)-a'\frac{d\theta'_{\prime\prime}}{dt}\sin(\quad)-a'n'\theta'_{\prime\prime}\cos(\quad);$$

$$(114)\ldots\qquad \frac{d\eta'}{dt}=\frac{dr'_{\prime\prime}}{dt}\sin(\quad)+n'r'_{\prime\prime}\cos(\quad)+a'\frac{d\theta'_{\prime\prime}}{dt}\cos(\quad)-a'n'\theta'_{\prime\prime}\sin(\quad);$$

$$(115)\ldots\quad \frac{d}{dt}\frac{d\xi'}{dt}=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}\cos\qquad -a'\frac{d}{dt}\frac{d\theta'_{\prime\prime}}{dt}\sin\qquad -2n'\frac{dr'_{\prime\prime}}{dt}\sin\qquad -2n'a'\frac{d\theta'_{\prime\prime}}{dt}\cos\qquad -n'^2\xi';$$

$$(116)\ldots\quad \frac{d}{dt}\frac{d\eta'}{dt}=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}\sin\qquad +a'\frac{d}{dt}\frac{d\theta'_{\prime\prime}}{dt}\cos\qquad +2n'\frac{dr'_{\prime\prime}}{dt}\cos\qquad -2n'a'\frac{d\theta'_{\prime\prime}}{dt}\sin\qquad -n'^2\eta';$$

$$(117)\ldots\quad \frac{d}{dt}\frac{d\xi'}{dt}\cos\qquad +\frac{d}{dt}\frac{d\eta'}{dt}\sin\qquad =\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}-2n'a'\frac{d\theta'_{\prime\prime}}{dt}-n'^2r'_{\prime\prime};$$

$$(118)\ldots\quad \frac{d}{dt}\frac{d\xi'}{dt}\sin\qquad -\frac{d}{dt}\frac{d\eta'}{dt}\cos\qquad =-a'\frac{d}{dt}\frac{d\theta'_{\prime\prime}}{dt}-2n'\frac{dr'_{\prime\prime}}{dt}+a'n'^2\theta'_{\prime\prime};$$

---

* [In this investigation the "unperturbed" orbit touches the orbit at the time $t=0$. Laplace makes the two orbits have the same mean angular velocity.]

∵ our differential equations become

$$(119)\ldots\quad 0=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}-3n'^2r'_{\prime\prime}-2n'a'\frac{d\theta'_{\prime\prime}}{dt}-2n'^2a\cos(nt-n't+\epsilon-\epsilon') \\ +\frac{a'-a\cos(nt-n't+\epsilon-\epsilon')}{\{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')\}^{\frac{3}{2}}};$$

$$(120)\ldots\quad 0=-a'\frac{d}{dt}\frac{d\theta'_{\prime\prime}}{dt}-2n'\frac{dr'_{\prime\prime}}{dt}-n'^2a\sin(nt-n't+\epsilon-\epsilon') \\ \left\{1-\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{3}{2}}\right\}.$$

The last gives, when we observe that we have supposed $r'_{\prime\prime}$, $\theta'_{\prime\prime}$, $\frac{dr'_{\prime\prime}}{dt}$, $\frac{d\theta'_{\prime\prime}}{dt}$ to vanish with t,

$$(121)\ldots\quad a'\frac{d\theta'_{\prime\prime}}{dt}=\psi'(t)-\psi'(0);$$

and therefore

$$(122)\ldots\quad a'\theta'_{\prime\prime}=\int_0^t\{\psi'(t)-\psi'(0)\}\,dt;$$

in which

$$(123)\ldots\quad \psi'(t)=-2n'r'_{\prime\prime}+\frac{n'^2a}{n-n'}\cos(nt-n't+\epsilon-\epsilon') \\ -\frac{n'^2a'}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(nt-n't+\epsilon-\epsilon')\right)^{-\frac{1}{2}};$$

also

$$(124)\ldots\quad \frac{a'-a\cos\omega}{(a^2+a'^2-2aa'\cos\omega)^{\frac{3}{2}}}=-\frac{d}{da'}(a^2+a'^2-2aa'\cos\omega)^{-\frac{1}{2}}$$

$$=\left(\frac{1}{a'}+\frac{a}{a'}\frac{d}{da}\right)(a^2+a'^2-2aa'\cos\omega)^{-\frac{1}{2}}$$

$$=\left(\frac{1}{a'^2}+\frac{a}{a'^2}\frac{d}{da}\right)\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos\omega\right)^{-\frac{1}{2}}=\frac{1}{a'^2}\sum_{(i)-\infty}^{\infty}\left(1+\frac{ad}{da}\right)\frac{b_{\frac{1}{2}}^{(i)}}{2}\cos i(nt-n't+\epsilon-\epsilon');$$

$$∵\ (125)\ldots\quad 0=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}+n'^2r'_{\prime\prime}+2n'\psi'(0)-\frac{2n'^2na}{n-n'}\cos(nt-n't+\epsilon-\epsilon')+\frac{n'^2a'}{2}\left(\frac{n+n'}{n-n'}\right) \\ \times\sum_{(i)-\infty}^{\infty}b_{\frac{1}{2}}^{(i)}\cos i(nt-n't+\epsilon-\epsilon')+\frac{n'^2a'a}{2}\sum_{(i)-\infty}^{\infty}\frac{db_{\frac{1}{2}}^{(i)}}{da}\cos i(nt-n't+\epsilon-\epsilon');$$

that is,

$$(126)\ldots\quad 0=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}+n'^2r'_{\prime\prime}+\tfrac{1}{2}\sum_{(i)-\infty}^{\infty}R'_i\cos i(nt-n't+\epsilon-\epsilon'),$$

in which

$$(127)\ldots\quad \tfrac{1}{2}R'_0=2n'\psi'(0)+\frac{n'^2a'}{2}\frac{n+n'}{n-n'}b_{\frac{1}{2}}^{(0)}+\frac{n'^2a'a}{2}\frac{db_{\frac{1}{2}}^{(0)}}{da};$$

$$(128)\ldots\quad \psi'(0)=\frac{n'^2a}{n-n'}\cos(\epsilon-\epsilon')-\frac{n'^2a'}{n-n'}\left(1+\frac{a^2}{a'^2}-\frac{2a}{a'}\cos(\epsilon-\epsilon')\right)^{-\frac{1}{2}};$$

$$(129)\ldots\quad R'_1=-\frac{2n'^2na}{n-n'}+\frac{n'^2a'(n+n')}{n-n'}b_{\frac{1}{2}}^{(1)}+n'^2a'a\frac{db_{\frac{1}{2}}^{(1)}}{da};$$

&, when $i > 1$,

(130)... $$R'_i = \frac{n'^2 a' (n + n')}{n - n'} b^{(i)}_{\frac{1}{2}} + n'^2 a' a \frac{db^{(i)}_{\frac{1}{2}}}{da}.$$

Hence, integrating,

(131)... $$r'_{,,} = -\frac{R'_0}{2n'^2} + \sum_{(i)1}^{\infty} \frac{R'_i \cos i (nt - n't + \epsilon - \epsilon')}{i^2 (n - n')^2 - n'^2} + R'_{,} \cos (n't + \epsilon') + R'_{,,} \sin (n't + \epsilon');$$

(132)... $$R'_{,} \cos \epsilon' + R'_{,,} \sin \epsilon' = \frac{R'_0}{2n'^2} - \sum_{(i)1}^{\infty} \frac{R'_i \cos i (\epsilon - \epsilon')}{i^2 (n - n')^2 - n'^2};$$

(133)... $$R'_{,} \sin \epsilon' - R'_{,,} \cos \epsilon' = -\frac{n - n'}{n'} \sum_{(i)1}^{\infty} \frac{i R'_i \sin i (\epsilon - \epsilon')}{i^2 (n - n')^2 - n'^2}.$$

Hence also, by (122) &c.,

(134)... $$\theta'_{,,} = -\frac{2R'_{,}}{a'} \left(\sin (n't + \epsilon') - \sin \epsilon'\right) + \frac{2R'_{,,}}{a'} \left(\cos (n't + \epsilon') - \cos \epsilon'\right)$$
$$+ \frac{t}{a'} \left\{2n' (R'_{,} \cos \epsilon' + R'_{,,} \sin \epsilon') - \sum_{(i)1}^{\infty} \Psi'_i \cos i (\epsilon - \epsilon')\right\}$$
$$+ \sum_{(i)1}^{\infty} \frac{\Psi'_i}{i} \frac{\sin i (nt - n't + \epsilon - \epsilon') - \sin i (\epsilon - \epsilon')}{(n - n') a'};$$

and

(135)... $$a' \frac{d\theta'_{,,}}{dt} = 2n' R'_{,} \left(\cos \epsilon' - \cos (n't + \epsilon')\right) + 2n' R'_{,,} \left(\sin \epsilon' - \sin (n't + \epsilon')\right)$$
$$+ \sum_{(i)1}^{\infty} \Psi'_i \{\cos i (nt - n't + \epsilon - \epsilon') - \cos i (\epsilon - \epsilon')\};$$

in which

(136)... $$\Psi'_1 = \frac{n'^2 a}{n - n'} - \frac{n'^2 a'}{n - n'} b^{(1)}_{\frac{1}{2}} - \frac{2n' R'_1}{(n - n')^2 - n'^2},$$

and, when $i > 1$,

(137)... $$\Psi'_i = -\frac{n'^2 a'}{n - n'} b^{(i)}_{\frac{1}{2}} - \frac{2n' R'_i}{i^2 (n - n')^2 - n'^2}.$$

We have ∵ completely integrated the differential equations (68), (69) and determined the perturbations of Saturn in radius vector and longitude referred to the centre of gravity of the system, namely $mr'_{,,}$ and $m\theta'_{,,}$. The perturbations, referred to the centre of the Sun, involve the additional terms $ma \cos (nt - n't + \epsilon - \epsilon')$ and $\frac{ma}{a'} \sin (nt - n't + \epsilon - \epsilon')$.

It might be worth while to compare the foregoing process of integration in which the coordinates are referred to the centre of gravity of the system with the usual processes referred to the centre of the Sun.

[*Approximate determination of the eight elements a, e, ϵ, ϖ, a', e', ϵ', ϖ' in terms of initial coordinates and initial velocities.*]

[**13.**] We have now obtained approximate integrals of the equations (1) (2) (3) (4), that is, approximate expressions for the coordinates of Jupiter & Saturn, of the form

(138)... $x = x_{,} + m'\xi$; (139)... $y = y_{,} + m'\eta$;

(140)... $x' = x'_{,} + m\xi'$; (141)... $y' = y'_{,} + m\eta'$.

These expressions involve, besides the planetary masses and their times, 8 elements or arbitrary constants a, e, ϵ, ϖ, a', e', ϵ', ϖ', which may be determined by the initial circumstances of the motion, that is, by the initial coordinates x_0, y_0, x'_0, y'_0 & their differential coefficients $\frac{dx_0}{dt}, \ldots \frac{dy'_0}{dt}$. In fact, we have, in our present order of approximation, by (48) etc.

(142)... $$x_0 = a\cos\epsilon + \frac{ae}{2}\cos\,(2\epsilon-\varpi) - 3\,\frac{ae}{2}\cos\varpi;$$

(143)... $$y_0 = a\sin\epsilon + \frac{ae}{2}\sin\,(2\epsilon-\varpi) - 3\,\frac{ae}{2}\sin\varpi;$$

(144)... $$x'_0 = a'\cos\epsilon' + \frac{a'e'}{2}\cos\,(2\epsilon'-\varpi') - 3\,\frac{a'e'}{2}\cos\varpi';$$

(145)... $$y'_0 = a'\sin\epsilon' + \frac{a'e'}{2}\sin\,(2\epsilon'-\varpi') - 3\,\frac{a'e'}{2}\sin\varpi';$$

(146)... $\frac{dx_0}{dt} = -an\sin\epsilon - ane\sin\,(2\epsilon-\varpi)$; (147)... $\frac{dy_0}{dt} = an\cos\epsilon + ane\cos\,(2\epsilon-\varpi)$;

(148)... $\frac{dx'_0}{dt} = -a'n'\sin\epsilon' - a'n'e'\sin\,(2\epsilon'-\varpi')$

(149)... $\frac{dy'_0}{dt} = a'n'\cos\epsilon' + a'n'e'\cos\,(2\epsilon'-\varpi')$;

or, putting

(150)... $x_0 = r_0\cos\theta_0$; (151)... $y_0 = r_0\sin\theta_0$;

(152)... $x'_0 = r'_0\cos\theta'_0$; (153)... $y'_0 = r'_0\sin\theta'_0$;

(154)... $\frac{dr_0}{dt} = \frac{dx_0}{dt}\cos\theta_0 + \frac{dy_0}{dt}\sin\theta_0$; (155)... $\frac{d\theta_0}{dt} = \frac{dy_0}{dt}\,\frac{\cos\theta_0}{r_0} - \frac{dx_0}{dt}\,\frac{\sin\theta_0}{}$;

(156)... $\frac{dr'_0}{dt} =$; (157)...

and neglecting e^2, e'^2, we have

(158)... $r_0 = a - ae\cos\,(\epsilon-\varpi)$; (159)... $\theta_0 = \epsilon + 2e\sin\,(\epsilon-\varpi)$;

(160)... $r'_0 = a' - a'e'\cos\,(\epsilon'-\varpi')$; (161)... $\theta'_0 = \epsilon' + 2e'\sin\,(\epsilon'-\varpi')$;

(162)... $\frac{dr_0}{dt} = ane\sin\,(\epsilon-\varpi)$; (163)... $\frac{d\theta_0}{dt} = n + 2ne\cos\,(\epsilon-\varpi)$;

(164)... $\frac{dr'_0}{dt} = a'n'e'\sin\,(\epsilon'-\varpi')$; (165)... $\frac{d\theta'_0}{dt} = n' + 2n'e'\cos\,(\epsilon'-\varpi')$.

These expressions may either be derived from (52) &c., or in the following manner from (150) &c.:

$$r_0^2 = x_0^2 + y_0^2 = a^2 + ae\cos\,(\epsilon-\varpi) - 3ae\cos\,(\epsilon-\varpi);$$

$$\because\quad r_0 = a - ae\cos\,(\epsilon-\varpi);$$

$$\sin\theta_0 = \sin\epsilon + \frac{e}{2}\{2\sin\epsilon\cos\,(\epsilon-\varpi) + \sin\,(2\epsilon-\varpi) - 3\sin\varpi\}$$

$$= \sin\epsilon + e\,(\sin\overline{2\epsilon-\varpi} - \sin\varpi) = \sin\epsilon + 2e\cos\epsilon\sin\,(\epsilon-\varpi) = \sin\,(\epsilon + 2e\sin\overline{\epsilon-\varpi});$$

$$\theta_0 = \epsilon + 2e\sin\overline{\epsilon-\varpi};$$

$$\because\quad \cos\theta_0 = \cos\epsilon - 2e\sin\epsilon\sin\overline{\epsilon-\varpi};$$

$$\frac{1}{an}\frac{dr_0}{dt} = -(\cos\epsilon - 2e\sin\epsilon\sin\overline{\epsilon-\varpi})(\sin\epsilon + e\sin\overline{2\epsilon-\varpi}) + (\sin\epsilon + 2e\cos\epsilon\sin\overline{\epsilon-\varpi})$$
$$\times(\cos\epsilon + e\cos\overline{2\epsilon-\varpi}) = e(2\sin\overline{\epsilon-\varpi} - \cos\epsilon\sin\overline{2\epsilon-\varpi} + \sin\epsilon\cos\overline{2\epsilon-\varpi}) = e\sin\overline{\epsilon-\varpi};$$

$$\frac{r_0}{an}\frac{d\theta_0}{dt} = (\cos\epsilon - 2e\sin\epsilon\sin\overline{\epsilon-\varpi})(\cos\epsilon + e\cos\overline{2\epsilon-\varpi}) + (\sin\epsilon + 2e\cos\epsilon\sin\overline{\epsilon-\varpi})$$
$$\times(\sin\epsilon + e\sin\overline{2\epsilon-\varpi}) = 1 + e(\cos\epsilon\cos\overline{2\epsilon-\varpi} + \sin\epsilon\sin\overline{2\epsilon-\varpi}) = 1 + e\cos(\epsilon-\varpi),$$

$$\frac{d\theta_0}{dt} = n + 2ne\cos(\epsilon-\varpi).$$

We have now, from the 5 equations (158) (159) (162) (163) & (60) to determine a, n, e, ϵ, ϖ, as functions of r_0, θ_0, $\frac{dr_0}{dt}$, $\frac{d\theta_0}{dt}$, & m, neglecting e^2 and quantities of the same order of smallness. In this approximation we find

(166)... $$r_0^2\frac{d\theta_0}{dt} = a^2 n = \frac{\sqrt{a}}{1+m};$$

(167)... $a = (1+m)^2 r_0^4\left(\frac{d\theta_0}{dt}\right)^2$; and (168)... $n = (1+m)^{-4} r_0^{-6}\left(\frac{d\theta_0}{dt}\right)^{-3}$;

$\therefore$ (169)... $an = (1+m)^{-2} r_0^{-2}\left(\frac{d\theta_0}{dt}\right)^{-1}$; (170)... $\frac{1}{an} = (1+m)^2 r_0^2\frac{d\theta_0}{dt}$;

$\therefore$ (171)... $$ae\sin(\epsilon-\varpi) = (1+m)^4 r_0^6\frac{dr_0}{dt}\left(\frac{d\theta_0}{dt}\right)^3;$$

(172)... $$ae\cos(\epsilon-\varpi) = (1+m)^2 r_0^4\left(\frac{d\theta_0}{dt}\right)^2 - r_0;$$

$\therefore$ (173)... $$\epsilon = \theta_0 - 2(1+m)^2 r_0^2\frac{dr_0}{dt}\frac{d\theta_0}{dt};$$

(& if we neglect m in the expression for e,

(174)... $e\sin(\epsilon-\varpi) = r_0^2\frac{dr_0}{dt}\frac{d\theta_0}{dt}$; (175)... $e\cos(\epsilon-\varpi) = r_0^3\left(\frac{d\theta_0}{dt}\right)^2 - 1$).

Hence the general expressions (52) (53), namely

$$r_{,} = a - ae\cos(nt+\epsilon-\varpi), \quad \theta_{,} = nt + \epsilon + 2e\sin(nt+\epsilon-\varpi),$$

become

(176)... $$r_{,} = (1+m)^2 r_0^4\left(\frac{d\theta_0}{dt}\right)^2(1-\cos nt) + r_0\cos nt + (1+m)^4 r_0^6\frac{dr_0}{dt}\left(\frac{d\theta_0}{dt}\right)^3\sin nt;$$

(177)... $$\theta_{,} = \theta_0 + nt - 2(1+m)^2 r_0^2\frac{dr_0}{dt}\frac{d\theta_0}{dt}(1-\cos nt) + 2\sin nt\left\{1 - (1+m)^{-2} r_0^{-3}\left(\frac{d\theta_0}{dt}\right)^{-2}\right\};$$

or, more concisely, but with equal accuracy, retaining the symbols a, n,

(178)... $$r_{,} = r_0\cos nt + a(1-\cos nt) + \frac{dr_0}{n\,dt}\sin nt;$$

(179)... $$\theta_{,} = \theta_0 + nt - \frac{2}{a}\frac{dr_0}{n\,dt}(1-\cos nt) + 2\left(1-\frac{r_0}{a}\right)\sin nt.$$

In fact, as a verification, these values not only reduce themselves to r_0, θ_0 when t vanishes but have then their differential coefficients equal to $\frac{dr_0}{dt}$ & $n+2n\left(1-\frac{r_0}{a}\right)=n+2n\cos(\epsilon-\varpi)$.

In like manner, for Saturn, we have

$$(180)\ldots\quad r'_{,}=r'_0\cos n't+a'(1-\cos n't)+\frac{dr'_0}{n'dt}\sin n't;$$

$$(181)\ldots\quad \theta'_{,}=\theta'_0+n't-\frac{2}{a'}\frac{dr'_0}{n'dt}(1-\cos n't)+2\left(1-\frac{r'_0}{a'}\right)\sin n't;$$

$$(182)\ldots\quad a'=(1+m')^2 r_0'^4\left(\frac{d\theta'_0}{dt}\right)^2;\qquad (183)\ldots\quad n'=(1+m')^{-4}r_0'^{-6}\left(\frac{d\theta'_0}{dt}\right)^{-3}.$$

And if we put

$$(184)\ldots\quad x=r\cos\theta;\qquad (185)\ldots\quad y=r\sin\theta;\qquad (186)\ldots\quad x'=r'\cos\theta';\qquad (187)\ldots\quad y'=r'\sin\theta';$$

we have, very nearly,

$$(188)\ldots\quad r=r_{,}+m'r_{,,};\qquad (189)\ldots\quad \theta=\theta_{,}+m'\theta_{,,};$$

$$(190)\ldots\quad r'=r'_{,}+mr'_{,,};\qquad (191)\ldots\quad \theta'=\theta'_{,}+m\theta'_{,,};$$

in which in the parts $m'r_{,,}$, $m'\theta_{,,}$, $mr'_{,,}$, $m\theta'_{,,}$, we may change a, a' to r_0, r'_0 & n, n' to $r_0^{-\frac{3}{2}}$, $r_0'^{-\frac{3}{2}}$ & $\epsilon-\epsilon'$ to $\theta_0-\theta'_0$.

[*Expressions for* vis viva *in form* $h=H_{,}+mm'H_{,,}$.]

[**14.**] Having thus expressed the 4 polar coordinates r, θ, r', θ' of Jupiter and Saturn, with some approximation, as functions of the time t, the planetary masses m, m', & the 8 initial quantities r_0, θ_0, r'_0, θ'_0, $\frac{dr_0}{dt}$, $\frac{d\theta_0}{dt}$, $\frac{dr'_0}{dt}$, $\frac{d\theta'_0}{dt}$, let us substitute these expressions in the equation of living force, & so not only verify our results, but obtain the relation between the constant h of living force, the masses and the 8 initial quantities.

In the equation (5) of living force change rectangular to polar coordinates; it will become

$$\begin{aligned}(192)\ldots\quad h=&\frac{m}{2}\left\{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right\}(1+m)+\frac{m'}{2}\left\{\left(\frac{dr'}{dt}\right)^2+r'^2\left(\frac{d\theta'}{dt}\right)^2\right\}(1+m')\\ &+mm'\left\{\left(\frac{dr}{dt}\frac{dr'}{dt}+rr'\frac{d\theta}{dt}\frac{d\theta'}{dt}\right)\cos(\theta-\theta')+\left(r'\frac{d\theta'}{dt}\frac{dr}{dt}-r\frac{d\theta}{dt}\frac{dr'}{dt}\right)\sin(\theta-\theta')\right.\\ &\quad 1+m\left\{r^2+\frac{2m'rr'}{1+m}\cos(\theta-\theta')+\left(\frac{m'r'}{1+m}\right)^2\right\}^{-\frac{1}{2}}\\ &-\frac{m'}{1+m'}\left\{r'^2+\frac{2mrr'}{1+m'}\cos(\theta-\theta')+\left(\frac{mr}{1+m'}\right)^2\right\}^{-\frac{1}{2}}\\ &-mm'\{r^2-2rr'\cos(\theta-\theta')+r'^2\}^{-\frac{1}{2}},\end{aligned}$$

neglecting nothing; or, neglecting terms of the 3rd dimension with respect to the masses and the excentricities,

$$\begin{aligned}(193)\ldots\quad h=&\frac{m(1+m)}{2}(r_{,}+m'r_{,,})^2\left(\frac{d\theta_{,}}{dt}+m'\frac{d\theta_{,,}}{dt}\right)^2+\frac{m'(1+m')}{2}(r'_{,}+mr'_{,,})^2\left(\frac{d\theta'_{,}}{dt}+m\frac{d\theta'_{,,}}{dt}\right)^2\\ &+mm'aa'nn'\cos(nt-n't+\epsilon-\epsilon')-mm'\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}}\\ &-\frac{mr^{-1}}{1+m}-\frac{m'r'^{-1}}{1+m'}+mm'\left(\frac{a'}{a^2}+\frac{a}{a'^2}\right)\cos(nt-n't+\epsilon-\epsilon'),\end{aligned}$$

that is,

(194)... $$h = H_{,} + mm' H_{\prime\prime},$$

if we put

(195)... $$H_{,} = \frac{m}{2}\frac{(1+m)^{-1}}{r_{,}}\left\{r_{,}^{3}(1+m)^{2}\left(\frac{d\theta_{,}}{dt}\right)^{2} - 2\right\} + \frac{m'}{2}\frac{(1+m')^{-1}}{r'_{,}}\left\{r'^{3}_{,}(1+m')^{2}\left(\frac{d\theta'_{,}}{dt}\right)^{2} - 2\right\},$$

(196)... $$H_{\prime\prime} = 2n^{2}ar_{\prime\prime} + na^{2}\frac{d\theta_{\prime\prime}}{dt} + 2n'^{2}a'r'_{\prime\prime} + n'a'^{2}\frac{d\theta'_{\prime\prime}}{dt}$$
$$+\left(aa'nn' + \frac{a'}{a^{2}} + \frac{a}{a'^{2}}\right)\cos(nt - n't + \epsilon - \epsilon') - \{a^{2} - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}}.$$

Now, by (178) & (179),

(197)... $$\frac{d\theta_{,}}{dt} = n - \frac{2}{a}\frac{dr_{0}}{dt}\sin nt + 2n\left(1 - \frac{r_{0}}{a}\right)\cos nt;$$

(198)... $$r_{,}^{2}(1+m)^{2}\left(\frac{d\theta_{,}}{dt}\right)^{2} - \frac{2}{r_{,}}$$
$$= \frac{1}{a}\left\{1 - \frac{2}{a}\frac{dr_{0}}{ndt}\sin nt + 2\left(1 - \frac{r_{0}}{a}\right)\cos nt\right\}^{2}\left\{1 + \frac{1}{a}\frac{dr_{0}}{ndt}\sin nt - \left(1 - \frac{r_{0}}{a}\right)\cos nt\right\}^{2}$$
$$- \frac{2}{a}\left\{1 + \frac{1}{a}\frac{dr_{0}}{ndt}\sin nt - \left(1 - \frac{r_{0}}{a}\right)\cos nt\right\}^{-1} = -\frac{1}{a};$$

and similarly

(199)... $$r'^{2}_{,}(1+m')^{2}\left(\frac{d\theta'_{,}}{dt}\right)^{2} - \frac{2}{r'_{,}} = -\frac{1}{a'},$$ $\therefore$ (200)... $$H_{,} = -\frac{m}{2(1+m)a} - \frac{m'}{2(1+m')a'};$$

also by (80)

(201)... $$na^{2}\frac{d\theta_{\prime\prime}}{dt} + 2n^{2}ar_{\prime\prime} = -\frac{n^{3}aa'}{n-n'}\cos(nt - n't + \epsilon - \epsilon')$$
$$+ \frac{n}{n-n'}\{a^{2} - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}} + \frac{n^{3}aa'}{n-n'}\cos(\epsilon - \epsilon')$$
$$- \frac{n}{n-n'}\{a^{2} - 2aa'\cos(\epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}};$$

& by (121) (123)

(202)... $$n'a'^{2}\frac{d\theta'_{\prime\prime}}{dt} + 2n'^{2}a'r'_{\prime\prime} = -\frac{n'^{3}aa'}{n-n'}\cos(nt - n't + \epsilon - \epsilon')$$
$$- \frac{n'}{n-n'}\{a^{2} - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}} - \frac{n'^{3}aa'}{n-n'}\cos(\epsilon - \epsilon')$$
$$+ \frac{n'}{n-n'}\{a^{2} - 2aa'\cos(\epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}};$$

$\therefore$ $\left[\text{since } \frac{n^{3} - n'^{3}}{n-n'}aa' = aa'(n^{2} + nn' + n'^{2}) = aa'nn' + \frac{a'}{a^{2}} + \frac{a}{a'^{2}}\right]$ we have by (196),

(203)... $$H_{\prime\prime} = aa'(n^{2} + nn' + n'^{2})\cos(\epsilon - \epsilon') - \{a^{2} - 2aa'\cos(\epsilon - \epsilon') + a'^{2}\}^{-\frac{1}{2}}.$$

We have $\therefore$ verified these expressions for the four polar coordinates, by showing that when they are substituted in the equation of living force, the time disappears. The resulting relation

(194), with the values (200) (203) for the coefficients, might however have been more simply obtained from (193), by making $t=0$, which would give

$$(204)\ldots \quad h=\frac{m(1+m)}{2} r_0^2 \left(\frac{d\theta_0}{dt}\right)^2 + \frac{m'(1+m')}{2} r_0'^2 \left(\frac{d\theta_0'}{dt}\right)^2 - \frac{m r_0^{-1}}{1+m} - \frac{m' r_0'^{-1}}{1+m'} + mm'H_{\prime\prime}.$$

$H_{\prime\prime}$ would have evidently the value (203) &

$$(205)\ldots \quad a(1+m)^2 r_0^2 \left(\frac{d\theta_0}{dt}\right)^2 - \frac{2a}{r_0} = (1-e\cos\overline{\epsilon-\varpi})^2 - 2(1-e\cos\overline{\epsilon-\varpi})^{-1} = -1,$$

so that

$$(206)\ldots \quad h - mm'H_{\prime\prime} = -\frac{m}{2(1+m)a} \quad \frac{m'}{2(1+m')a'} = H,$$

as before.

[**15.**] Without neglecting any powers of the excentricities, if we only neglect m^2m', mm'^2, we may approximate as follows. In the first place we may employ as rigorous the polar expressions $x=r\cos\theta$, ... $y'=r'\sin\theta'$; we may also consider as rigorous the expressions

$$r=r_{,}+m'r_{\prime\prime}, \quad \theta=\theta_{,}+m'\theta_{\prime\prime}, \quad r'=r'_{,}+mr'_{\prime\prime}, \quad \theta'=\theta'_{,}+m\theta'_{\prime\prime},$$

observing only that, if so, we must change a little the meanings of $r_{\prime\prime}$, $\theta_{\prime\prime}$, $r'_{\prime\prime}$, $\theta'_{\prime\prime}$ & consider the foregoing values of these four functions as only approximations to their true values; in particular these functions will now involve the excentricities, of which they were before independent. They will involve the masses also, in their new rigorous conception, but in using them these masses will still be neglected, since we neglect m^2m', mm'^2. In this manner the rigorous equation (192) will still give an approximate equation of the form (194), namely $h=H_{,}+mm'H_{\prime\prime}$, but we shall now have, by not neglecting excentricities,

$$(207)\ldots \quad H_{,}=\frac{m(1+m)}{2}\left\{\left(\frac{dr_{,}}{dt}\right)^2 + r_{,}^2\left(\frac{d\theta_{,}}{dt}\right)^2\right\} + \frac{m'(1+m')}{2}\left\{\left(\frac{dr'_{,}}{dt}\right)^2 + r'^2_{,}\left(\frac{d\theta'_{,}}{dt}\right)^2\right\} - \frac{mr_{,}^{-}}{1+m} \quad 1+m'$$

$$(208)\ldots \quad H_{\prime\prime}=\frac{dr_{,}}{dt}\frac{dr_{\prime\prime}}{dt} + r_{,}^2\frac{d\theta_{,}}{dt}\frac{d\theta_{\prime\prime}}{dt} + r_{\prime\prime}\left\{r_{,}\left(\frac{d\theta_{,}}{dt}\right)^2 + \frac{1}{r_{,}^2}\right\} + \left(\frac{r'_{,}}{r_{,}^2} + \frac{r_{,}}{r'^2_{,}}\right)\cos(\theta_{,}-\theta'_{,})$$
$$+\frac{dr'_{,}}{dt}\frac{dr'_{\prime\prime}}{dt} + r'^2_{,}\frac{d\theta'_{,}}{dt}\frac{d\theta'_{\prime\prime}}{dt} + r'_{\prime\prime}\left\{r'_{,}\left(\frac{d\theta'_{,}}{dt}\right)^2 + \frac{1}{r'^2_{,}}\right\} - \{r_{,}^2 - 2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,}) + r'^2_{,}\}^{-}$$
$$+\left(\frac{dr_{,}}{dt}\frac{dr'_{,}}{dt} + r_{,}r'_{,}\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)\cos(\theta_{,}-\theta'_{,}) + \left(r'_{,}\frac{d\theta'_{,}}{dt}\frac{dr_{,}}{dt} - r_{,}\frac{d\theta_{,}}{dt}\frac{dr'_{,}}{dt}\right)\sin(\theta_{,}-\theta'_{,}).$$

Now $r_{,}$, $\theta_{,}$ are rigorously the polar coordinates of one focal ellipse and $r'_{,}$, $\theta'_{,}$ of another; in such a manner that we have rigorously the following differential equations of the first order between those coordinates

$$(209)\ldots \quad \frac{1}{2}\left\{\left(\frac{dr_{,}}{dt}\right)^2 + r_{,}^2\left(\frac{d\theta_{,}}{dt}\right)^2\right\}(1+m)^2 - \frac{1}{r_{,}} = \text{const.} = -\frac{1}{2a}; \quad (210)\ldots \quad r_{,}^2\frac{d\theta_{,}}{dt} = \frac{\sqrt{a(1-e^2)}}{1+m};$$

$$(211)\ldots \quad \frac{1}{2}\left\{\left(\frac{dr'_{,}}{dt}\right)^2 + r'^2_{,}\left(\frac{d\theta'_{,}}{dt}\right)^2\right\}(1+m')^2 - \frac{1}{r'_{,}} = \text{const.} = -\frac{1}{2a'}; \quad (212)\ldots \quad r'^2_{,}\frac{d\theta'_{,}}{dt} = \frac{\sqrt{a'(1-e'^2)}}{1+m'}.$$

the constants a, e depending on m, r_0, $\frac{dr_0}{dt}$, $\frac{d\theta_0}{dt}$, & a', e' depending on m', r'_0, $\frac{dr'_0}{dt}$, $\frac{d\theta'_0}{dt}$, by the analogous equations

$$(213)\ldots\quad \frac{1}{2}\left\{\left(\frac{dr_0}{dt}\right)^2 + r_0^2\left(\frac{d\theta_0}{dt}\right)^2\right\}(1+m)^2 - \frac{1}{r_0} = -\frac{1}{2a};\qquad (214)\ldots\quad r_0^2\frac{d\theta_0}{dt} = \frac{\sqrt{a(1-e^2)}}{1+m};$$

$$(215)\ldots\quad \frac{1}{2}\left\{\left(\frac{dr'_0}{dt}\right)^2 + \qquad\qquad\right\}\qquad\qquad\qquad ;\qquad (216)\ldots\qquad\qquad\qquad .$$

Hence the equation $H_{,} = -\frac{m}{2(1+m)a} - \frac{m'}{2(1+m')a'}$ holds *rigorously*, the excentricities not being neglected. And since, without neglecting the excentricities, $\frac{dr_{,,}}{dt}$, $\frac{d\theta_{,,}}{dt}$, $\frac{dr'_{,,}}{dt}$, $\frac{d\theta'_{,,}}{dt}$ & also $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$ vanish with t, we have *rigorously*, $H_{,,}$ being independent of t,

$$(217)\ldots\quad H_{,,} = \left(\frac{r'_0}{r_0^2} + \frac{r_0}{r_0'^2}\right)\cos(\theta_0 - \theta'_0) - \{r_0^2 - 2r_0 r'_0\cos(\theta_0 - \theta'_0) + r_0'^2\}^{-\frac{1}{2}}$$
$$+ \left(\frac{dr_0}{dt}\frac{dr'_0}{dt} + r_0 r'_0\frac{d\theta_0}{dt}\frac{d\theta'_0}{dt}\right)\cos(\theta_0 - \theta'_0) + \left(r'_0\frac{d\theta'_0}{dt}\frac{dr_0}{dt} - r_0\frac{d\theta_0}{dt}\frac{dr'_0}{dt}\right)\sin(\theta_0 - \theta'_0).$$

If the 4 functions $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$ were determined accurately with respect to the excentricities, neglecting only the masses, we should have a verification of those determinations by substituting the four functions in the equation (208), and seeing that they reduce it to the form (217); a verification which we have already made, neglecting excentricities.

[*Expressions for perturbations* $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$.]

[**16.**] To effect this verification of $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$, without neglecting any powers of the excentricities, we may transform the fundamental differential equations (1)...(4). Observing that

$$(218)\ldots\quad x\frac{d}{dt}\frac{dx}{dt} + y\frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}\left(r\frac{dr}{dt}\right) - \left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\theta}{dt}\right)^2 = r\left\{\frac{d}{dt}\frac{dr}{dt} - r\left(\frac{d\theta}{dt}\right)^2\right\},$$

$$(219)\ldots\quad x\frac{d}{dt}\frac{dy}{dt} - y\frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d}{dt}\frac{d\theta}{dt} + 2r\frac{dr}{dt}\frac{d\theta}{dt},$$

$$(220)\ldots\quad xx' + yy' = rr'\cos(\theta - \theta'),\qquad (221)\ldots\quad yx' - xy' = rr'\sin(\theta - \theta'),$$

we find the rigorous transformations,

$$(222)\ldots\quad 0 = \frac{d}{dt}\frac{dr}{dt} - r\left(\frac{d\theta}{dt}\right)^2 + \{(1+m)r + m'r'\cos(\theta - \theta')\}\{(1+m)^2r^2$$
$$+ 2m'(1+m)rr'\cos(\theta - \theta') + m'^2r'^2\}^{-\frac{3}{2}}$$
$$+ m'\{r - r'\cos(\theta - \theta')\}\{r^2 - 2rr'\cos(\theta - \theta') + r'^2\}^{-\frac{3}{2}};$$

$$(223)\ldots\quad 0 = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + m'rr'\sin(\theta - \theta')[\{r^2 - 2rr'\cos(\theta - \theta') + r'^2\}^{-\frac{3}{2}}$$
$$- \{(1+m)^2r^2 + 2m'(1+m)rr'\cos(\theta - \theta') + m'^2r'^2\}^{-\frac{3}{2}}];$$

$$(224)\ldots\quad 0 = \frac{d}{dt}\frac{dr'}{dt} - r'\left(\frac{d\theta'}{dt}\right)^2 + \{(1+m')r' + mr\cos(\theta - \theta')\}\{(1+m')^2r'^2$$
$$+ 2m(1+m')rr'\cos(\theta - \theta') + m^2r^2\}^{-\frac{3}{2}}$$
$$+ m\{r' - r\cos(\theta - \theta')\}\{r^2 - 2rr'\cos(\theta - \theta') + r'^2\}^{-\frac{3}{2}};$$

$$(225)\ldots\quad 0=\frac{d}{dt}\left(r'^2\frac{d\theta'}{dt}\right)-mrr'\sin(\theta-\theta')\,[\{r^2-2rr'\cos(\theta-\theta')+r'^2\}^{-\frac{3}{2}}$$
$$-\{(1+m')^2r'^2+2m(1+m')rr'\cos(\theta-\theta')+m^2r^2\}^{-\frac{3}{2}}].$$

Putting then, as before, $r=r_{,}+m'r_{,,}$, $\theta=\theta_{,}+m'\theta_{,,}$, $r'=r'_{,}+mr'_{,,}$, $\theta'=\theta'_{,}+m\theta'_{,,}$, we have, without neglecting anything, the elliptic equations

$$(226)\ldots\quad 0=\frac{d}{dt}\frac{dr_{,}}{dt}-r_{,}\left(\frac{d\theta_{,}}{dt}\right)^2+\frac{(1+m)^{-2}}{r_{,}^2};\qquad (227)\ldots\quad 0=\frac{d}{dt}\left(r_{,}^2\frac{d\theta_{,}}{dt}\right);$$

$$(228)\ldots\quad 0=\frac{d}{dt}\frac{dr'_{,}}{dt}-r'_{,}\left(\frac{d\theta'_{,}}{dt}\right)^2+\frac{(1+m')^{-2}}{r'^2_{,}};\qquad (229)\ldots\quad 0=\frac{d}{dt}\left(r'^2_{,}\frac{d\theta'_{,}}{dt}\right);$$

and neglecting only the mass m' in the functions $r_{,,}$, $\theta_{,,}$ and the mass m in $r'_{,,}$, $\theta'_{,,}$, we have

$$(230)\ldots\quad 0=\frac{d}{dt}\frac{dr_{,,}}{dt}-r_{,,}\left(\frac{d\theta_{,}}{dt}\right)^2-2r_{,}\frac{d\theta_{,}}{dt}\frac{d\theta_{,,}}{dt}-\frac{2(1+m)^{-2}}{r_{,}^3}\left(r_{,,}+r'_{,}\frac{\cos(\theta_{,}-\theta'_{,})}{1+m}\right)$$
$$+\frac{r_{,}-r'_{,}\cos(\theta_{,}-\theta'_{,})}{\{r_{,}^2-2rr'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{\frac{3}{2}}};$$

$$(231)\ldots\quad 0=\frac{d}{dt}\left(r_{,}^2\frac{d\theta_{,,}}{dt}+2r_{,}r_{,,}\frac{d\theta_{,}}{dt}\right)+r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})\,[\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{-\frac{3}{2}}$$
$$-(1+m)^{-3}r_{,}^{-3}];$$

$$(232)\ldots\quad 0=$$

(233)...

Hence

$$(234)\ldots\quad \frac{d\theta_{,,}}{dt}=-\frac{2r_{,,}}{r_{,}}\frac{d\theta_{,}}{dt}-\frac{1}{r_{,}^2}\int_0^t r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})\left\{\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{-\frac{3}{2}}-\frac{r_{,}^{-3}}{(1+m)^3}\right\}dt;$$

$$(235)\ldots\quad 0=\frac{d}{dt}\frac{dr_{,,}}{dt}+r_{,,}\left\{3\left(\frac{d\theta_{,}}{dt}\right)^2-\frac{2r_{,}^{-3}}{(1+m)^2}\right\}$$
$$+\frac{2}{r_{,}}\frac{d\theta_{,}}{dt}\int_0^t r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})\left\{\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{-\frac{3}{2}}-\frac{r_{,}^{-3}}{(1+m)^3}\quad dt\right.$$
$$\frac{2r'_{,}\cos(\theta_{,}-\theta'_{,})}{r_{,}^3(1+m)}+\frac{r_{,}-r'_{,}\cos(\theta_{,}-\theta'_{,})}{\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{\frac{3}{2}}},$$

$$(236)\ldots\quad \frac{d\theta'_{,,}}{dt}=-\frac{2r'_{,,}}{r'_{,}}\frac{d\theta'_{,}}{dt}+\frac{1}{r'^2_{,}}\int_0^t r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})$$
$$\times\left\{\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{-\frac{3}{2}}-\frac{r'^{-3}_{,}}{(1+m')^3}\right\}dt;$$

$$(237)\ldots\quad 0=\frac{d}{dt}\frac{dr'_{,,}}{dt}+r'_{,,}\left\{3\left(\frac{d\theta'_{,}}{dt}\right)^2-\frac{2r'^{-3}_{,}}{(1+m')^2}\right\}-\frac{2}{r'_{,}}\frac{d\theta'_{,}}{dt}\int_0^t$$
$$\frac{2r_{,}\cos(\theta_{,}-\theta'_{,})}{r'^3_{,}(1+m')}+\frac{r'_{,}-r_{,}\cos(\theta_{,}-\theta'_{,})}{\{r_{,}^2-2r_{,}r'_{,}\cos(\theta_{,}-\theta'_{,})+r'^2_{,}\}^{\frac{3}{2}}}.$$

Substituting the values (234) (236) in (208), and suppressing the masses we find, by (226) (228),

$$(238)\ldots\quad H_{''}=\frac{dr_{,}}{dt}\frac{dr_{''}}{dt}-r_{''}\frac{d}{dt}\frac{dr_{,}}{dt}+\frac{dr'_{,}}{dt}\frac{dr'_{''}}{dt}-r'_{''}\frac{d}{dt}\frac{dr'_{,}}{dt}+\left(\frac{r'_{,}}{r_{,}^{2}}+\frac{r_{,}}{r'^{2}_{,}}\right)\cos(\theta_{,}-\theta'_{,})$$

$$-\left(\frac{d\theta_{,}}{dt}-\frac{d\theta'_{,}}{dt}\right)\int_{0}^{t}r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})\left(r_{,}^{2}-2r_{,}r'_{,}\cos\overline{\theta_{,}-\theta'_{,}}+r'^{2}_{,}\right)^{-\frac{3}{2}}dt$$

$$+\frac{d\theta_{,}}{dt}\int_{0}^{t}\frac{r'_{,}}{r_{,}^{2}}\sin(\theta_{,}-\theta'_{,})\,dt-\frac{d\theta'_{,}}{dt}\int_{0}^{t}\frac{r_{,}}{r'^{2}_{,}}\sin(\theta_{,}-\theta'_{,})\,dt-\left(r_{,}^{2}-2r_{,}r'_{,}\cos\overline{\theta_{,}-\theta'_{,}}+r'^{2}_{,}\right)^{-\frac{1}{2}}$$

$$+\left(\frac{dr_{,}}{dt}\frac{dr'_{,}}{dt}+r_{,}r'_{,}\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)\cos(\theta_{,}-\theta'_{,})+\left(r'_{,}\frac{d\theta'_{,}}{dt}\frac{dr_{,}}{dt}-r_{,}\frac{d\theta_{,}}{dt}\frac{dr'_{,}}{dt}\right)\sin(\theta_{,}-\theta'_{,});$$

∵ by (235) (237),

$$(239)\ldots\quad \frac{dH_{''}}{dt}=-r_{''}\left\{\left(\frac{d}{dt}\right)^{3}r_{,}+\left(3\left(\frac{d\theta_{,}}{dt}\right)^{2}-\frac{2}{r_{,}^{3}}\right)\frac{dr_{,}}{dt}\right\}-r'_{''}\left\{\left(\frac{d}{dt}\right)^{3}r'_{,}+\left(3\left(\frac{d\theta'_{,}}{dt}\right)^{2}-\frac{2}{r'^{3}_{,}}\right)\frac{dr'_{,}}{dt}\right\}$$

$$+\left\{\frac{dr'_{,}}{dt}\left(\frac{d}{dt}\frac{dr_{,}}{dt}+\frac{1}{r_{,}^{2}}\right)+\frac{dr_{,}}{dt}\left(\frac{d}{dt}\frac{dr'_{,}}{dt}+\frac{1}{r'^{2}_{,}}\right)+\frac{d}{dt}\left(r_{,}r'_{,}\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)\right.$$

$$\left.+\left(\frac{d\theta_{,}}{dt}-\frac{d\theta'_{,}}{dt}\right)\left(r'_{,}\frac{d\theta'_{,}}{dt}\frac{dr_{,}}{dt}-r_{,}\frac{d\theta_{,}}{dt}\frac{dr'_{,}}{dt}\right)\right\}\cos(\theta_{,}-\theta'_{,})$$

$$+\left\{\frac{r'_{,}}{r_{,}^{2}}\frac{d\theta'_{,}}{dt}-\frac{r_{,}}{r'^{2}_{,}}\frac{d\theta_{,}}{dt}-\left(\frac{d\theta_{,}}{dt}-\frac{d\theta'_{,}}{dt}\right)\left(\frac{dr_{,}}{dt}\frac{dr'_{,}}{dt}+r_{,}r'_{,}\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)+\frac{d}{dt}\left(r'_{,}\frac{d\theta'_{,}}{dt}\frac{dr_{,}}{dt}-r_{,}\frac{d\theta_{,}}{dt}\frac{dr'_{,}}{dt}\right)\right\}\sin(\theta_{,}-\theta'_{,})$$

$$-\left\{\frac{2}{r_{,}}\frac{dr_{,}}{dt}\frac{d\theta_{,}}{dt}-\frac{2}{r'_{,}}\frac{dr'_{,}}{dt}\frac{d\theta'_{,}}{dt}+\frac{d}{dt}\left(\frac{d\theta_{,}}{dt}-\frac{d\theta'_{,}}{dt}\right)\right\}\int_{0}^{t}r_{,}r'_{,}\sin(\theta_{,}-\theta'_{,})\left(r_{,}^{2}-2r_{,}r'_{,}\cos\overline{\theta_{,}-\theta'_{,}}+r'^{2}_{,}\right)^{-\frac{3}{2}}dt$$

$$\left\{\frac{d}{dt}\frac{d\theta_{,}}{dt}+\frac{2}{r_{,}}\frac{dr_{,}}{dt}\frac{d\theta_{,}}{dt}\right\}\int_{0}^{t}\frac{r'_{,}}{r_{,}^{2}}\sin(\theta_{,}-\theta'_{,})\,dt-\left\{\frac{d}{dt}\frac{d\theta'_{,}}{dt}+\frac{2}{r'_{,}}\frac{dr'_{,}}{dt}\frac{d\theta'_{,}}{dt}\right\}\int_{0}\frac{r_{,}}{r'^{2}_{,}}\sin(\theta_{,}-\theta'_{,})\,dt;$$

& I suppose that of these 7 parts each vanishes separately by the laws of elliptic motion. In fact these equations make the coefficients of the 3 integrals vanish by the law of equal areas; also

$$(240)\ldots\quad \left(\frac{d}{dt}\right)^{3}r_{,}+\left(3\left(\frac{d\theta_{,}}{dt}\right)^{2}-\frac{2}{r_{,}^{3}}\right)\frac{dr_{,}}{dt}=4\left(\frac{d\theta_{,}}{dt}\right)^{2}\frac{dr_{,}}{dt}+r_{,}\frac{d}{dt}\left(\frac{d\theta_{,}}{dt}\right)^{2}=\frac{2}{r_{,}}\frac{d\theta_{,}}{dt}\frac{d}{dt}\left(r_{,}^{2}\frac{d\theta_{,}}{dt}\right)=0;\ \&c.;$$

and it only remains ∵ to show that the coefficients of $\cos(\theta_{,}-\theta'_{,})$ and of $\sin(\theta_{,}-\theta'_{,})$ vanish.

Now the coefficient of $\sin(\theta-\theta_{,})$ is

$$(241)\ldots\quad \frac{d\theta'_{,}}{dt}\left\{\frac{r'_{,}}{r_{,}^{2}}+2\frac{dr_{,}}{dt}\frac{dr'_{,}}{dt}+r'_{,}\frac{d}{dt}\frac{dr_{,}}{dt}-r_{,}r'_{,}\left(\frac{d\theta_{,}}{dt}\right)^{2}\right\}+\frac{d}{dt}\frac{d\theta'_{,}}{dt}r'_{,}\frac{dr_{,}}{dt}$$

$$-\frac{d\theta_{,}}{dt}\left\{\frac{r_{,}}{r'^{2}_{,}}+2\frac{dr_{,}}{dt}\frac{dr'_{,}}{dt}+r_{,}\frac{d}{dt}\frac{dr'_{,}}{dt}-r_{,}r'_{,}\left(\frac{d\theta'_{,}}{dt}\right)^{2}\right\}-\frac{d}{dt}\frac{d\theta'_{,}}{dt}r_{,}\frac{dr'_{,}}{dt}$$

$$=\frac{d\theta'_{,}}{dt}\left\{\frac{r'_{,}}{r_{,}^{2}}+r'_{,}\frac{d}{dt}\frac{dr_{,}}{dt}-r_{,}r'_{,}\left(\frac{d\theta_{,}}{dt}\right)^{2}\right\}-\frac{d\theta_{,}}{dt}\left\{\frac{r_{,}}{r'^{2}_{,}}+r_{,}\frac{d}{dt}\frac{dr'_{,}}{dt}-r_{,}r'_{,}\left(\frac{d\theta'_{,}}{dt}\right)^{2}\right\}=0;$$

and finally the coefficient of $\cos(\theta_{,}-\theta'_{,})$ is

$$(242)\ldots\quad 2\frac{d}{dt}(r_{,}r'_{,})\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}+r_{,}r'_{,}\frac{d}{dt}\left(\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)=\frac{1}{r_{,}r'_{,}}\frac{d}{dt}\left(r_{,}^{2}r'^{2}_{,}\frac{d\theta_{,}}{dt}\frac{d\theta'_{,}}{dt}\right)=0.$$

We have ∵ verified the constancy of the expression (208) for $H_{''}$, not neglecting excentricities.

[17.] To integrate approximately the equations (234)...(237) for the perturbations, neglecting squares and products of the excentricities, we may put (235) (237) under the form

(243)... $$0=\frac{d}{dt}\frac{dr_{\prime\prime}}{dt}+n^2r_{\prime\prime}+6n^2er_{\prime\prime}\cos(nt+\epsilon-\varpi)+L+e\frac{\delta L}{\delta e}+e'\frac{\delta L}{\delta e'};^*$$

(244)... $$0=\frac{d}{dt}\frac{dr'_{\prime\prime}}{dt}+n'^2r'_{\prime\prime}+6n'^2e'r'_{\prime\prime}\cos(n't+\epsilon'-\varpi')+L'+e\frac{\delta L'}{\delta e}+e'\frac{\delta L'}{\delta e'};$$

in which, by foregoing investigations,

(245)... $$L=\tfrac{1}{2}\sum_{(i)-\infty}R_i\cos i(nt-n't+\epsilon-\epsilon'),$$

(246)... $$L'=\tfrac{1}{2}\sum_{(i)-\infty}^{\infty}R'_i\cos i(nt-n't+\epsilon-\epsilon'),$$

[see (85) (126)]; while $\frac{\delta L}{\delta e}$, $\frac{\delta L}{\delta e'}$, $\frac{\delta L'}{\delta e}$, $\frac{\delta L'}{\delta e'}$ can be developed in other less simple series, proceeding according to cosines (& sines?) of linear functions of the time and involving anomalies as well as longitudes. In this manner the equations (243) (244) can be integrated by the known methods under the form

(247)... $$0=\frac{d}{dt}\frac{d\Delta r_{\prime\prime}}{dt}+n^2\Delta r_{\prime\prime}+6n^2e\,(r_{\prime\prime})\cos(nt+\epsilon-\varpi)+e\frac{\delta L}{\delta e}+e'\frac{\delta L}{\delta e'},$$

(248)... $$0=\frac{d}{dt}\frac{d\Delta r'_{\prime\prime}}{dt}+n'^2\Delta r'_{\prime\prime}+6n'^2e'\,(r'_{\prime\prime})\cos(n't+\epsilon'-\varpi')+e\frac{\delta L'}{\delta e}+e'\frac{\delta L'}{\delta e'},$$

in which we have put

(249)... $r_{\prime\prime}=(r_{\prime\prime})+\Delta r_{\prime\prime}$, (250)... $r'_{\prime\prime}=(r'_{\prime\prime})+\Delta r'_{\prime\prime}$,

$\Delta r_{\prime\prime}$, $\Delta r'_{\prime\prime}$ being the sought parts depending on the excentricities & $(r_{\prime\prime})$, $(r'_{\prime\prime})$ being the parts independent of them, so that

(251)... $0=\frac{d}{dt}\frac{d(r_{\prime\prime})}{dt}+n^2(r_{\prime\prime})+L$, (252)... $0=\frac{d}{dt}\frac{d(r'_{\prime\prime})}{dt}+n'^2(r'_{\prime\prime})+L'$,

and integrating,

(253)... $$(r_{\prime\prime})=-\frac{R_0}{2n^2}+\sum_{(i)1}^{\infty}\frac{R_i\cos i(nt-n't+\epsilon-\epsilon')}{i^2(n-n')^2-n^2}+R_{,}\cos(nt+\epsilon)+R_{\prime\prime}\sin(nt+\epsilon),$$

(254)... $$(r'_{\prime\prime})=-\frac{R'_0}{2n'^2}+\sum_{(i)1}^{\infty}\frac{R'_i\cos i(nt-n't+\epsilon-\epsilon')}{i^2(n-n')^2-n^2}+R'_{,}\cos(n't+\epsilon')+R'_{\prime\prime}\sin(n't+\epsilon'),$$

R_0, R_i, R'_0, R'_i having the same meanings as before, but the four constants $R_{,}$, $R_{\prime\prime}$, $R'_{,}$, $R'_{\prime\prime}$, which will evidently melt with the 4 introduced by integrating (247) (248), being now determined so that $(r_{\prime\prime})+\Delta r_{\prime\prime}$, $(r'_{\prime\prime})+\Delta r'_{\prime\prime}$ & their 1st differential coefficients taken with respect to the time t may vanish when $t=0$. We may however give the symbols $R_{,}$, $R_{\prime\prime}$, $R'_{,}$, $R'_{\prime\prime}$ their former values by using additional symbols $\Delta R_{,}, \ldots \Delta R'_{\prime\prime}$ of the same kind. The similar approximate determination of the functions

(255)... $\theta_{\prime\prime}=(\theta_{\prime\prime})+\Delta\theta_{\prime\prime}$, (256)... $\theta'_{\prime\prime}=(\theta'_{\prime\prime})+\Delta\theta'_{\prime\prime}$,

* [L is not a function of e or e'. $\frac{\delta L}{\delta e}$, $\frac{\delta L}{\delta e'}$ are the coefficients of e, e' on the right-hand side of (243). Similarly for (244).]

will then be easily effected by integrating the equations (234) (236). Of course these determinations of the parts of the perturbations of the *centrobaric* coordinates depending on the 1st powers of the excentricities must, when converted into the perturbations of the *heliocentric* coordinates, agree with the results of No. 50 of the IInd Book of the *Méc. Cél.*, with the exception of those differences which arise from a different choice of the 4 arbitrary constants. Perhaps I shall find it worth while to return hereafter to the integrations here described.

[*Putting* $r=(r_{,})+\Delta r_{,}+m'(r_{,,})+m'\Delta r_{,,}$, $\Delta H_{,,}$, $\Delta V_{,,}$ *and* δV *are calculated.*]

[18.] We have now found approximate expressions for the centrobaric coordinates of Jupiter and Saturn of the form

(257)... $r=(r_{,})+\Delta r_{,}+m'(r_{,,})+m'\Delta r_{,,}$; (258)... $\theta=(\theta_{,})+\Delta\theta_{,}+m'(\theta_{,,})+m'\Delta\theta_{,,}$;

(259)... $r'=(r'_{,})+\Delta r'_{,}+m(r'_{,,})+m\Delta r'_{,,}$; (260)... $\theta'=(\theta'_{,})+\Delta\theta'_{,}+m(\theta'_{,,})+m\Delta\theta'_{,,}$.

The parts $\Delta r_{,}$ &c. depend on the 1st powers of the excentricities and the other parts are independent of the excentricities, so that

(261)... $(r_{,})=a$; (262)... $(\theta_{,})=nt+\epsilon$;

(263)... $(r'_{,})=a'$; (264)... $(\theta'_{,})=n't+\epsilon'$;

(265)... $\Delta r_{,}=-ae\cos(nt+\epsilon-\varpi)$; (266)... $\Delta\theta_{,}=2e\sin(nt+\epsilon-\varpi)$;

(267)... $\Delta r'_{,}=-a'e'\cos(n't+\epsilon'-\varpi')$; (268)... $\Delta\theta'_{,}=2e'\sin(n't+\epsilon'-\varpi')$:

while $(r_{,,})$, $(\theta_{,,})$, $(r'_{,,})$, $(\theta'_{,,})$ are the 1st approximate coefficients of perturbation, namely the functions (90) (102) (131) and $\Delta r_{,,}$, ... $\Delta\theta'_{,,}$ are the corrections of these coefficients of perturbation depending on the 1st powers of the excentricities. In like manner, by [**14.**] [**15.**],

(269)... $$h=H_{,}+mm'H_{,,}=H_{,}+mm'(H_{,,})+mm'\Delta H_{,,},$$

in which $H_{,}=-\dfrac{m}{2(1+m)a}-\dfrac{m'}{2(1+m')a'}$, and, by (217),

(270)... $$\begin{aligned}(H_{,,})+\Delta H_{,,}&=H_{,,}\\ &=\left\{\frac{a'+\Delta r'_0}{a^2+2a\Delta r_0}+\frac{a+\Delta r_0}{a'^2+2a'\Delta r'_0}+aa'nn'\left(1+\frac{\Delta r_0}{a}\right)\left(1+\frac{\Delta r'_0}{a'}\right)\left(1+\frac{d\Delta\theta_0}{ndt}\right)\left(1+\frac{d\Delta\theta'_0}{n'dt}\right)\right\}\\ &\quad\times\cos(\epsilon-\epsilon'+\Delta\theta_0-\Delta\theta'_0)+nn'aa'\left\{\frac{d\Delta r_0}{andt}-\frac{d\Delta r'_0}{a'n'dt}\right\}\sin(\epsilon-\epsilon')\\ &\quad-\{a^2-2aa'\cos(\epsilon-\epsilon')+a'^2+2(a-a'\cos\overline{\epsilon-\epsilon'})\Delta r_0+2(a'-a\cos\overline{\epsilon-\epsilon'})\Delta r'_0\\ &\quad+2aa'\sin(\epsilon-\epsilon')(\Delta\theta_0-\Delta\theta'_0)\}^{-\frac{1}{2}};\end{aligned}$$

∴ (271)... $$(H_{,,})=aa'(n^2+nn'+n'^2)\cos(\epsilon-\epsilon')-\{a^2-2aa'\cos(\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}};$$

(272)... $$\begin{aligned}\Delta H_{,,}&=\cos(\epsilon-\epsilon').aa'\\ &\quad\times\left\{n^2\frac{\Delta r'_0}{a'}+n'^2\frac{\Delta r_0}{a}-2n^2\frac{\Delta r_0}{a}-2n'^2\frac{\Delta r'_0}{a'}+nn'\left(\frac{\Delta r_0}{a}+\frac{d\Delta\theta_0}{ndt}+\frac{\Delta r'_0}{a'}+\frac{d\Delta\theta'_0}{n'dt}\right)\right\}\\ &\quad+\sin(\epsilon-\epsilon').aa'\left\{\left(\frac{d\Delta r_0}{andt}-\frac{d\Delta r'_0}{a'n'dt}+\Delta\theta'_0-\Delta\theta_0\right)nn'+(n^2+n'^2)(\Delta\theta'_0-\Delta\theta_0)\right\}\\ &\quad+\{a^2-2aa'\cos(\epsilon-\epsilon')+a'^2\}^{-\frac{3}{2}}\\ &\quad\times[\{a-a'\cos(\epsilon-\epsilon')\}\Delta r_0+\{a'-a\cos(\epsilon-\epsilon')\}\Delta r'_0+aa'\sin(\epsilon-\epsilon')(\Delta\theta_0-\Delta\theta'_0)],\end{aligned}$$

that is,

$$(273)\ldots\qquad \Delta H_{\prime\prime}=e\frac{\delta\Delta H_{\prime\prime}}{\delta e}+e'\frac{\delta\Delta H_{\prime\prime}}{\delta e'}$$

in which

$$(274)\ldots\qquad \frac{\delta\Delta H_{\prime\prime}}{\delta e}=aa'\cos(\epsilon-\epsilon')\cos(\epsilon-\varpi)(2n^2+nn'-n'^2)$$

$$-aa'\sin(\epsilon-\epsilon')\sin(\epsilon-\varpi)(2n^2+nn'+2n'^2)$$

$$+a(a^2-2aa'\cos\overline{\epsilon-\epsilon'}+a'^2)^{-\frac{3}{2}}\{-(a-a'\cos\overline{\epsilon-\epsilon'})\cos(\epsilon-\varpi)+2a'\sin(\epsilon-\epsilon')\sin(\epsilon-\varpi)\};$$

$$(275)\ldots\qquad \frac{\delta\Delta H_{\prime\prime}}{\delta e'}=aa'\cos(\epsilon-\epsilon')\cos(\epsilon'-\varpi')(2n'^2+nn'-n^2)$$

$$+aa'\sin(\epsilon-\epsilon')\sin(\epsilon'-\varpi')(2n^2+2nn'+2n'^2)$$

$$+a'(a^2-2aa'\cos\overline{\epsilon-\epsilon'}+a'^2)^{-\frac{3}{2}}\{-(a'-a\cos\overline{\epsilon-\epsilon'})\cos(\epsilon'-\varpi')-2a\sin(\epsilon-\epsilon')\sin(\epsilon'-\varpi')\}.$$

Thus the five constants r_0, θ_0, r'_0, θ'_0, h are expressed as functions of the 8 elements a, e, ϵ, ϖ, a', e', ϵ', ϖ' and of the two masses m, m'; the 4 variable coordinates r, θ, r', θ' are expressed as functions of the time t and of the same 10 quantities. We can ∵ reciprocally determine, or at least conceive determined, the 9 quantities a, e, ϵ, ϖ, a', e', ϵ', ϖ', t as functions of the 2 masses, the constant h & the 8 final and initial coordinates; and if we can express the action V as a function of these 9 functions, we shall have it also expressed as a function of h, m, m', r, θ, r', θ', r_0, θ_0, r'_0, θ'_0.

[19.] Now

$$(276)\ldots\qquad V=\int_0^t \Sigma mv^2\,dt,$$

in which, by the equation of living forces, we have *rigorously*

$$(277)\ldots\qquad \frac{1}{2}\frac{dV}{dt}=\tfrac{1}{2}\Sigma mv^2=h+\frac{mr^{-1}}{1+m}\left\{1+\frac{2m'r'}{r}\frac{\cos(\theta-\theta')}{1+m}+\left(\frac{m'r'r^{-1}}{1+m}\right)^2\right\}^{-\frac{1}{2}}$$

$$+\frac{m'r'^{-1}}{1+m'}\left\{1+\frac{2mr}{r'}\frac{\cos(\theta-\theta')}{1+m'}+\left(\frac{mrr'^{-1}}{1+m'}\right)^2\right\}^{-\frac{1}{2}}+mm'\{r^2-2rr'\cos(\theta-\theta')+r'^2\}^{-\frac{1}{2}};$$

& ∵ approximately

$$(278)\ldots\qquad \frac{1}{2}\frac{dV}{dt}=h+\frac{mr^{-1}}{1+m}+\frac{m'r'^{-1}}{1+m'}-mm'\left(\frac{r'}{r^2}+\frac{r}{r'^2}\right)\cos(\theta-\theta')+\frac{mm'}{\sqrt{r^2-2rr'\cos(\theta-\theta')+r'^2}},$$

so that we may put

$$(279)\ldots\qquad V=(V_{,})+\Delta V_{,}+\{(V_{\prime\prime})+\Delta V_{\prime\prime}\}mm';$$

in which

$$(280)\ldots\qquad (V_{,})=\frac{mt}{(1+m)a}+\frac{m't}{(1+m')a'}$$

$$(281)\ldots\qquad \Delta V_{,}=\frac{2me}{(1+m)an}\{\sin(nt+\epsilon-\varpi)-\sin(\epsilon-\varpi)\}$$

$$+\frac{2m'e'}{(1+m')a'n'}\{\sin(n't+\epsilon'-\varpi')-\sin(\epsilon'-\varpi')\};$$

$$(282)\ldots\qquad \frac{1}{2}\frac{d(V_{\prime\prime})}{dt}=(H_{\prime\prime})-\frac{(r_{\prime\prime})}{a^2}-\frac{(r'_{\prime\prime})}{a'^2}-\left(\frac{a'}{a^2}+\frac{a}{a'^2}\right)\cos(nt-n't+\epsilon-\epsilon')$$

$$+\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}},$$

or by (196),

$$(283)\ldots\quad \frac{d(V_{\prime\prime})}{dt} = (H_{\prime\prime}) + \frac{d(\theta_{\prime\prime})}{an\,dt} + \frac{d(\theta'_{\prime\prime})}{a'n'\,dt} + aa'(nn' - n^2 - n'^2)\cos(nt - n't + \epsilon - \epsilon') + \{a^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^2\}^{-\frac{1}{2}};$$

$$\because\ (284)\ldots\quad (V_{\prime\prime}) = (H_{\prime\prime})\,t + \frac{(\theta_{\prime\prime})}{an} + \frac{(\theta'_{\prime\prime})}{a'n'} + \frac{aa'(nn' - n^2 - n'^2)}{n - n'}\{\sin(nt - n't + \epsilon - \epsilon') - \sin(\epsilon - \epsilon')\} + \int_0^t \{a^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^2\}^{-\frac{1}{2}}\,dt;$$

omitting here the initial values of $(\theta_{\prime\prime})$ and $(\theta'_{\prime\prime})$, which may however affect the remaining coefficient $\Delta V_{\prime\prime}$: & with respect to this last coefficient, if we put

$$(285)\ldots\quad V_{\prime\prime} = (V_{\prime\prime}) + \Delta V_{\prime\prime},$$

we have

$$(286)\ldots\quad \frac{1}{2}\frac{dV_{\prime\prime}}{dt} = H_{\prime\prime} - \frac{r_{\prime\prime}}{r_{,}^2} - \frac{r'_{\prime\prime}}{r_{,}'^2} - \left(\frac{r'_{,}}{r_{,}^2} + \frac{r_{,}}{r_{,}'^2}\right)\cos(\theta - \theta') + \frac{1}{\sqrt{\{r_{,}^2 - 2r_{,}r'_{,}\cos(\theta - \theta') + r_{,}'^2\}}};$$

while by (208) & by the equations of elliptic motion, neglecting only squares of excentricities, we have

$$(287)\ldots\quad H_{\prime\prime} = \frac{2r_{\prime\prime}}{r_{,}^2} + \frac{d}{dt}.r_{\prime\prime}\frac{dr_{,}}{dt} + \sqrt{a}\frac{d\theta_{\prime\prime}}{dt} + \frac{2r'_{\prime\prime}}{r_{,}'^2} + \frac{d}{dt}.r'_{\prime\prime}\frac{dr'_{,}}{dt} + \sqrt{a'}\frac{d\theta'_{\prime\prime}}{dt} + \left(\frac{r'_{,}}{r_{,}^2} + \frac{r_{,}}{r_{,}'^2} + \frac{\sqrt{aa'}}{r_{,}r'_{,}}\right)\cos(\theta_{,} - \theta'_{,}) + aa'nn'\left(\frac{dr_{,}}{an\,dt} - \frac{dr'_{,}}{a'n'\,dt}\right)\sin(nt - n't + \epsilon - \epsilon') - \{r_{,}^2 - 2r_{,}r'_{,}\cos(\theta_{,} - \theta'_{,}) + r_{,}'^2\}^{-\frac{1}{2}};$$

we have $\because$

$$(288)\ldots\quad \frac{dV_{\prime\prime}}{dt} = H_{\prime\prime} + \frac{d}{dt}\left(r_{\prime\prime}\frac{dr_{,}}{dt} + r'_{\prime\prime}\frac{dr'_{,}}{dt} + na^2\theta_{\prime\prime} + n'a'^2\theta'_{\prime\prime}\right) + (r_{,}^2 - 2r_{,}r'_{,}\cos\theta_{,} - \theta'_{,} + r_{,}'^2)^{-\frac{1}{2}} \left(\frac{\sqrt{aa'}}{r_{,}r'_{,}} - \frac{r'_{,}}{r_{,}^2} - \frac{r_{,}}{r_{,}'^2}\right)\cos(\theta_{,} - \theta'_{,}) + nn'aa'\left(\frac{dr_{,}}{an\,dt} - \frac{dr'_{,}}{a'n'\,dt}\right)\sin(nt - n't + \epsilon - \epsilon'),$$

and

$$(289)\ldots\quad V_{\prime\prime} = H_{\prime\prime}t + r_{\prime\prime}\frac{dr_{,}}{dt} + r'_{\prime\prime}\frac{dr'_{,}}{dt} + \frac{\theta_{\prime\prime}}{an} + \frac{\theta'_{\prime\prime}}{a'n'} + \int_0^t (r_{,}^2 - 2r_{,}r'_{,}\cos\theta_{,} - \theta'_{,} + r_{,}'^2)^{-\frac{1}{2}}\,dt + \int_0^t \left(\frac{\sqrt{aa'}}{r_{,}r'_{,}} - \frac{r'_{,}}{r_{,}^2} - \frac{r_{,}}{r_{,}'^2}\right)\cos(\theta_{,} - \theta'_{,})\,dt + nn'aa'\int_0^t \left(\frac{dr_{,}}{an\,dt} - \frac{dr'_{,}}{a'n'\,dt}\right)\sin(nt - n't + \epsilon - \epsilon')\,dt;$$

and*

$$(290)\ldots\quad \Delta V_{\prime\prime} = t\Delta H_{\prime\prime} + aen(r_{\prime\prime})\sin(nt + \epsilon - \varpi) + a'e'n'(r'_{\prime\prime})\sin(n't + \epsilon' - \varpi') + \frac{\Delta\theta_{\prime\prime}}{an} + \frac{\Delta\theta'_{\prime\prime}}{a'n'}$$
$$+ aa'\int_0^t \{e\sin(nt + \epsilon - \varpi) - e'\sin(n't + \epsilon' - \varpi')\}(2n^2 + 2n'^2 - nn')\sin(nt - n't + \epsilon - \epsilon')\,dt$$
$$+ aa'\int_0^t \{(2n^2 + n'^2 + nn')e\cos(nt + \epsilon - \varpi) + (2n'^2 + n^2 + nn')e'\cos(n't + \epsilon' - \varpi')\} \times\cos(nt - n't + \epsilon - \epsilon')\,dt$$
$$+ \int_0^t [\{r_{,}^2 - 2r_{,}r'_{,}\cos(\theta_{,} - \theta'_{,}) + r_{,}'^2\}^{-\frac{1}{2}} - \{a^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^2\}^{-\frac{1}{2}}]\,dt,$$

* [In the second integral in (290) $2n^2$, $2n'^2$ should be $-2n^2$, $-2n'^2$.]

an expression which we may put under the form

$$(291)\ldots \qquad \Delta V_{\prime\prime} = e\,\frac{\delta \Delta V_{\prime\prime}}{\delta e} + e'\,\frac{\delta \Delta V_{\prime\prime}}{\delta e'},$$

$$(292)\ldots \qquad \frac{\delta \Delta V_{\prime\prime}}{\delta e} = t\,\frac{\delta \Delta H_{\prime\prime}}{\delta e} + an\,(r_{\prime\prime}) \sin(nt+\epsilon-\varpi) + a^2 n\,\frac{\delta \Delta \theta_{\prime\prime}}{\delta e} + a'^2 n'\,\frac{\delta \Delta \theta'_{\prime\prime}}{\delta e}$$

$$+ (2n^2+2n'^2-nn')\,aa' \int_0^t \sin(nt+\epsilon-\varpi)\sin(nt-n't+\epsilon-\epsilon')\,dt$$

$$+ (2n^2+n'^2+nn')\,aa' \int_0^t \cos(nt+\epsilon-\varpi)\cos(nt-n't+\epsilon-\epsilon')\,dt$$

$$+ a \int_0^t \frac{\{a-a'\cos(nt-n't+\epsilon-\epsilon')\}\cos(nt+\epsilon-\varpi) - 2a'\sin(nt-n't+\epsilon-\epsilon')\sin(nt+\epsilon-\varpi)}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}}\,dt;$$

$$(293)\ldots \qquad \frac{\delta \Delta V_{\prime\prime}}{\delta e'} = t\,\frac{\delta \Delta H_{\prime\prime}}{\delta e'} + a'n'\,(r'_{\prime\prime}) \sin(n't+\epsilon'-\varpi') + a^2 n\,\frac{\delta \Delta \theta_{\prime\prime}}{\delta e'} + a'^2 n'\,\frac{\delta \Delta \theta'_{\prime\prime}}{\delta e'}$$

$$- (2n'^2+2n^2-nn')\,aa' \int_0^t \sin(n't+\epsilon'-\varpi')\sin(nt-n't+\epsilon-\epsilon')\,dt$$

$$+ (2n'^2+n^2+nn')\,aa' \int_0^t \cos(n't+\epsilon'-\varpi')\cos(nt-n't+\epsilon-\epsilon')\,dt$$

$$+ a' \int_0^t \frac{\{a'-a\cos(nt-n't+\epsilon-\epsilon')\}\cos(n't+\epsilon'-\varpi') + 2a\sin(nt-n't+\epsilon-\epsilon')\sin(n't+\epsilon'-\varpi')}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}}\,dt.$$

We may ∵ consider V as expressed as a function of a, e, ϵ, ϖ, a', e', ϵ', ϖ', t.

[**20.**] This expression for V may be put under the form*

$$(294)\ldots \qquad V = m\sqrt{a}\,(\theta-\theta_0) + m'\sqrt{a'}\,(\theta'-\theta'_0) + mm'\left(V_{\prime\prime} - \frac{\theta_{\prime\prime}}{an} - \frac{\theta'_{\prime\prime}}{a'n'}\right);$$

& if as a 1st approximation we neglect the terms multiplied by mm', we have

$$(295)\ldots \qquad V = m\sqrt{a}\,(\theta-\theta_0) + m'\sqrt{a'}\,(\theta'-\theta'_0);$$

& we are to try to express a, a' as functions of h, m, m', r, θ, r', θ', r_0, θ_0, r'_0, θ'_0 by the four equations (158)...(161) & the analogous approximate equations

$(296)\ldots \quad r = a - ae\cos(nt+\epsilon-\varpi)$, $\qquad (297)\ldots \quad \theta = nt+\epsilon+2e\sin(nt+\epsilon-\varpi)$,

$(298)\ldots \quad r' = a' - a'e'\cos(n't+\epsilon'-\varpi')$, $\qquad (299)\ldots \quad \theta' = n't+\epsilon'+2e'\sin(n't+\epsilon'-\varpi')$,

together with the following other approximate equation

$$(300)\ldots \qquad 0 = 2h + \frac{ma^{-1}}{1+m} + \frac{m'a'^{-1}}{1+m'}.$$

The equations (159) (297) give

$$(301)\ldots \qquad \frac{\theta-\theta_0-nt}{} - e\cos\left(\frac{nt}{2}+\epsilon-\varpi\right)\sin\frac{nt}{2};$$

[From (279).]

& (158) (296) give

(302)... $$\frac{r+r_0}{2a} = 1 - e\cos\left(\frac{nt}{2}+\epsilon-\varpi\right)\cos\frac{nt}{2}.$$

∵ (303)... $$nt = \theta-\theta_0+4\left(\frac{r+r_0}{2a}-1\right)\tan\frac{nt}{2};$$

∵ nearly

(304)... $$t = (1+m)\,a^{\frac{3}{2}}\left\{\theta-\theta_0+4\left(\frac{r+r_0}{2a}-1\right)\tan\frac{\theta-\theta_0}{2}\right\}.$$

In like manner the other ellipse gives, nearly,

(305)... $$t = (1+m')\,a'^{\frac{3}{2}}\left\{\theta'-\theta'_0+4\left(\frac{r'+r'_0}{2a'}-1\right)\tan\frac{\theta'-\theta'_0}{2}\right\}$$

We have then the approximate equation

(306)... $$(1+m)\,a^{\frac{3}{2}}\left\{\theta-\theta_0+4\left(\frac{r+r_0}{2a}-1\right)\tan\frac{\theta-\theta_0}{2}\right\}$$
$$= (1+m')\,a'^{\frac{3}{2}}\left\{\theta'-\theta'_0+4\left(\frac{r'+r'_0}{2a'}-1\right)\tan\frac{\theta'-\theta'_0}{2}\right\}$$

to be combined with the approximate equation $0 = 2h + \frac{m(1+m)^{-1}}{a} + \frac{m'(1+m')^{-1}}{a'}$ in order to determine a, a' as functions of h, m, m', r, r', θ, θ', r_0, θ_0, r'_0, θ'_0. Now supposing the initial coordinates & a, a' to vary, the equation (306) gives

(307)... $$\frac{3t}{4}\left(\frac{\delta a}{a}-\frac{\delta a'}{a'}\right) - \frac{(1+m)(r+r_0)}{\sqrt{a}}\tan\frac{\theta-\theta_0}{2}\,.\,\delta a + \frac{(1+m')(r'+r'_0)}{\sqrt{a'}}\tan\frac{\theta'-\theta'_0}{2}\,.\,\delta a'$$
$$= (1+m)\,a^{\frac{3}{2}}\,\delta\theta_0\left\{\frac{1}{2}+\left(\frac{r+r_0}{2a}-1\right)\left(\sec\frac{\theta-\theta_0}{2}\right)^2\right\} - (1+m')\,a'^{\frac{3}{2}}\,\delta\theta'_0\left\{\frac{1}{2}+\left(\frac{r'+r'_0}{2a'}-1\right)\left(\sec\frac{\theta'-\theta'_0}{2}\right)^2\right\}$$
$$-(1+m)\sqrt{a}\,\delta r_0\tan\frac{\theta-\theta_0}{2} + (1+m')\sqrt{a'}\,\delta r'_0\tan\frac{\theta'-\theta'_0}{2};$$

while (300) gives

(308)... $$0 = \frac{m}{1+m}\frac{\delta a}{a^2} + \frac{m'}{1+m'}\frac{\delta a'}{a'^2};$$

& if we use the approximation (295) for V, which does not involve the squares of the excentricities, we must omit the excentricities themselves in the coefficients of δV; we must ∵ by (308) reduce the variation δV to the following, in which we shall make the final coordinates vary,

(309)... $$\delta V = m\sqrt{a}\,(\delta\theta-\delta\theta_0) + m'\sqrt{a'}\,(\delta\theta'-\delta\theta'_0),$$

because the part

(310)... $$\frac{m(\theta-\theta_0)}{2}\frac{\delta a}{\sqrt{a}} + \frac{m'(\theta'-\theta'_0)}{2}\frac{\delta a'}{\sqrt{a'}} = 0,$$

observing that $(\theta-\theta_0)$ & $(\theta'-\theta'_0)$ are to be considered as proportional to $\frac{a^{-\frac{3}{2}}}{1+m}$, $\frac{a'^{-\frac{3}{2}}}{1+m'}$ when we neglect the excentricities. The approximate expression (309) of δV corresponds to circular

& uniform motion. For, in general, the fundamental property of the variation of V may be expressed rigorously by the following polar equation

$$
\begin{aligned}
(311)\ldots\quad \delta V = {} & m(1+m)\left(\frac{dr}{dt}\delta r + r^2\frac{d\theta}{dt}\delta\theta\right) + m'(1+m')\left(\frac{dr'}{dt}\delta r' + r'^2\frac{d\theta'}{dt}\delta\theta'\right) \\
& - m(1+m)\left(\frac{dr_0}{dt}\delta r_0 + r_0^2\frac{d\theta_0}{dt}\delta\theta_0\right) - m'(1+m')\left(\frac{dr'_0}{dt}\delta r'_0 + r'^2_0\frac{d\theta'_0}{dt}\delta\theta'_0\right) \\
& + mm'\left\{\left(\frac{dr}{dt}\delta r' + \frac{dr'}{dt}\delta r + rr'\frac{d\theta}{dt}\delta\theta' + rr'\frac{d\theta'}{dt}\delta\theta\right)\cos(\theta-\theta')\right. \\
& \left. + \left(r'\frac{dr}{dt}\delta\theta' - r\frac{dr'}{dt}\delta\theta - r\frac{d\theta}{dt}\delta r' + r'\frac{d\theta'}{dt}\delta r\right)\sin(\theta-\theta')\right\} \\
& - mm'\left\{\left(\frac{dr_0}{dt}\delta r'_0 + \frac{dr'_0}{dt}\delta r_0 + r_0 r'_0\frac{d\theta_0}{dt}\delta\theta'_0 + r_0 r'_0\frac{d\theta'_0}{dt}\delta\theta_0\right)\cos(\theta_0-\theta'_0)\right. \\
& \left. + \left(r'_0\frac{dr_0}{dt}\delta\theta'_0 - r_0\frac{dr'_0}{dt}\delta\theta_0 - r_0\frac{d\theta_0}{dt}\delta r'_0 + r'_0\frac{d\theta'_0}{dt}\delta r_0\right)\sin(\theta_0-\theta'_0)\right\};
\end{aligned}
$$

if then we neglect mm', (309) gives here

$$(312)\ldots\quad \frac{dr}{dt}=0, \qquad (313)\ldots\quad \frac{dr'}{dt}=0,$$

$$\therefore\ (314)\ldots\quad r=r_0, \qquad (315)\ldots\quad r'=r'_0;$$

and

$$(316)\ldots\quad \frac{d\theta}{dt}=\frac{\sqrt{a}\,r_0^{-2}}{1+m}=\frac{na^2}{r_0^2}, \qquad (317)\ldots\quad \frac{d\theta'}{dt}=\frac{\sqrt{a'}\,r_0'^{-2}}{1+m'}=\frac{n'a'^2}{r_0'^2},$$

denoting by n, n' the functions (60) (61) of a, m, a', m' and supposing that a, a' are functions of $\theta-\theta_0$, $\theta'-\theta'_0$, determined by the equation (300) and by the following

$$(318)\ldots\quad (1+m)\,a^{\frac{3}{2}}(\theta-\theta_0)=(1+m')\,a'^{\frac{3}{2}}(\theta'-\theta'_0),$$

which gives

$$(319)\ldots\quad \frac{\theta-\theta_0}{n}=\frac{\theta'-\theta'_0}{n'}.$$

Hence we see that

$$(320)\ldots\quad \frac{a}{r_0}=\frac{a'}{r'_0};$$

& to determine this common ratio we have

$$(321)\ldots\quad \frac{V}{t}=\frac{mna^{\frac{5}{2}}}{r_0^2}+\frac{m'n'a'^{\frac{5}{2}}}{r_0'^2}=\frac{mar_0^{-2}}{1+m}+\frac{m'a'r_0'^{-2}}{1+m'}=\frac{a}{r_0}\left\{\frac{mr_0^{-1}}{1+m}+\frac{m'r_0'^{-1}}{1+m'}\right\};$$

but also

$$(322)\ldots\quad \frac{V}{t}-\frac{dV}{dt}=\Sigma mv^2=2h+\frac{2mr_0^{-1}}{1+m}+\frac{2m'r_0'^{-1}}{1+m'}$$

&

$$(323)\ldots\quad 2h=-\frac{r_0}{a}\left(\frac{mr_0^{-1}}{1+m}+\frac{m'r_0'^{-1}}{1+m'}\right);$$

$$\therefore\ (324)\ldots\quad \frac{a}{r_0}+\frac{r_0}{a}=2, \qquad (325)\ldots\quad r_0=a.$$

Thus, from the approximate expression (295)... $V = m\sqrt{a}(\theta-\theta_0) + m'\sqrt{a'}(\theta'-\theta'_0)$, in which a, a' are approximately such functions of the initial and final coordinates as to satisfy the conditions (300) (318), we might conclude by my general methods (including the employment of the law of living forces) that the 2 planets move nearly in circular orbits with constant radii a, a' & with uniform angular velocities $\frac{a^{-\frac{3}{2}}}{1+m}, \frac{a'^{-\frac{3}{2}}}{1+m'}$. And, in like manner, from the approximate expression (294), it must be possible to determine the perturbations of these nearly circular orbits.

[*Perturbations of nearly circular orbits.*]

[21.] For this determination, we shall employ a notation analogous to that already introduced, and shall put

(326)... $a = a_{,} + m'a_{,,}$; (327)... $a' = a'_{,} + ma'_{,,}$;

(328)... $n = n_{,} + m'n_{,,}$; (329)... $n' = n'_{,} + mn'_{,,}$; (330)... $t = t_{,} + mm't_{,,}$;

(331)... $\epsilon = \epsilon_{,} + m'\epsilon_{,,}$; (332)... $\epsilon' = \epsilon'_{,} + m\epsilon'_{,,}$;

& so for the other elements e, e', ϖ, ϖ' we shall put

(333)... $a_{,} = (a_{,}) + \Delta a_{,}$; (334)... $a'_{,} = (a'_{,}) + \Delta a'_{,}$; &c.;

& then we shall have, for the parts which may be called elliptic, the equations

(335)... $$0 = 2h + \frac{m(1+m)^{-1}}{(a_{,}) + \Delta a_{,}} + \frac{m'(1+m')^{-1}}{(a'_{,}) + \Delta a'_{,}};$$

(336)... $$(t_{,}) + \Delta t_{,} = (1+m)$$
$$\times \left\{\{(a_{,}) + \Delta a_{,}\}(\theta-\theta_0) + 4\left(\frac{r+r_0}{2} - (a_{,}) - \Delta a_{,}\right)\tan\frac{\theta-\theta_0}{2}\right\}\sqrt{(a_{,}) + \Delta a_{,}};$$

(337)... $$(t_{,}) + \Delta t_{,} = (1+m')$$
$$\times \left\{\{(a'_{,}) + \Delta a'_{,}\}(\theta'-\theta'_0) + 4\left(\frac{r'+r'_0}{2} - (a'_{,}) - \Delta a'_{,}\right)\tan\frac{\theta'-\theta'_0}{2}\right\}\sqrt{(a'_{,}) + \Delta a'_{,}};$$

that is, we are first to determine the circular parts $(a_{,})$, $(a'_{,})$, $(t_{,})$ by the 3 equations

(338)... $$0 = 2h + \frac{m(1+m)^{-1}}{(a_{,})} + \frac{m'(1+m')^{-1}}{(a'_{,})}$$

(339)... $(t_{,}) = (1+m)(a_{,})^{\frac{3}{2}}(\theta-\theta_0)$, (340)... $(t_{,}) = (1+m')(a'_{,})^{\frac{3}{2}}(\theta'-\theta'_0)$,

and then their elliptic corrections $\Delta a_{,}$, $\Delta a'_{,}$, $\Delta t_{,}$ by these 3 other relations

(341)... $$0 = \frac{m}{1+m}\frac{\Delta a_{,}}{(a_{,})^2} + \frac{m'}{1+m'}\frac{\Delta a'_{,}}{(a'_{,})^2},$$

(342)... $$\Delta t_{,} = \frac{3}{2}(1+m)(a_{,})^{\frac{1}{2}}\left(\theta-\theta_0-4\tan\frac{\theta-\theta_0}{2}\right)\Delta a_{,}$$
$$+ (1+m)(a_{,})^{\frac{1}{2}}\left((r+r_0)\left(2+\frac{\Delta a_{,}}{(a_{,})}\right) - 4(a_{,})\right)\tan \theta-\theta$$

(343)... $$\Delta t_{,} = (1+m')(a'_{,})^{\frac{1}{2}}\left\{\left(\frac{3}{2}(\theta'-\theta'_0) - 6\tan\frac{\theta'-\theta'_0}{2}\right)\Delta a'_{,}\right.$$
$$\left. + \left((r'+r'_0)\left(2+\frac{\Delta a'_{,}}{(a'_{,})}\right) - 4(a'_{,})\right)\tan\frac{\theta'-\theta'_0}{2}\right\};$$

which will give $\Delta a_{,}$, $\Delta a'_{,}$, $\Delta t_{,}$ as linear functions each of $r + r_0 - 2(a')$ & of $r' + r'_0 - 2(a'_{,})$.

The relation (341) may be put under the form

(344)... $0=\frac{m\,(n_{,})\,\Delta a_{,}}{\sqrt{(a_{,})}}+\frac{m'\,(n'_{,})\,\Delta a'_{,}}{\sqrt{(a'_{,})}}$; $\because$ (345)... $\Delta a_{,}=m'\,(n'_{,})\,\sqrt{(a_{,})}\,\Delta A$,

(346)... $$\Delta a'_{,}=-m\,(n_{,})\,\sqrt{(a'_{,})}\,\Delta A,$$

ΔA being a new sought quantity. To determine this, we have by (342) (343), if we put

(347)... $$B=\frac{3}{2}\frac{m'\,(n'_{,})}{(n_{,})\sqrt{(a_{,})}}\left(\theta-\theta_0-4\tan\frac{\theta-\theta_0}{2}\right)+\frac{3}{2}\frac{m\,(n_{,})}{(n'_{,})\sqrt{(a'_{,})}}\left(\theta'-\theta'_0-4\tan\frac{\theta'-\theta'_0}{2}\right)$$
$$\frac{m'\,(n'_{,})}{(n_{,})\sqrt{(a_{,})}}\frac{r+r_0}{(a_{,})}\tan\frac{\theta-\theta_0}{2}+\frac{m\,(n_{,})}{(n'_{,})\sqrt{(a'_{,})}}\frac{r'+r'_0}{(a'_{,})}\tan$$

(348)... $$\Delta A=-\frac{2}{B\,(n_{,})}\tan\frac{\theta-\theta_0}{2}\left(\frac{r+r_0}{(a_{,})}-2\right)+\frac{2}{B\,(n'_{,})}\tan\frac{\theta'-\theta'_0}{2}\left(\frac{r'+r'_0}{(a'_{,})}-2\right).$$

Hence also

(349)... $$B\Delta t_{,}=\frac{2}{(n_{,})}\tan\frac{\theta-\theta_0}{2}\left(\frac{r+r_0}{(a_{,})}-2\right)$$
$$\times\left\{B-\frac{3m'\,(n'_{,})}{2\,(n_{,})\sqrt{(a_{,})}}\left(\theta-\theta_0-4\tan\frac{\theta-\theta_0}{2}+\frac{2}{3}\frac{r+r_0}{(a_{,})}\tan\frac{\theta-}{2}\right.\right.$$
$$\frac{2}{(n'_{,})}\tan\frac{\theta'-\theta'_0}{2}\left(\frac{r'+r'_0}{(a'_{,})}-2\right)\times\frac{3m'\,(n'_{,})}{2\,(n_{,})\sqrt{(a_{,})}}\left(\theta-\theta_0-4\tan\frac{\theta-\theta_0}{2}+\frac{2}{3}\frac{r+r_0}{(a_{,})}\tan\frac{\theta-\theta_0}{2}\right),$$

that is,

(350)... $$\Delta t_{,}=\frac{3m}{B\,(n'_{,})\sqrt{(a'_{,})}}\tan\frac{\theta-\theta_0}{2}\left(\frac{r+r_0}{(a_{,})}-2\right)\left\{\theta'-\theta'_0-4\tan\frac{\theta'-\theta'_0}{2}+\frac{2}{3}\frac{r'+r'_0}{(a'_{,})}\tan\frac{\theta'-\theta'_0}{2}\right\}$$
$$+\frac{3m'}{B\,(n_{,})\sqrt{(a_{,})}}\tan\frac{\theta'-\theta'_0}{2}\left(\frac{r'+r'_0}{(a'_{,})}-2\right)\left\{\theta-\theta_0-4\tan\frac{\theta-\theta_0}{2}+\frac{2}{3}\frac{r+r_0}{(a_{,})}\tan\frac{\theta-\theta_0}{2}\right\}.$$

And, since we neglect the squares and products of excentricities, we may reduce B to its circular value

(351)... $$(B)=\frac{m'\,(n'_{,})}{(n_{,})\sqrt{(a_{,})}}\left\{\frac{3}{2}(\theta-\theta_0)-4\tan\frac{\theta-\theta_0}{2}\right\}+\frac{m\,(n_{,})}{(n'_{,})\sqrt{(a'_{,})}}\left\{\frac{3}{2}(\theta'-\theta'_0)-4\tan\frac{\theta'-\theta'_0}{2}\right\};$$

& may put

(352)... $$\Delta t_{,}=\frac{m}{(B)\,(n'_{,})\sqrt{(a_{,})}}\left(\frac{r+r_0}{(a_{,})}-2\right)\tan\frac{\theta-\theta_0}{2}\left\{3\,(\theta'-\theta'_0)-8\tan\frac{\theta'-\theta'_0}{}\right.$$
$$\frac{m'}{(B)\,(n_{,})\sqrt{(a'_{,})}}\left(\frac{r'+r'_0}{(a'_{,})}-2\right)\tan\frac{\theta'-\theta'_0}{2}\left\{3\,(\theta-\theta_0)-8\tan\frac{\theta-\theta}{2}\right.$$

(353)... $$\Delta a_{,}=\frac{-2m'\,(n'_{,})\sqrt{(a_{,})}}{(B)}\left\{\frac{1}{(n_{,})}\tan\frac{\theta-\theta_0}{2}\left(\frac{r+r_0}{(a_{,})}-2\right)-\frac{1}{(n'_{,})}\tan\frac{\theta'-\theta'_0}{2}\left(\frac{r'+r'_0}{(a'_{,})}-2\right)\right\};$$

(354)... $$\Delta a'_{,}=\frac{2m\,(n_{,})\sqrt{(a'_{,})}}{(B)}\left\{\frac{1}{(n_{,})}\tan\frac{\theta-\theta_0}{2}\left(\frac{r+r_0}{(a_{,})}-2\right)-\frac{1}{(n'_{,})}\tan\frac{\theta'-\theta'_0}{2}\left(\frac{r'+r'_0}{(a'_{,})}-2\right)\right\}.$$

Thus, neglecting squares and products of excentricities,

(355)... $$m\Delta\sqrt{a_{,}}\,.\,(\theta-\theta_0)+m'\Delta\sqrt{a'_{,}}\,.\,(\theta'-\theta'_0)=0;$$

if ∵ we neglect also the product of the masses we have

(356)... $$V = m\sqrt{(a_{,})}\,.\,(\theta-\theta_0) + m'\sqrt{(a'_{,})}\,.\,(\theta'-\theta'_0).$$

Taking account ∵ of the product of the masses, but still neglecting squares and products of excentricities, the characteristic function V may be put under the following form, deduced from the form (294),

(357)... $$V = m\sqrt{a_{,}}\,.\,(\theta-\theta_0) + m'\sqrt{a'_{,}}\,.\,(\theta'-\theta'_0) + mm'W;$$

in which we are to try to determine the *disturbing function*

(358)... $$W = V_{,,} + \frac{a_{,,}}{2\sqrt{a_{,}}}(\theta-\theta_0) + \frac{a'_{,,}}{2\sqrt{a'_{,}}}(\theta'-\theta'_0) - \frac{\theta_{,,}}{an} - \frac{\theta'_{,,}}{a'n'};$$

or rather its variation δW for the case of evanescent excentricities and of evanescent planetary masses.

[*Expressions for variation of constants* a, e, ϵ, ... *in terms of* δr, δr_0, $\delta\theta$, $\delta\theta_0$, ...]

[22.] For this purpose it seems convenient to put

(359)... $e\cos(\epsilon-\varpi) = k$;

(360)... $e\sin(\epsilon-\varpi) = l$;

(361)... $e'\cos(\epsilon'-\varpi') = k'$;

(362)... $e'\sin(\epsilon'-\varpi') = l'$;

& ∵, neglecting perturbations,

(363)... $\frac{r}{a} = 1 - k\cos nt + l\sin nt$;

(364)... $\theta = nt + \epsilon + 2k\sin nt + 2l\cos nt$;

(365)... $\frac{r'}{a'} = 1 - k'\cos n't + l'\sin n't$;

(366)... $\theta' = n't + \epsilon' + 2k'\sin n't + 2l'\cos n't$;

also, independent of perturbations

(367)... $\frac{r_0}{a} = 1 - k$;

(368)... $\theta_0 = \epsilon + 2l$;

(369)... $\frac{r_0}{a'} = 1 - k'$;

(370)... $\theta'_0 = \epsilon' + 2l'$.

Hence

(371)... $$k = 1 - \frac{r_0}{(a)} + \frac{\Delta a}{a};$$

(372)... $$l = \left(\frac{r}{(a)} - 1 - \frac{\Delta a}{a}\right)\operatorname{cosec}(\theta-\theta_0) + \left(1 - \frac{r_0}{(a)} + \frac{\Delta a}{a}\right)\operatorname{cotan}(\theta-\theta_0);$$

(373)... $$\epsilon = \theta_0 - 2l;$$

&, as a verification,

(374)... $$\Delta t = \frac{3}{2}t\frac{\Delta a}{a} + \frac{4l}{n}\left(\sin\frac{\theta-\theta_0}{2}\right)^2 - \frac{4k}{n}\sin\frac{\theta-\theta_0}{2}\cos\frac{\theta-\theta_0}{2}$$
$$= \frac{3}{2}\frac{\theta-\theta_0}{n}\frac{\Delta a}{a} + \frac{2}{n}\left(\frac{r+r_0}{(a)} - 2 - \frac{2\Delta a}{a}\right)\tan\frac{\theta-\theta_0}{2};$$

which will agree with (342) if we neglect, as we may, $\frac{\Delta a}{an}\left(\frac{r+r_0}{(a)} - 2\right)\tan\frac{\theta-\theta_0}{2}$

In like manner

(375)... $$k' = 1 - \frac{r'_0}{(a')} + \frac{\Delta a'}{a'};$$

(376)... $$l' = \left(\frac{r'}{(a')} - 1 - \frac{\Delta a'}{a'}\right) \operatorname{cosec}(\theta' - \theta'_0) + \left(1 - \frac{r'_0}{(a')} + \frac{\Delta a'}{a'}\right) \operatorname{cotan}(\theta' - \theta'_0);$$

(377)... $$\epsilon' = \theta'_0 - 2l':$$

in which, if we put

(378)... $$B = \frac{m'n'}{n\sqrt{a}} \left\{\frac{3}{2}(\theta - \theta_0) - 4\tan\frac{\theta - \theta_0}{2}\right\} + \frac{mn}{n'\sqrt{a'}} \left\{\frac{3}{2}(\theta' - \theta'_0) - 4\tan\frac{\theta' - \theta'_0}{2}\right.$$

we shall have

(379)... $$\Delta a = -\frac{2m'n'\sqrt{a}}{B} \left\{\frac{1}{n}\tan\frac{\theta - \theta_0}{2} \cdot \left(\frac{r + r_0}{(a)} - 2\right) - \frac{1}{n'}\tan\frac{\theta' - \theta'_0}{2} \cdot \left(\frac{r' + r'_0}{(a')} - 2\right)\right\},$$

(380)... $$\Delta a' = \quad 2mn\sqrt{a'}$$

Hence

(381)... $$\delta\Delta a = -\frac{2m'n'\sqrt{a}}{B} \left\{\tan\frac{\theta - \theta_0}{2} \cdot \frac{\delta r + \delta r_0}{an} - \tan\frac{\theta' - \theta'_0}{2} \cdot \frac{\delta r' + \delta r'_0}{a'n'}\right\};$$

(382)... $$\delta\Delta a' = \frac{2mn\sqrt{a'}}{B}$$

(383)... $$\delta k = \frac{-\delta r_0 + \delta\Delta a}{a};$$ (384)... $$\delta l = \frac{\delta r - \delta\Delta a}{a} \operatorname{cosec}(\theta - \theta_0) + \frac{\delta\Delta a - \delta r_0}{a} \operatorname{cotan}(\theta - \theta_0);$$

(385)... $$\delta k' = \frac{-\delta r'_0 + \delta\Delta a'}{a'};$$

(386)... $$\delta l' = \frac{\delta r' - \delta\Delta a'}{a'} \operatorname{cosec}(\theta' - \theta'_0) + \frac{\delta\Delta a' - \delta r'_0}{a'} \operatorname{cotan}(\theta' - \theta'_0);$$

(387)... $$\delta\epsilon = \delta\theta_0 - 2\delta l;$$ (388)... $$\delta\epsilon' = \delta\theta'_0 - 2\delta l'.$$

What are $\delta(t_{,})$, $\delta(a_{,})$, $\delta(a'_{,})$?

By (339) (340),

(389)... $$\delta(t_{,}) = \frac{\delta\theta - \delta\theta_0}{(n_{,})} + \frac{3}{2}(t_{,})\frac{\delta(a_{,})}{(a_{,})};$$ (390)... $$\delta(t_{,}) = \frac{\delta\theta' - \delta\theta'_0}{(n'_{,})} + \frac{3}{2}(t_{,})\frac{\delta(a'_{,})}{(a'_{,})};$$

also, by (338),

(391)... $$0 = \frac{m(n_{,})}{\sqrt{(a_{,})}}\delta(a_{,}) + \frac{m'(n'_{,})}{\sqrt{(a'_{,})}}\delta(a'_{,});$$

∵ (392)... $$\delta(t_{,}) = \frac{m\sqrt{(a_{,})}(\delta\theta - \delta\theta_0) + m'\sqrt{(a'_{,})}(\delta\theta' - \delta\theta'_0)}{m(n_{,})\sqrt{(a_{,})} + m'(n'_{,})\sqrt{(a'_{,})}};$$

(393)... $$\delta(a_{,}) = \frac{2m'}{3(t_{,})}\frac{(a_{,})\sqrt{(a'_{,})}}{(n_{,})}\frac{(n_{,})(\delta\theta' - \delta\theta'_0) - (n'_{,})(\delta\theta - \delta\theta_0)}{m(n_{,})\sqrt{(a_{,})} + m'(n'_{,})\sqrt{(a'_{,})}};$$

(394)... $$\delta(a'_{,}) = -\frac{2m}{3(t_{,})}\frac{(a'_{,})\sqrt{(a_{,})}}{(n'_{,})}\frac{(n_{,})(\delta\theta' - \delta\theta'_0) - (n'_{,})(\delta\theta - \delta\theta_0)}{m(n_{,})\sqrt{(a_{,})} + m'(n'_{,})\sqrt{(a'_{,})}}.$$

Also, neglecting perturbations, we have by (352)

$$(395)\ldots\quad \delta\Delta t = \frac{m}{Bn'a\sqrt{a'}}(\delta r + \delta r_0)\tan\frac{\theta-\theta_0}{2}\left\{3(\theta'-\theta'_0) - 8\tan\frac{\theta'-\theta'_0}{2}\right.$$

$$\left. + \frac{m}{Bna'\sqrt{a}}(\delta r' + \delta r'_0)\tan\frac{\theta'-\theta'_0}{2}\left\{3(\theta-\theta_0) - 8\tan\frac{\theta-\theta_0}{2}\right\}\right. .$$

We have therefore expressed the *elliptic variations* of the 9 quantities a, ϵ, k, l, a', ϵ', k', l', t, in terms of the variations of the 8 extreme polar coordinates, neglecting excentricities in the coefficients of these variations; that is, we have expressed the variations of the 9 elliptic quantities $a_{,}$, $\epsilon_{,}$, $k_{,}$, $l_{,}$, $a'_{,}$, $\epsilon'_{,}$, $k'_{,}$, $l'_{,}$, t, and they may be thus recapitulated:

$$(396)\ldots\quad \delta a_{,} = \frac{2}{3}\frac{m'a\sqrt{a'}}{nt}\,\frac{n(\delta\theta'-\delta\theta'_0) - n'(\delta\theta-\delta\theta_0)}{mn\sqrt{a}+m'n'\sqrt{a'}} - \frac{2m'n'\sqrt{a}}{B}$$

$$\times\left\{\frac{\delta r+\delta r_0}{an}\tan\frac{\theta-\theta_0}{2} - \frac{\delta r'+\delta r'_0}{a'n'}\tan\frac{\theta'-\theta'_0}{2}\right\};$$

$$(397)\ldots\quad \delta a'_{,} = -\frac{2}{3}\frac{ma'\sqrt{a}}{n't}\cdot\frac{n(\delta\theta'-\delta\theta'_0) - n'(\delta\theta-\delta\theta_0)}{mn\sqrt{a}+m'n'\sqrt{a'}} + \frac{2mn\sqrt{a'}}{B}$$

$$\times\left\{\frac{\delta r+\delta r_0}{an}\tan\frac{\theta-\theta_0}{2} - \frac{\delta r'+\delta r'_0}{a'n'}\tan\frac{\theta'-\theta'_0}{2}\right\};$$

$$(398)\ldots\quad \delta t_{,} = \frac{m\sqrt{a}(\delta\theta-\delta\theta_0) + m'\sqrt{a'}(\delta\theta'-\delta\theta'_0)}{mn\sqrt{a}+m'n'\sqrt{a'}} + \frac{m(\delta r+\delta r_0)}{Bn'a\sqrt{a'}}\tan\frac{\theta-\theta_0}{2}$$

$$\times\left\{3(\theta'-\theta'_0) - 8\tan\frac{\theta'-\theta'_0}{2}\right\} + \frac{m'(\delta r'+\delta r'_0)}{Bna'\sqrt{a}}\tan\frac{\theta'-\theta'_0}{2}\left\{3(\theta-\theta_0) - 8\tan\frac{\theta-\theta_0}{2}\right.$$

together with the values (383)...(388), combined with (381) (382), for $\delta k_{,}$, $\delta k'_{,}$, $\delta l_{,}$, $\delta l'_{,}$, $\delta\epsilon_{,}$, $\delta\epsilon'_{,}$; B having the value (378) & the quantities a, a', n, n', t in the coefficients of these variations having their 1st approximate or circular values.

[*The disturbing function W*.]

[**23.**] Now in the *disturbing function* W, (358), the quantities $a_{,,}$, $a'_{,,}$ are connected by the relation resulting from (269), (326), (327), and from (335), or from the following

$$(399)\ldots\qquad h = \frac{-m}{2(1+m)a_{,}} - \frac{m'}{2(1+m')a'_{,}},$$

namely

$$(400)\ldots\qquad \frac{n_{,}a_{,,}}{\sqrt{a_{,}}} + \frac{n'_{,}a'_{,,}}{\sqrt{a'_{,}}} = -2H_{,,};$$

in the second member of which we may substitute elliptic values.

By a similar substitution

$$(401)\ldots\qquad \theta-\theta_0 = n_{,}t_{,} + 2k_{,}\sin n_{,}t_{,} - 2l_{,}\,\mathrm{vers}\,n_{,}t_{,};$$

$$(402)\ldots\qquad \theta'-\theta'_0 = n'_{,}t_{,} + 2k'_{,}\sin n'_{,}t_{,} - 2l'_{,}\,\mathrm{vers}\,n'_{,}t_{,};$$

we may ∵ put

(403)... $$\frac{a_{\prime\prime}}{2\sqrt{a_{,}}}(\theta-\theta_0)+\frac{a'_{\prime\prime}}{2\sqrt{a'_{,}}}(\theta'-\theta'_0)$$
$$=-tH_{\prime\prime}+\frac{a_{\prime\prime}}{\sqrt{a_{,}}}(k\sin nt-l\operatorname{vers}nt)+\frac{a'_{\prime\prime}}{\sqrt{a'_{,}}}(k'\sin n't-l'\operatorname{vers}n't),$$

and we obtain an expression for the disturbing function,* which may be put under the form

(404)... $$W=(W)+\Delta W,$$

(W) being the circular part, and ΔW the correction for excentricities; or, in full,

(405)... $$W=\frac{a_{,}a'_{,}(n_{,}n'_{,}-n_{,}^2-n'^2_{,})}{n_{,}-n'_{,}}\{\sin(n_{,}t_{,}-n'_{,}t_{,}+\epsilon_{,}-\epsilon'_{,})-\sin(\epsilon_{,}-\epsilon'_{,})\}$$
$$+\int_0^{t_{,}}\{a_{,}^2-2a_{,}a'_{,}\cos(n_{,}t_{,}-n'_{,}t_{,}+\epsilon_{,}-\epsilon'_{,})+a'^2_{,}\}^{-\frac{1}{2}}dt_{,}+kK+lL+k'K'+l'L',$$

in which

(406)... $$K=\frac{r_{\prime\prime}+a_{\prime\prime}}{\sqrt{a}}\sin nt+aa'\left(2n^2+\frac{3}{2}n'^2\right)\int_0^t\cos(n't+\epsilon'-\epsilon)\,dt$$
$$a'\left(nn'-\frac{n'^2}{2}\right)\int\cos(2nt-n't+\epsilon-\epsilon')\,dt$$
$$+a\int_0^t\frac{(a-a'\cos\overline{nt}-n't+\epsilon-\epsilon')\cos nt-2a'\sin(nt-n't+\epsilon-\epsilon')\sin nt}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}}dt,$$

that is,

(407)... $$K=\frac{r_{\prime\prime}+a_{\prime\prime}}{\sqrt{a}}\sin nt+\frac{aa'}{2n'}\{(4n^2+3n'^2)\sin(n't+\epsilon'-\epsilon)+n'^2\sin(2nt-n't+\epsilon-\epsilon')$$
$$+(4n^2+2n'^2)\sin(\epsilon-\epsilon')\}+a\int^t\frac{(a-a'\cos nt-\overline{n't+\epsilon-\epsilon'})\cos nt-2a'\sin(nt-n't+\epsilon-\epsilon')\sin nt}{\{}dt;$$

(408)... $$L=\frac{(r_{\prime\prime}+a_{\prime\prime})\cos nt-a_{\prime\prime}}{\sqrt{a}}$$
$$+\frac{aa'}{2n'}\{(4n^2+3n'^2)\cos(n't+\epsilon'-\epsilon)+n'^2\cos(2nt-n't+\epsilon-\epsilon')-4(n^2+n'^2)\cos(\epsilon-\epsilon')\}$$
$$-a\int_0^t\frac{(a-a'\cos\overline{nt-n't+\epsilon-\epsilon'})\sin nt+2a'\sin(nt-n't+\epsilon-\epsilon')\cos nt}{\{}dt;$$

(409)... $$K'=\frac{r'_{\prime\prime}+a'_{\prime\prime}}{\sqrt{a'}}\sin n't$$
$$+\frac{aa'}{2n}\{(4n'^2+3n^2)\sin(nt+\epsilon-\epsilon')+n^2\sin(2n't-nt+\epsilon'-\epsilon)+(4n'^2+2n^2)\sin(\epsilon'-\epsilon)\}$$
$$+a'\int_0^t\frac{(a'-a\cos\overline{nt-n't+\epsilon-\epsilon'})\cos n't+2a\sin(nt-n't+\epsilon-\epsilon')\sin n't}{\{\qquad\qquad\}^{\frac{3}{2}}}dt;$$

---

* [From (289) and (363)–(366).]

$$(410)\ldots\quad L'=\frac{(r'_{\prime\prime}+a'_{\prime\prime})\cos n't-a'_{\prime\prime}}{\sqrt{a'}}$$

$$+\frac{aa'}{2n}\left\{(4n'^2+3n^2)\cos(nt+\epsilon-\epsilon')+n^2\cos(2n't-nt+\epsilon'-\epsilon)-4(n^2+n'^2)\cos(\epsilon-\epsilon')\right\}$$

$$-a'\int_0^t\frac{(a'-a\cos\overline{nt-n't+\epsilon-\epsilon'})\sin n't-2a\sin(nt-n't+\epsilon-\epsilon')\cos n't}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}}\,dt.$$

And

$$(411)\ldots\quad \delta W=C\,\delta t_{,}+D\,\delta a_{,}+D'\,\delta a'_{,}+E\,(\delta\epsilon-\delta\epsilon')+K\,\delta k+L\,\delta l+K'\,\delta k'+L'\,\delta l',$$

if we put

$$(412)\ldots\quad C=aa'(nn'-n^2-n'^2)\cos(nt-n't+\epsilon-\epsilon')+\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}};$$

$$(413)\ldots\quad D=\frac{a'(nn'-n^2-n'^2)}{n-n'}\{\sin(nt-n't+\epsilon-\epsilon')-\sin(\epsilon-\epsilon')\}$$

$$+\frac{\delta}{\delta a}\int_0^t\frac{dt}{\sqrt{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')}}-\frac{3Fn}{2a};$$

$$(414)\ldots\quad D'=\frac{a(nn'-n^2-n'^2)}{n-n'}\{\sin(nt-n't+\epsilon-\epsilon')-\sin(\epsilon-\epsilon')\}$$

$$+\frac{\delta}{\delta a'}\int_0^t\frac{dt}{\sqrt{a^2+a'^2-2aa'\cos(nt-n't+\epsilon-\epsilon')}}-\frac{3F'n'}{2a'};$$

F, F' being the coefficients of δn, $\delta n'$ when n, n' are treated as independent of a, a'; so that, because

$$(415)\ldots\quad \frac{nn'-n^2-n'^2}{n-n'}=-\frac{3}{4}(n-n')-\frac{1}{4}\frac{(n+n')^2}{n-n'},$$

$$(416)\ldots\quad \frac{F-F'}{2}=\frac{aa't(nn'-n^2-n'^2)}{n-n'}\cos(nt-n't+\epsilon-\epsilon')$$

$$+\frac{aa'}{2}\,\frac{4nn'-n^2-n'^2}{(n-n')^2}\{\sin(nt-n't+\epsilon-\epsilon')-\sin(\epsilon-\epsilon')\}$$

$$-aa'\int_0^t\frac{t\sin(nt-n't+\epsilon-\epsilon')\,dt}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}};$$

$$(417)\ldots\quad \frac{F+F'}{2}=-\frac{aa'}{2}\,\frac{n+n'}{n-n'}\{\sin(nt-n't+\epsilon-\epsilon')-\sin(\epsilon-$$

and finally

$$(418)\ldots\quad E=\frac{aa'(nn'-n^2-n'^2)}{n-n'}\{\cos(nt-n't+\epsilon-\epsilon')-\cos(\epsilon-\epsilon')\}$$

$$-aa'\int_0^t\frac{\sin(nt-n't+\epsilon-\epsilon')\,dt}{\{a^2-2aa'\cos(nt-n't+\epsilon-\epsilon')+a'^2\}^{\frac{3}{2}}}.$$

In all these coefficients of δW we are to change a, a', n, n', t, ϵ, ϵ' to their circular values; and then we may put, by (311) (357), observing that

$$(419)\ldots\quad m(\theta-\theta_0)\,\delta\sqrt{a_{,}}+m'(\theta'-\theta'_0)\,\delta\sqrt{a'_{,}}=0,$$

and supposing only the initial coordinates to vary,

$$(420)\ldots\quad \boxed{\begin{aligned}\delta W = (an\,\delta r_0' - a'n'\delta r_0)\sin(\theta_0-\theta_0') - aa'\,(n\,\delta\theta_0' + n'\delta\theta_0)\cos(\theta_0-\theta_0')\\ -\frac{a_{\prime\prime}}{2\sqrt{a}}\,\delta\theta_0 - \frac{a_{\prime\prime}'}{2\sqrt{a'}}\,\delta\theta_0'.\end{aligned}}$$

The terms $-\frac{\omega_{\prime\prime}}{2\sqrt{a}}\delta\theta_0$, $-\frac{\omega_{\prime\prime}}{2\sqrt{a'}}\delta\theta_0'$ are here obtained from $m\left\{\sqrt{(a_{\prime})}-(1+m)r_0^2\frac{d\theta_0}{dt}\right\}\delta\theta_0+\&c.$ by observing that

$$(421)\ldots\qquad (1+m)\,r_0^2\frac{d\theta_0}{dt} = \sqrt{a} = \sqrt{(a_{\prime})+m'a_{\prime\prime}},\ \&c.$$

We have ∵ the four following equations:

$$(422)\ldots\quad 0=\frac{\delta W}{\delta\theta_0}+aa'n'\cos(\epsilon-\epsilon')+\frac{a_{\prime\prime}}{2\sqrt{a}};\qquad (423)\ldots\quad 0=\frac{\delta W}{\delta r_0}+a'n'\sin(\epsilon-\epsilon');$$

$$(424)\ldots\quad 0=\frac{\delta W}{\delta\theta_0'}+aa'n\cos(\epsilon-\epsilon')+\frac{a_{\prime\prime}'}{2\sqrt{a'}};\qquad (425)\ldots\quad 0=\frac{\delta W}{\delta r_0'}-an\sin(\epsilon-\epsilon');$$

in which the 4 partial differential coefficients of W are functions of the circular elements a, a', n, n', ϵ, ϵ', the time t, the coefficients of perturbation $r_{\prime\prime}$, $a_{\prime\prime}$, $r_{\prime\prime}'$, $a_{\prime\prime}'$ and the true centrobaric longitudes θ, θ', which may evidently in the present relations be changed to the circular quantities $nt+\epsilon$, $n't+\epsilon'$: in which n, n' are functions of a, a'. If ∵ the four relations are not equivalent to only three, they will enable us to determine the 4 coefficients $r_{\prime\prime}$, $a_{\prime\prime}$, $r_{\prime\prime}'$, $a_{\prime\prime}'$ as functions of the time t and of the four circular elements a, a', ϵ, ϵ'. And though there should be, as I think there will, a mutual dependence between these four relations arising from the initial state of the equation of living forces, yet that same equation will furnish a fifth relation, namely (400), or the following:

$$(426)\ldots\qquad \frac{a_{\prime\prime}}{a^2}+\frac{a_{\prime\prime}'}{a'^2} = -2\,(H_{\prime\prime});$$

& this, I am pretty sure, will enable us to complete the determination of $r_{\prime\prime}$, $a_{\prime\prime}$, $r_{\prime\prime}'$, $a_{\prime\prime}'$ as functions of t, a, a', ϵ, ϵ': after which it will only remain to express $\theta_{\prime\prime}$, $\theta_{\prime\prime}'$ by means of $r_{\prime\prime}$, $a_{\prime\prime}$, $r_{\prime\prime}'$, $a_{\prime\prime}'$, t, a, a', ϵ, ϵ'.

[*Difficulties arising from the assumption* $t=t_{\prime}+mm't_{\prime\prime}$.*]

[**24.**] For this last purpose, we shall seek to express $a_{\prime\prime}$, $a_{\prime\prime}'$ as functions of $r_{\prime\prime}$, $r_{\prime\prime}'$, $\theta_{\prime\prime}$, $\theta_{\prime\prime}'$, a, a', ϵ, ϵ', t, by the conditions

$$(427)\ldots\quad a_{\prime\prime}=\frac{a-(a_{\prime})-\Delta a_{\prime}}{m'};\qquad (428)\ldots\quad a_{\prime\prime}'=\frac{a'-(a_{\prime}')-\Delta a_{\prime}'}{m};$$

combined with the equations (338) (339) (340) (378) (379) (380), or (335) (336) (337), and with the following:

$$(429)\ldots\quad r=a+m'r_{\prime\prime};\qquad (430)\ldots\quad r'=a'+mr_{\prime\prime}';$$

$$(431)\ldots\quad \theta=nt+\epsilon+m'\theta_{\prime\prime};\qquad (432)\ldots\quad \theta'=n't+\epsilon'+m\theta_{\prime\prime}'.$$

---

* [See Appendix, Note 1, p. 612.]

Substituting these last values for θ, θ' in (336) (337) and changing $(a_{\prime}) + \Delta a_{\prime}$ to $a - m'a_{\prime\prime}$ & $(a'_{\prime}) + \Delta a'_{\prime}$ to $a' - ma'_{\prime\prime}$, we find

$$(433)\ldots \quad \left(1 - \frac{m'a_{\prime\prime}}{a}\right)^{\frac{3}{2}} \left(t + \frac{m'\theta_{\prime\prime}}{n}\right) + 4\sqrt{a} \,.\, \tan\frac{nt}{2} \,.\, m'\left(a_{\prime\prime} + \frac{1}{2} r_{\prime\prime}\right)$$
$$= \left(1 - \frac{ma'_{\prime\prime}}{a}\right)^{\frac{3}{2}} \left(t + \frac{m\theta'_{\prime\prime}}{n'}\right) + 4\sqrt{a'} \,.\, \tan\frac{n't}{2} \,.\, m\left(a'_{\prime\prime} + \frac{1}{2} r'_{\prime\prime}\right),$$

observing that the initial values of $r_{\prime\prime}$, $\theta_{\prime\prime}$, $r'_{\prime\prime}$, $\theta'_{\prime\prime}$ vanish; that is,

$$(434)\ldots \quad m'\sqrt{a}\left\{a\theta_{\prime\prime} + 2r_{\prime\prime} \tan\frac{nt}{2} + a_{\prime\prime}\left(4\tan\frac{nt}{2} - \frac{3nt}{2}\right)\right\}$$
$$= m\sqrt{a'}\left\{a'\theta'_{\prime\prime} + 2r'_{\prime\prime} \tan\frac{n't}{2} + a'_{\prime\prime}\left(4\tan\frac{n't}{2} - \frac{3n't}{2}\right.\right.$$

If then we put, by a simplification of (378),

$$(435)\ldots \qquad B = \left\{\frac{m'\sqrt{a}}{a'^2}\left(\frac{3}{2}nt - 4\tan\frac{nt}{2}\right) + \frac{m\sqrt{a'}}{a^2}\left(\frac{3}{2}n't - 4\tan\frac{n't}{2}\right)\right\}\sqrt{aa'},$$

we shall have by (434) (426),

$$(436)\ldots \quad \frac{Ba_{\prime\prime}}{\sqrt{aa'}} = \frac{1}{a'^2}\left\{m'\sqrt{a}\left(a\theta_{\prime\prime} + 2r_{\prime\prime}\tan\frac{nt}{2}\right) - m\sqrt{a'}\left(a'\theta'_{\prime\prime} + 2r'_{\prime\prime}\tan\frac{n't}{2}\right)\right\}$$
$$- 2m\sqrt{a'}\,(H_{\prime\prime})\left(\frac{3}{2}n't - 4\tan\frac{n't}{2}\right),$$

that is,

$$(437)\ldots \quad a_{\prime\prime} = \frac{n'\sqrt{a}}{B}\left\{m'\sqrt{a}\left(a\theta_{\prime\prime} + 2r_{\prime\prime}\tan\frac{nt}{2}\right) - m\sqrt{a'}\left(a'\theta'_{\prime\prime} + 2r'_{\prime\prime}\tan\frac{n't}{2}\right)\right\}$$
$$\frac{2ma'\sqrt{a}}{B}(H_{\prime\prime})\left(\frac{3}{2}n't - 4\tan\frac{n't}{2}\right);$$

and similarly

$$(438)\ldots \quad a'_{\prime\prime} = -\frac{n\sqrt{a'}}{B}\left\{m'\sqrt{a}\left(a\theta_{\prime\prime} + 2r_{\prime\prime}\tan\frac{nt}{2}\right) - m\sqrt{a'}\left(a'\theta'_{\prime\prime} + 2r'_{\prime\prime}\tan\frac{n't}{2}\right)\right\}$$
$$\frac{2m'a\sqrt{a'}}{B}(H_{\prime\prime})\left(\frac{3}{2}nt - 4\tan\frac{nt}{2}\right).$$

These values ∵ of $a_{\prime\prime}$, $a'_{\prime\prime}$ are to be substituted in the 4 relations (422)...(425). However, as these values were deduced from the relation (426), that relation cannot give any additional information respecting the four functions $r_{\prime\prime}$, $r'_{\prime\prime}$, $\theta_{\prime\prime}$, $\theta'_{\prime\prime}$; & perhaps it will be impossible to determine completely these four functions without an integration relatively to the time, as in the general theory of my V.

But before having recourse to such a supplementary integration, it would be proper to consider whether the condition expressed by the equation (330), namely that the true time differs from the elliptic time only by a quantity of the order of the product of the masses, is not a true condition (on account of this difference vanishing when either mass vanishes) and whether this condition does not give the means of completing the solution of our perturbation problem without effecting the integration which would give $t_{\prime\prime}$ in the expression $t = t_{\prime} + mm't_{\prime\prime}$.

Admitting for the present the truth of this last expression, both sides of (434) will vanish & we shall have

$$(439)\ldots\quad a_{,,}=\frac{a\theta_{,,}+2r_{,,}\tan\dfrac{nt}{2}}{\dfrac{3}{2}nt-4\tan\dfrac{nt}{2}};\qquad (440)\ldots\quad a'_{,,}=\frac{a'\theta'_{,,}+2r'_{,,}\tan\dfrac{n't}{2}}{\dfrac{3}{2}n't-4\tan\dfrac{n't}{2}}.\quad \text{(Not true.)}$$

Another reason for adopting these last expressions is that otherwise the coefficients $a_{,,}$, $a'_{,,}$ would involve the ratio of the masses $\frac{m'}{m}$, which it is hard to think that they should do. And if these expressions be true, we shall have a curious relation between the perturbation coefficients $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$, the circular elements a, a', $\epsilon-\epsilon'$, & the time t, by substituting in (426) these expressions for $a_{,,}$, $a'_{,,}$.

But this last relation is certainly not true, because the expressions (439) (440) vanish with t while the constant $(H_{,,})$ does not in general vanish. I must ∵ conclude that the two expressions (434) for the excess $t_{,}-t$ of elliptic time over true do not separately vanish; and that although this excess has the product of the masses for a factor & vanishes with either of those masses, yet it is not of the order of this product but of a lower order: no doubt by its coefficient $t_{,,}$ having for divisor the quantity B which is of the order of the masses. And, in fact, from (437) (438), which we know to be true whether they can be reduced to (439) (440) or not, we find

$$(441)\ldots\quad B(t-t_{,})=Bm'a_{,,}\sqrt{a}\left(\frac{3}{2}nt-4\tan\frac{nt}{2}\right)-Bm'\sqrt{a}\left(a\theta_{,,}+2r_{,,}\tan\frac{nt}{2}\right),$$

that is,

$$(442)\ldots\quad -Bt_{,,}=\frac{a}{a'}\left(a'\theta'_{,,}+2r'_{,,}\tan\frac{n't}{2}\right)\left(\frac{3}{2}nt-4\tan\frac{nt}{2}\right)$$
$$+\frac{a'}{a}\left(a\theta_{,,}+2r_{,,}\tan\frac{nt}{2}\right)\left(\frac{3}{2}n't-4\tan\frac{n't}{2}\right)+2aa'(H_{,,})\left(\frac{3}{2}nt-4\tan\frac{nt}{2}\right)\left(\frac{3}{2}n't-4\tan\frac{n't}{2}\right);$$

so that $t_{,,}$ has B for a divisor.

[**25.**] It seems then impossible to dispense with an integration for the time, such as is required in my general method. But the four relations (422)...(425) between $r_{,,}$, $\theta_{,,}$, $r'_{,,}$, $\theta'_{,,}$, t, equivalent thus to only 3 distinct relations and involving the 3 circular constants a, a', $\epsilon-\epsilon'$, cannot involve really, though they do apparently, the ratio $\frac{m'}{m}$ of the planetary masses: they may therefore be simplified by supposing this ratio $=0$. By this supposition we find

$$(443)\ldots\quad B=mna'\left(\frac{3}{2}n't-4\tan\frac{n't}{2}\right);$$

$a'_{,,}$ is expressed by (440), & we have

$$(444)\ldots\quad a_{,,}=-2a^2(H_{,,})-\frac{a^2}{a'^2}\,\frac{a'\theta'_{,,}+2r'_{,,}\tan\dfrac{n't}{2}}{\dfrac{3}{2}n't-4\tan\dfrac{n't}{2}},\qquad (445)\ldots\quad \delta a_{,}=0;$$

&, attending only to initial variations,

(446)... $$\delta a'_{,} = \frac{2a'}{3nn't}(n\,\delta\theta'_0 - n'\delta\theta_0) + \frac{2\sqrt{\dfrac{a}{a'}}\,\delta r_0 \,.\, \tan\dfrac{nt}{2} - 2\delta r'_0 \,.\, \tan\dfrac{n't}{2}}{\dfrac{3}{2}n't - 4\tan\dfrac{n't}{2}}$$

(447)... $\delta t_{,} = -\frac{1}{n}\delta\theta_0 + \frac{2\delta r_0}{na}\tan\frac{nt}{2}$; (448)... $\delta\Delta a_{,} = 0$;

(449)... $$\delta\Delta a'_{,} = \frac{2\sqrt{\dfrac{a'}{a}}\,\delta r_0 \,.\, \tan\dfrac{nt}{2} - 2\delta r'_0 \,.\, \tan\dfrac{n't}{2}}{\dfrac{3}{2}n't - 4\tan\dfrac{n't}{2}}:$$

(450)... $\delta k = -\frac{1}{a}\delta r_0$;

(385) gives $\delta k'$;

(451)... $\delta l = -\frac{\delta r_0}{a}\,\text{cotan}\,nt$; (452)... $\delta l' = -\frac{\delta r'_0}{a'}\,\text{cotan}\,n't - \frac{\delta\Delta a'_{,}}{a'}\tan\frac{n't}{2}$;

(387) (388) give $\delta\epsilon$, $\delta\epsilon'$; thus in the two equations (422) (424) we have

(453)... $\frac{\delta W}{\delta\theta_0} = -\frac{C}{n} - \frac{2a'D'}{3nt} + E$; (454)... $\frac{\delta W}{\delta\theta'_0} = \frac{2a'D'}{3n't} - E$;

& since by (426)

(455)... $$0 = n\frac{\delta W}{\delta\theta_0} + n'\frac{\delta W}{\delta\theta'_0} + 2aa'nn'\cos(\epsilon - \epsilon') - (H_{,,}),$$

we ought to have

(456)... $$(H_{,,}) = 2aa'nn'\cos(\epsilon - \epsilon') + E(n - n') - C:$$

which, if true, as it seems that it must be, is the condition connecting the 4 relations (422)...(425). To verify this last condition we are to calculate the 2nd side by (412) (418), and to compare it with the expression (271) for 1st side.

By (418),

(457)... $$E(n - n') + 2aa'nn'\cos(\epsilon - \epsilon')$$
$$= aa'(n^2 + nn' + n'^2)\cos(\epsilon - \epsilon') - aa'(n^2 - nn' + n'^2)\cos(nt - n't + \epsilon - \epsilon')$$
$$+ \{a^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon') + a'^2\}^{-\frac{1}{2}} - \{a^2 - 2aa'\cos(\epsilon - \epsilon') + a'^2\}^{-\frac{1}{2}};$$

& in fact this is $= C + (H_{,,})$ by (412) & (271): the verification ∵ is complete.

Continuing to express the partial differential coefficients of the disturbing function W, we have

(458)... $$\frac{\delta W}{\delta r_0} = \frac{2C}{na}\tan\frac{nt}{2} + \left\{D' + \frac{K'}{a'} - \frac{2E + L'}{a'}\tan\frac{n't}{2}\right\} . \frac{2\sqrt{\dfrac{a}{a'}}\tan\dfrac{nt}{2}}{\dfrac{3}{2}n't - 4\tan\dfrac{n't}{2}}$$
$$- \frac{K}{a} + \frac{2E - L}{a}\,\text{cotan}\,nt;$$

(459)... $$\frac{\delta W}{\delta r'_0} = -\left\{D' + \frac{K'}{a'} - \frac{2E + L'}{a'}\tan\frac{n't}{2}\right\} . \frac{2\tan\dfrac{n't}{2}}{\dfrac{3}{2}n't - 4\tan\dfrac{n't}{2}} - \frac{2E + L'}{a'}\,\text{cotan}\,n't - \frac{K'}{a'};$$

and since by (423) (425),

$$(460)\ldots\qquad 0=\sqrt{a'}\,\frac{\delta W}{\delta r_0}+\sqrt{a}\,\frac{\delta W}{\delta r_0'},$$

we have

$$(461)\ldots\qquad 0=2C\tan\frac{nt}{2}-Kn-K'n'+n\,(2E-L)\operatorname{cotan} nt-n'\,(2E+L')\operatorname{cotan} n't$$
$$+\,2a'n'\left\{D'+\frac{K'}{a'}-\frac{2E+L'}{a'}\tan\frac{n't}{2}\right\}\frac{\tan\dfrac{nt}{2}-\tan\dfrac{n't}{2}}{\dfrac{3}{2}n't-4\tan\dfrac{n't}{2}}.$$

We ought to try to verify this equation. When $t=0$, it reduces itself to the following:

$$(462)\ldots\qquad \frac{dL}{dt}+\frac{dL'}{dt}=0.$$

Now, when $t=0$, we have by (408),

$$(463)\ldots\qquad \frac{dL}{dt}=aa'\left[(2n^2+2n'^2-nn')-2\,\{a^2+a'^2-2aa'\cos(\epsilon-\epsilon')\}^{-\frac{3}{2}}\right]\sin(\epsilon-\epsilon');$$

and similarly,

$$(464)\ldots\qquad \frac{dL'}{dt}=aa'\left[(2n^2+2n'^2-nn')-2\,\{a^2+a'^2-2aa'\cos(\epsilon-\epsilon')\}^{-\frac{3}{2}}\right]\sin(\epsilon'-\epsilon);$$

∵ (462) is true.

We have therefore verified (461) for the case $t=0$, employing the conditions that $r_{\prime\prime}$, $r'_{\prime\prime}$, $\theta_{\prime\prime}$, $\theta'_{\prime\prime}$ & their differential coefficients then vanish. But it will be important to verify it, or transform it generally.

[**26.**] Let us examine the equation (424). If the transformation (454) be true, when $t=0$, we must then have

$$(465)\ldots\qquad \frac{dD'}{dt}=-\frac{3}{2}ann'\cos(\epsilon-\epsilon').$$

But, by (414), when $t=0$,

$$(466)\ldots\qquad \frac{dD'}{dt}=a\,(nn'-n^2-n'^2)\cos(\epsilon-\epsilon')+\frac{\delta}{\delta a'}\{a^2+a'^2-2aa'\cos(\epsilon-\epsilon')\}^{-\frac{1}{2}}-\frac{3n'}{2a'}\frac{dF'}{dt};$$

$$\&\quad(467)\ldots\qquad \frac{dF'}{dt}=-aa'\cos(\epsilon-\epsilon')\left\{\frac{n+n'}{2}+\frac{nn'-n^2-n'^2}{n-n'}+\frac{2nn'-\dfrac{n^2+n'^2}{2}}{n-n'}\right\}$$
$$=-aa'\cos(\epsilon-\epsilon')\left\{\frac{n+n'}{2}-\frac{3}{2}(n-n')\right\}=aa'\cos(\epsilon-\epsilon')\,(n-2n');$$

$$∵\ (468)\ldots\qquad \frac{dD'}{dt}=\frac{a}{2}\cos(\epsilon-\epsilon')\,(-nn'-2n^2+4n'^2)+\frac{\delta}{\delta a'}\{a^2+a'^2-2aa'\cos(\epsilon-\epsilon')\}^{-\frac{1}{2}};$$

but this does not agree with (465). It seems then that some error has been introduced, perhaps by the unsymmetric supposition of $m'=0$. Let me therefore avoid this supposition, & try whether, without it, the equation (424) holds good when $t=0$.

When $t=0$, we have

(469)... $$\frac{dB}{dt} = \frac{\ }{2\sqrt{aa'}}(mn\sqrt{a}+m'n'\sqrt{a'});$$

(470)... $$\frac{\delta t_{,}}{\delta\theta'_0} = -\frac{m'\sqrt{a}}{mn\sqrt{a}+m'n'\sqrt{a'}};$$

which last equation holds for all values of t;

(471)... $$t\frac{\delta a_{,}}{\delta\theta'_0} = -\frac{2}{3}\frac{m'a\sqrt{a'}}{mn\sqrt{a}+m'n'\sqrt{a'}},$$ (472)... $$t\frac{\delta a'_{,}}{\delta\theta'_0} = \frac{2n}{3n'}\cdot\frac{ma'\sqrt{a}}{mn\sqrt{a}+m'n'\sqrt{a'}}$$

(these 2 are also general);

(473)... $$C = aa'(nn'-n^2-n'^2)\cos(\epsilon-\epsilon')+\{a^2-2aa'\cos(\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}};$$

(474)... $$\frac{dD}{dt} = a'(nn'-n^2-n'^2)\cos(\epsilon-\epsilon')+\frac{\delta}{\delta a}\{a^2-2aa'\cos(\epsilon-\epsilon')+a'^2\}^{-\frac{1}{2}}-\frac{3n}{2a}\frac{dF}{dt};$$

& as in (466) $$\frac{dD'}{dt} = a(\qquad\qquad)\cos(\epsilon-\epsilon')+\frac{\delta}{\delta a'}\{\qquad\qquad\}^{-\frac{1}{2}}-\frac{3n'}{2a'}\frac{dF'}{dt};$$

(475)... $$\frac{1}{2}\left(\frac{dF}{dt}+\frac{dF'}{dt}\right) = -\frac{1}{2}aa'(n+n')\cos(\epsilon-\epsilon');$$

(476)... $$\frac{1}{2}\left(\frac{dF}{dt}-\frac{dF'}{dt}\right) = -\frac{3}{2}aa'(n-n')\cos(\epsilon-\epsilon');$$

∵ (477)... $$\frac{dF}{dt} = aa'(n'-2n)\cos(\epsilon-\epsilon'),$$

and as in (467) $$\frac{dF'}{dt} = aa'(n-2n')\cos(\epsilon-\epsilon');$$

& since all the terms of (411) except the three first disappear in $\frac{\delta W}{\delta\theta'_0}$, we ought to have when $t=0$, by (424),

(478)... $$0 = C\frac{\delta t_{,}}{\delta\theta'_0}+t\frac{\delta a_{,}}{\delta\theta'_0}\frac{dD}{dt}+t\frac{\delta a'_{,}}{\delta\theta'_0}\frac{dD'}{dt}+aa'n\cos(\epsilon-\epsilon')+\frac{m'an(H_{,,})}{\frac{dB}{dt}}.$$

or, multiplying by $mn\sqrt{a}+m'n'\sqrt{a'}$ and equating separately to 0 the coefficients of m, m' in a result of the form

(479)... $$Mm+M'm'=0,$$

we get (465), and the following (dividing by $\sqrt{a'}$)

(480)... $$0 = -C-\frac{2}{3}a\frac{dD}{dt}+aa'nn'\cos(\epsilon-\epsilon')-(H_{,,});$$

but it does not appear that either of these coefficients M, M' vanishes.

There seems ∴ to have been some mistake in the foregoing calculations*; and I suppose it originated in some wrong substitution of the elliptic time for the true. But omitting for the present the search for such a slip, I shall resume the problem of the determination of the disturbing function W by a method more peculiarly my own.

* [See § [**33.**], pp. 58–62, and also Appendix, note 1, p. 612.]

[*Restatement of problem with new definitions of a, e, etc.*]

[27.] We have seen that for nearly circular orbits, neglecting the squares and products of the excentricities, the Characteristic Function V may be thus expressed (see equation (357))

$$(481)\qquad V = m\sqrt{\mathrm{a}}\,(\theta-\theta_0) + m'\sqrt{\mathrm{a}'}\,(\theta'-\theta'_0) + mm'W;$$

a, a′ denoting here the same quantities as $(a_,)$, $(a'_,)$ denoted formerly, so that they are functions of $\theta-\theta_0$, $\theta'-\theta'_0$ and of m, m', h satisfying the 2 following conditions:

$$(482)\ldots\quad 0 = 2h + \frac{m\mathrm{a}^{-1}}{1+m} + \frac{m'\mathrm{a}'^{-1}}{1+m'};\qquad (483)\ldots\quad (1+m)\,\mathrm{a}^{-\frac{3}{2}}(\theta-\theta_0) = (1+m')\,\mathrm{a}'^{-\frac{3}{2}}(\theta'-\theta'_0).$$

Hence

$$(484)\ldots\qquad \delta V = m\sqrt{\mathrm{a}}\,(\delta\theta-\delta\theta_0) + m'\sqrt{\mathrm{a}'}\,(\delta\theta'-\delta\theta'_0) + mm'\delta W.$$

$$\because\ (485)\ldots\quad \frac{\delta V}{\delta x} = -\frac{m\sqrt{\mathrm{a}}}{r}\sin\theta + mm'\frac{\delta W}{\delta x};\qquad (486)\ldots\quad \frac{\delta V}{\delta y} = \frac{m\sqrt{\mathrm{a}}}{r}\cos\theta + mm'\frac{\delta W}{\delta y};$$

$$(487)\ldots\quad \frac{\delta V}{\delta x'} = -\frac{m'\sqrt{\mathrm{a}'}}{r'}\sin\theta' + mm'\frac{\delta W}{\delta x'},\qquad (488)\ldots\quad \frac{\delta V}{\delta y'} = \frac{m'\sqrt{\mathrm{a}'}}{r'}\cos\theta' + mm'\frac{\delta W}{\delta y'};\ \&\text{c.}$$

But I propose to go still further back and to take up the problem more generally.

[28.] Let a, ϵ, k, l, a', ϵ', k', l' (new meanings of elements a, ϵ, &c.) be elliptic elements such that with them, in an elliptic time T, the two planets would have described, round the centre of gravity of the 3 bodies, the actual arcs $\theta-\theta_0$, $\theta'-\theta'_0$, having also the actual final and initial distances from that centre r, r_0, r', r'_0; & such besides as to satisfy *rigorously* equation (300), $0 = \frac{ma^{-1}}{1+m} + \frac{m'a'^{-1}}{1+m'} + 2h$, m, m', h being still the planetary masses and the constant of living force; & the quantities n, n' being still defined to satisfy equations (60) (61).

With these *new meanings*, the 8 elements a, ϵ, … l' will not be strictly constant nor determined solely by the circumstances of initial motion; they will, by perturbation, be functions of the time, & will gradually alter a little. And if we employ u, u' to denote the excentric anomalies of the 2 planets in these slowly varying ellipses, we shall have certain rigorous elliptic equations which may approximately be expressed, neglecting squares of excentricities but not now neglecting perturbations, by the equations (363)…(370), if only we change t to the elliptic time T. The rigorous equations may be put under the form

$$(489)\ldots\quad r = a - a\,e\cos u;\qquad (490)\ldots\quad \tan\frac{\theta-\varpi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2};\qquad (491)\ldots\quad nT+\epsilon = u - e\sin u;$$

$$(492)\ldots\quad r_0 = a - a\,e\cos u_0;\qquad (493)\ldots\quad \tan\frac{\theta_0-\varpi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u_0}{2};\qquad (494)\ldots\quad \epsilon = u_0 - e\sin u_0;$$

$$(495)\ldots\quad r' = a' - a'e'\cos u';\qquad (496)\ldots\qquad ;\qquad (497)\ldots\qquad ;$$

$$(498)\ldots\quad r'_0 = a' - a'e'\cos u'_0;\qquad (499)\ldots\qquad ;\qquad (500)\ldots\quad \epsilon' = u'_0 - e'\sin u'_0;$$

They are in number 12, and when combined with the rigorous equation (300) and with the definitions (60) (61), they give 15 equations to determine the 15 quantities a, e, ϵ, ϖ, n, u, u_0, a', e', ϵ', ϖ', n', u', u'_0, T as functions of the eleven following r, θ, r_0, θ_0, m, r', θ', r'_0, θ'_0, m', h.

If the planets actually moved in the ellipses thus determined, we should have for all excentricities

(501)... $$m(1+m)\int_0^T v^2 dT + m'(1+m')\int_0^T v'^2 dT$$
$$= m\sqrt{a}\{u-u_0+e(\sin u-\sin u_0)\}+m'\sqrt{a'}\{u'-u_0'+e'(\sin u'-\sin u_0')\};$$

because we should have

(502)... $$v^2=(1+m)^{-2}\left(\frac{2}{r}-\frac{1}{a}\right),$$

(observing that $a^3n^2=(1+m)^{-2}$),

(503)... $$dT=(1+m)\sqrt{a}\,r\,du,$$

and

(504)... $$\int_{u_0}^{u}\left(2-\frac{r}{a}\right)du=\int_{u_0}^{u}(1+e\cos u)\,du=u-u_0+e(\sin u-\sin u_0).$$

If then we put

(505)... $$V_,=\sqrt{a}\{u-u_0+e(\sin u-\sin u_0)\};$$

(506)... $$V'_,=\sqrt{a'}\{u'-u_0'+e'(\sin u'-\sin u_0')\};$$

the quantities $(1+m)\int_0^t v^2 dt$, $(1+m')\int_0^t v'^2\,dt$ will only differ by perturbations from $V_,$, $V'_,$, and their differences will bear some finite limiting ratios to m', m respectively; from which it seems just to conclude that my characteristic function V may be put under the form

(507)... $$V=mV_,+m'V'_,+mm'W,$$

$V_,$ $V'_,$ being functions of r, θ, r_0, θ_0, r', θ', r_0', θ_0', m, m', h determined as above and W being a sought *disturbing function* which tends to some finite limit when the masses m, m' tend to 0 along with h, preserving finite ratios $\frac{m}{h}$, $\frac{m'}{h}$.

The function $V_,$ involves, by former investigations of mine on separate sheets* (series begun Aug. 1833), only the sum $r+r_0$ of the extreme radii vectores, the semiaxis a, & the elliptic chord; because calling this chord χ,

(508)... $\chi^2=r^2+r_0^2-2rr_0\cos(\theta-\theta_0)$, (509)... $\chi'^2=r'^2+r_0'^2-2r'r_0'\cos(\theta'-\theta_0')$.

We have

(510)... $$\chi=2a\sin\frac{u-u_0}{2}\sqrt{1-e^2\left(\cos\frac{u+u_0}{2}\right)^2},$$

(511)... $$\chi'=2a'\sin\frac{u'-u_0'}{2}\sqrt{1-e'^2\left(\cos\frac{u'+u_0'}{2}\right)^2},$$

(512)... $$r+r_0=2a\left(1-e\cos\frac{u+u_0}{2}\cos\frac{u-u_0}{2}\right),$$

(513)... $$r'+r_0'=2a'\left(1-e'\cos\frac{u'+u_0'}{2}\cos\frac{u'-u_0'}{2}\right),$$

---

* [We have not been able to trace these sheets. See pp. 135–140 and also Appendix 5, p. 624.]

(514)... $$V_, = \sqrt{a}\left\{u - u_0 + 2e\cos\frac{u+u_0}{2}\sin\frac{u-u_0}{2}\right\},$$

(515)... $$V'_, = \sqrt{a'}\left\{u' - u'_0 + 2e'\cos\frac{u'+u'_0}{2}\sin\frac{u'-u'_0}{2}\right\};$$

eliminating ∵ $u - u_0$ & $e\cos\frac{u+u_0}{2}$ between (510) (512) (514), we shall get $V_,$ as a function of a, $r + r_0$ & χ; and similarly for $V'_,$. It is to be observed that the equations (510) (511) suppose the elliptic chords capable of changing sign.

The equations (512) (514) give

(516)... $$\frac{V_,}{\sqrt{a}} = u - u_0 + \left(2 - \frac{r+r_0}{a}\right)\tan\frac{u-u_0}{2};$$

&, if we neglect e^2, (510) gives

(517)... $$\chi = 2a\sin\frac{u-u_0}{2};$$

& (516) gives

(518)... $$\frac{u-u_0}{2} = \frac{V_,}{2\sqrt{a}} + \left(\frac{r+r_0}{2a} - 1\right)\tan\frac{V_,}{2\sqrt{a}},$$

∵ (519)... $$\chi = (r + r_0)\sin\frac{V_,}{2\sqrt{a}};$$

accurately the elliptic chord χ would be this + terms × $\left(\frac{r+r_0}{2a} - 1\right)^2$. Hence by (508),

(520)... $$(r+r_0)^2\left(\cos\frac{V_,}{2\sqrt{a}}\right)^2 = (r+r_0)^2 - \chi^2 = 4rr_0\left(\cos\frac{\theta-\theta_0}{2}\right)^2;$$

but, if we neglect squares of excentricities, we may put

(521)... $\frac{4rr_0}{(r+r_0)^2} = 1$, & ∵ (522)... $V_, = \sqrt{a}\,(\theta - \theta_0)$.

However, it may be better for some purposes to retain the form (520), in which we have only neglected $\left(\frac{r+r_0}{2a} - 1\right)^2$, but not $(r - r_0)^2$; & in which ∵ we have kept closer to the conception of $V_,$ as a function of a, $r + r_0$ and χ. Thus the form (522) will not give *identically*

(523)... $$\left(\frac{\delta V_,}{\delta r}\right)^2 + \left(\frac{\delta V_,}{r\,\delta\theta}\right)^2 = \frac{2}{r} - \frac{1}{a};$$

but perhaps (520) will do so. It gives or at least it permits us to put

(524)... $$\cos\frac{V_,}{2\sqrt{a}} = \frac{2\sqrt{rr_0}}{r+r_0}\cos\frac{\theta-\theta_0}{2},$$

& differentiating this we find

$$(525)\ldots\qquad -\frac{1}{2\sqrt{a}}\sin\frac{V_{,}}{2\sqrt{a}}\cdot\frac{\delta V_{,}}{\delta r}=\frac{r^{-\frac{1}{2}}\sqrt{r_0}\,(r_0-r)}{(r+r_0)^2}\cos\frac{\theta-\theta_0}{2};$$

$$(526)\ldots\qquad -\frac{1}{2\sqrt{a}}\sin\frac{V_{,}}{2\sqrt{a}}\cdot\frac{\delta V_{,}}{r\,\delta\theta}=-\frac{r^{-\frac{1}{2}}\sqrt{r_0}}{r+r_0}\sin\frac{\theta-\theta_0}{2};$$

$$\because\ (527)\ldots\qquad \left(\frac{\delta V_{,}}{\delta r}\right)^2+\left(\frac{\delta V_{,}}{r\,\delta\theta}\right)^2=\frac{4a\left\{\dfrac{r_0}{r\,(r+r_0)^2}-\dfrac{4r_0^2}{(r+r_0)^4}\left(\cos\dfrac{\theta-\theta_0}{2}\right)^2\right\}}{1-\dfrac{4rr_0}{(r+r_0)^2}\left(\cos\dfrac{\theta-\theta_0}{2}\right)^2}=\frac{4ar_0}{r\,(r+r_0)^2}$$

$$=\frac{4a}{r\,(r+r_0)}-\frac{4a}{(r+r_0)^2},$$

which in fact becomes $=\frac{2}{r}-\frac{1}{a}$ when we change $r+r_0$ to $2a$; but still the equation (523) is not *identically* satisfied. It may ∵ be well after all to confine ourselves to (522), as a simple approximate integral of the partial differential equation (523), neglecting the square of the excentricity.

[*The characteristic function of elliptic motion and the equation* $\frac{\delta V}{\delta h}=t$.]

[29.] Meanwhile I know, from former investigations, that the function $V_{,}$, determined by (510) (512) (514), does in fact satisfy identically the partial differential equation (523). But to establish this important point anew and at the same time to prepare for other investigations, let us put for abbreviation

$$(528)\ldots\qquad e\cos\frac{u+u_0}{2}=\zeta;\qquad\qquad (529)\ldots\qquad e'\cos\frac{u'+u'_0}{2}=\zeta'.$$

Then

$$(530)\ldots\qquad V_{,}=\sqrt{a}\left(u-u_0+2\zeta\sin\frac{u-u_0}{2}\right);\qquad (531)\ldots\qquad V'_{,}=\sqrt{a'}\left(u'-u'_0+2\zeta'\sin\frac{u'-u'_0}{2}\right);$$

$$(532)\ldots\qquad \frac{r+r_0}{2a}=1-\zeta\cos\frac{u-u_0}{2};\qquad (533)\ldots\qquad \chi=2a\sin\frac{u-u_0}{2}\sqrt{1-\zeta^2};$$

∵, considering a as constant,

$$(534)\ldots\qquad \frac{\delta V_{,}}{\sqrt{a}}=\left(1+\zeta\cos\frac{u-u_0}{2}\right)\delta\,(u-u_0)+2\,\delta\zeta\sin\frac{u-u_0}{2};$$

$$(535)\ldots\qquad \frac{\delta\,(r+r_0)}{a}=\zeta\sin\frac{u-u_0}{2}\,\delta\,(u-u_0)-2\,\delta\zeta\cos\frac{u-u_0}{2};$$

$$(536)\ldots\qquad \frac{\delta\chi}{\chi}=\frac{1}{2}\cot\!\operatorname{an}\frac{u-u_0}{2}\,\delta\,(u-u_0)-\frac{\zeta\,\delta\zeta}{1-\zeta^2};$$

$$(537)\ldots\qquad \zeta\frac{\delta\chi}{\chi}\left(\sin\frac{u-u_0}{2}\right)^2-\frac{\delta\,(r+r_0)}{2a}\cos\frac{u-u_0}{2}=\frac{\delta\zeta}{1-\zeta^2}\left\{\left(\cos\frac{u-u_0}{2}\right)^2-\zeta^2\right\};$$

$$(538)\ldots\qquad (1-\zeta^2)\frac{\delta\chi}{\chi}\sin\,(u-u_0)-\zeta\sin\frac{u-u_0}{2}\,\frac{\delta\,(r+r_0)}{a}=\delta\,(u-u_0)\left\{\left(\cos\frac{u-u_0}{2}\right)^2-\zeta^2\right\};$$

$$\because\ (539)\ldots\quad \frac{\left(\cos\frac{u-u_0}{2}\right)^2-\zeta^2}{(1-\zeta^2)\sin\frac{u-u_0}{2}}\frac{\delta V_,}{\sqrt{a}}=\left(1+\zeta\cos\frac{u-u_0}{2}\right)\left\{2\frac{\delta\chi}{\chi}\cos\frac{u-u_0}{2}-\frac{\zeta}{1-\zeta^2}\frac{\delta(r+r_0)}{a}\right\}$$

$$+\frac{2\zeta\delta\chi}{\chi}\left(\sin\frac{u-u_0}{2}\right)^2-\frac{\delta(r+r_0)}{a}\cos\frac{u-u_0}{2}$$

$$=\frac{2\delta\chi}{\chi}\left(\cos\frac{u-u_0}{2}+\zeta\right)-\frac{\delta(r+r_0)}{a(1-\zeta^2)}\left(\cos\frac{u-u_0}{2}+\zeta\right),$$

that is,

$$(540)\ldots\quad \delta V_,=\frac{\sqrt{a}\sin\frac{u-u_0}{2}}{\cos\frac{u-u_0}{2}-\zeta}\left\{\frac{2(1-\zeta^2)\delta\chi}{\chi}-\frac{1}{a}\delta(r+r_0)\right\}\qquad\frac{\sqrt{1-\zeta^2}\,.\,\delta\chi-\sin\frac{u-u_0}{2}\delta(r+r_0)}{\sqrt{a}\left(\cos\frac{u-u_0}{2}-\zeta\right)}$$

we know $\because$ $\frac{\delta V_,}{\delta(r+r_0)}$ and $\frac{\delta V_,}{\delta\chi}$. Now

$$(541)\ldots\quad \frac{\delta V_,}{\delta r}=\frac{\delta V_,}{\delta(r+r_0)}+\frac{\delta V_,}{\delta\chi}\frac{\delta\chi}{\delta r};\qquad (542)\ldots\quad \frac{\delta V_,}{r\delta\theta}=\frac{\delta V_,}{\delta\chi}\frac{\delta\chi}{r\delta\theta};$$

also

$$(543)\ldots\quad \frac{\delta\chi}{\delta r}=\frac{r-r_0\cos(\theta-\theta_0)}{\chi};\qquad (544)\ldots\quad \frac{\delta\chi}{r\delta\theta}=\frac{r_0\sin(\theta-\theta_0}{\chi}$$

$$(545)\ldots\quad \left(\frac{\delta\chi}{\delta r}\right)^2+\left(\frac{\delta\chi}{r\delta\theta}\right)^2=1;$$

we ought $\because$ to have

$$(546).\quad 1-\zeta^2+\left(\sin\frac{u-u_0}{2}\right)^2+\frac{r_0\cos(\theta-\theta_0)-r}{a}=\left(\frac{2a}{r}-1\right)\left(\cos\frac{u-u_0}{2}-\zeta\right)^2,$$

that is, by (512),

$$(547)\ldots\quad \frac{rr_0}{a^2}=\left(\frac{\cos\frac{u-u_0}{2}-\zeta}{\cos\frac{\theta-\theta_0}{2}}\right)^2,\quad\text{or,}\quad (548)\ldots\quad \left(\cos\frac{u-u_0}{2}-\zeta\right)^2=\frac{(r+r_0)^2-\chi^2}{4a^2};$$

& this last condition is satisfied, by (532) (533). (See 10th sheet of series begun Aug. 1833 on elliptic motion.)

It is worth observing that, by (532) (533), we have not only $\left(\cos\frac{u-u_0}{2}-\zeta\right)^2=\frac{(r+r_0)^2-\chi^2}{4a^2}$, but also

$$(549)\ldots\quad \left(\cos\frac{u-u_0}{2}+\zeta\right)^2=\left(2-\frac{r+r_0}{2a}\right)^2-\frac{\chi^2}{4a^2}.$$

Hence

$$(550)\ldots\quad \zeta^2+\left(\cos\frac{u-u_0}{2}\right)^2=2-\frac{r+r_0}{a}+\frac{(r+r_0)^2-\chi^2}{4a^2},$$

$$(551)\ldots\quad 1-\zeta^2+\left(\sin\frac{u-u_0}{2}\right)^2=\frac{r+r_0}{a}-\frac{(r+r_0)^2-\chi^2}{4a^2}.$$

so that

$$(552)\ldots \qquad \left(\frac{\delta V_,}{\delta\chi} \pm \frac{\delta V_,}{\delta(r+r_0)}\right)^2 = \frac{4(r+r_0\mp\chi)}{(r+r_0)^2-\chi^2} - \frac{1}{a} = \frac{4}{r+r_0\pm\chi} - \frac{1}{a},$$

and ∴

$$(553)\ldots \qquad \frac{\delta V_,}{\delta\chi} + \frac{\delta V_,}{\delta(r+r_0)} = \pm\surd\ r+r_0+\chi \quad a'$$

$$(554)\ldots \qquad \frac{\delta V_,}{\delta\chi} - \frac{\delta V_,}{\delta(r+r_0)} = \pm\surd\ r+r_0-\chi \quad a'$$

$$(555)\ldots \qquad \frac{\delta V_,}{\delta\chi} = \pm\surd\ \frac{1}{r+r_0+\chi} - \frac{1}{4a} \pm \surd\ \frac{1}{r+r_0-\chi} \quad 4a,$$

$$(556)\ldots \qquad \frac{\delta V}{\delta(r+r_0)} = \pm \ / \qquad\qquad \mp$$

When we may take the upper sign in these 2 last expressions, we have*

$$(557)\ldots \qquad V_, = \int_{-\chi}^{\chi} \sqrt{\frac{1}{r+r_0+\chi} - \frac{1}{4a}}\, d\chi,$$

an expression of which I made great use in former investigations. In general we may put

$$(558)\ldots \qquad V_, = \int_{-\chi}^{\chi} \pm\sqrt{\frac{1}{r+r_0+\chi} - \frac{1}{4a}}\, d\chi.$$

If we wish to avoid considering a as constant, we have only to change in (540)

$$\delta V_, \text{ to } \sqrt{a}\,\delta\left(\frac{V_,}{\sqrt{a}}\right);\ \delta\chi \text{ to } a\,\delta\left(\frac{\chi}{a}\right);\ \&\ \delta(r+r_0) \text{ to } a\,\delta\left(\frac{r+r_0}{a}\right);$$

& we find

$$(559)\ldots \qquad \frac{\delta V_,}{\delta a} = \frac{1}{a}\left(\frac{V_,}{2} - \chi\frac{\delta V_,}{\delta\chi} - (r+r_0)\frac{\delta V_,}{\delta(r+r_0)}\right),$$

which also follows from $V_,$ being a homogeneous function of the dimension $\frac{1}{2}$, of a, $r+r_0$, χ. We have ∴

$$(560)\ldots \qquad \frac{\delta V_,}{\delta a} = \frac{1}{\sqrt{a}}\left\{\frac{u-u_0}{2} + \zeta\sin\frac{u-u_0}{2} \cdot\cdot \frac{2\sin\frac{u-u_0}{2}}{\cos\frac{u-u_0}{2} - \zeta}\left(1-\zeta^2 - 1 - \zeta\cos\frac{u-u_0}{2}\right)\right\}$$

$$= \frac{1}{\sqrt{a}}\left\{\frac{u-u_0}{2} - \zeta\sin\frac{u-u_0}{2}\right\} = \frac{1}{\sqrt{a}}\left\{\frac{u-u_0}{2} - e\cos\frac{u+u_0}{2}\sin\frac{u-u_0}{2}\right.$$

$$= \frac{1}{2\sqrt{a}}\{u-u_0-e(\sin u - \sin u_0)\} = \frac{nT}{2\sqrt{a}} = \frac{T}{2(1+m)a^2},$$

---

* [This formula for $V_,$, which appears in the *First Essay*, equation (W^3.), p. 140, has been erroneously attributed to Jacobi. Cf. Appell, *Mécanique Rationnelle*, Tome I, § 304, p. 577.]

T being elliptic time. This last expression shows that although we must consider in the present method the semiaxes a, a' as functions of the coordinates, yet their variations disappear from the variation of the part $mV_{,} + m'V'_{,}$ of V; because

$$(561)\ldots \qquad m\frac{\delta V_{,}}{\delta a}\delta a + m'\frac{\delta V'_{,}}{\delta a'}\delta a' = \frac{T}{2}\left(\frac{m\,\delta a}{(1+m)\,a^2} + \frac{m'\,\delta a'}{(1+m')\,a'^2}\right) = 0,$$

by (300). We may ∵ in differentiating $mV_{,} + m'V'_{,}$ treat a, a' as constant, & ∵ may differentiate $V_{,}$ with respect to the coordinates of m only & $V'_{,}$ with respect to those of m'.

We see also that if we suppose $m' = 0$, & so reduce V to $mV_{,}$ & ∵ to a function of $r + r_0$, χ, a, m, which may be considered as a function of $r + r_0$, χ, h, m, we have

$$(562)\ldots \qquad \frac{\delta V}{\delta h} = m\frac{\delta V_{,}}{\delta a}\frac{\delta a}{\delta h} = \frac{mT}{2\,(1+m)\,a^2}\left(\frac{\delta h}{\delta a}\right)^{-1} = T = t.$$

IT WOULD BE AN EXTREMELY CURIOUS THEOREM IF WE SHOULD FIND THAT IN GENERAL $\frac{\delta V}{\delta h} = t$. For thus we should complete our solution of the problem of 3 bodies or any other system, by means of the action function V, without employing any integration after this function has once been determined.

Meanwhile, without supposing m or m' to vanish, we see that the part of $\frac{\delta V}{\delta h}$ which is not small, namely

$$m\frac{\delta V_{,}}{\delta h} + m'\frac{\delta V'_{,}}{\delta h} \quad \text{or} \quad m\frac{\delta V_{,}}{\delta a}\frac{\delta a}{\delta h} + m'\frac{\delta V'_{,}}{\delta a'}\frac{\delta a'}{\delta h} \quad \text{or} \quad \frac{mTa^{-2}}{2\,(1+m)}\frac{\delta a}{\delta h} + \frac{m'Ta'^{-2}}{2\,(1+m')}\frac{\delta a'}{\delta h},$$

is in fact equal to T, by (300)*; which is, so far, a verification of my general conjecture.

(Jan. 8. 1834.) Three pages hence I give a general proof of the truth of this theorem, $t = \frac{\delta V}{\delta h}$. By this theorem the integration of the differential equations of motion of any system of bodies (including problems of rotation) is reduced to finding the form of V, differentiating it with respect to the initial coordinates and h, and determining the final coordinates as functions of these partial differential coefficients so obtained & of the initial coordinates. The expressions thus determined for the final coordinates ought not to involve h.

[*Expression for disturbing function and comparison with Laplace.*

General proof of $\frac{\delta V}{\delta h} = t$.

[**30.**] By what precedes, we have

$$(563)\ldots \qquad \frac{\delta V}{\delta x} = m\frac{\delta V_{,}}{\delta x} + mm'\frac{\delta W}{\delta x}; \qquad (564)\ldots \qquad \frac{\delta V}{\delta y} = m\frac{\delta V_{,}}{\delta y} + mm'\frac{\delta W}{\delta y};$$

$$(565)\ldots \qquad \frac{\delta V}{\delta x'} = m'\frac{\delta V'_{,}}{\delta x'} + mm'\frac{\delta W}{\delta x'}; \qquad (566)\ldots \qquad \frac{\delta V}{\delta y'} = m'\frac{\delta V'_{,}}{\delta y'} + mm'\frac{\delta W}{\delta y'};$$

* [**If we differentiate (300) with respect to h, we obtain** $2 - \frac{ma^{-2}}{1+m}\frac{\delta a}{\delta h} - \frac{m'a'^{-2}}{1+m'}\frac{\delta a'}{\delta h} = 0.$]

a being treated as constant in $V_{,}$ and a' in $V'_{,}$; we have also, when a, a' are thus treated,

$$(567)\ldots\quad \left(\frac{\delta V_{,}}{\delta x}\right)^2+\left(\frac{\delta V_{,}}{\delta y}\right)^2=\frac{2}{r}-\frac{1}{a}; \qquad (568)\ldots\quad \left(\frac{\delta V'_{,}}{\delta x'}\right)^2+\left(\frac{\delta V'_{,}}{\delta y'}\right)^2=\frac{2}{r'}-\frac{1}{a'};$$

no powers of excentricities being neglected here. Attending $\because$ to (300), the partial differential equation (38) becomes

$$(569)\ldots\quad \frac{\delta V_{,}}{\delta x}\frac{\delta W}{\delta x}+\frac{\delta V_{,}}{\delta y}\frac{\delta W}{\delta y}+\frac{\delta V'_{,}}{\delta x'}\frac{\delta W}{\delta x'}+\frac{\delta V'_{,}}{\delta y'}\frac{\delta W}{\delta y'}=\frac{\delta V_{,}}{\delta x}\frac{\delta V'_{,}}{\delta x'}+\frac{\delta V_{,}}{\delta y}\frac{\delta V'_{,}}{\delta y'}$$
$$+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-(xx'+yy')(r^{-3}+r'^{-3});$$

which may easily be put under the following form, when we neglect the planetary masses in W,

$$(570)\ldots\quad \frac{dW}{dt}=\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-(xx'+yy')(r^{-3}+r'^{-3}),$$

the coordinates and time in this expression being replaced by their elliptic values. And since W varies with the time, we have

$$(571)\ldots\quad W=\int_0^t Q\,dt,$$

if $$(572)\ldots\quad Q=\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-(xx'+yy')(r^{-3}+r'^{-3}).$$

Again, by the differential equations of elliptic motion, neglecting planetary masses, we have

$$(573)\ldots\quad -\frac{x}{r^3}=\frac{d}{dt}\frac{dx}{dt}; \qquad (574)\ldots\quad -\frac{y}{r^3}=\frac{d}{dt}\frac{dy}{dt};$$

$$(575)\ldots\quad -\frac{x'}{r'^3}=\frac{d}{dt}\frac{dx'}{dt}; \qquad (576)\ldots\quad -\frac{y'}{r'^3}=\frac{d}{dt}\frac{dy'}{dt};$$

$$\because\ (577)\ldots\quad -(xx'+yy')(r^{-3}+r'^{-3})=\left(\frac{d}{dt}\right)^2(xx'+yy')-2\left(\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}\right);$$

$$\because\ (578)\ldots\quad Q=\left(\frac{d}{dt}\right)^2(xx'+yy')-\left(\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}\right)+\{(x-x')^2+(y-y'^2)\}^{-\frac{1}{2}},$$

& also

$$(579)\ldots\quad Q=\frac{1}{2}\left(\frac{d}{dt}\right)^2(xx'+yy')-\tfrac{1}{2}(xx'+yy')(r^{-3}+r'^{-3})+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}};$$

also

$$(580)\ldots\quad Q=\frac{d}{dt}\left(x'\frac{dx}{dt}+y'\frac{dy}{dt}\right)-\frac{xx'+yy'}{r'^3}+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}},$$

& $$(581)\ldots\quad Q=\frac{d}{dt}\left(x\frac{dx'}{dt}+y\frac{dy'}{dt}\right)-\frac{xx'+yy'}{r^3}+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}.$$

When the orbit of m' is exterior to that of m, the form (580) appears to be the most convenient; it gives

$$(582)\ldots\quad W=x'\frac{dx}{dt}+y'\frac{dy}{dt}-x'_0\frac{dx_0}{dt}-y'_0\frac{dy_0}{dt}+\int_0^t\left\{\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-\frac{xx'+yy'}{r'^3}\right.\ dt,$$

that is,

$$(583)\ldots\quad W=x'\frac{\delta V_{,}}{\delta x}+y'\frac{\delta V_{,}}{\delta y}+x'_0\frac{\delta V_{,}}{\delta x_0}+y'_0\frac{\delta V_{,}}{\delta y_0}+\int_0^t\left\{\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-\frac{xx'+yy'}{r'^3}\right\}dt.$$

It is important to observe that the quantity under the integral sign, in this expression for my disturbing function W, is what Laplace* calls $-\frac{R}{m}$. Indeed, I have here neglected the inclinations of the orbits; but I know from former investigations that the general expressions, including all powers of those inclinations and of the excentricities, would give the analogous result

$$(584)\ldots\quad W = x'\frac{\delta V_{,}}{\delta x} + y'\frac{\delta V_{,}}{\delta y} + z'\frac{\delta V_{,}}{\delta z} + x_0'\frac{\delta V_{,}}{\delta x_0} + y_0'\frac{\delta V_{,}}{\delta y_0} + z_0'\frac{\delta V_{,}}{\delta z_0}$$
$$+\int_0^t \left\{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}} - \frac{xx'+yy'+zz'}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}\right\} dt;$$

the quantity under the integral sign being still what Laplace calls $-\frac{R}{m'}$. Adopting ∵ this notation, we have

$$(585)\ldots\quad mm'W = mm'\left(x'\frac{\delta V_{,}}{\delta x} + y'\frac{\delta V_{,}}{\delta y} + z'\frac{\delta V_{,}}{\delta z} + x_0'\frac{\delta V_{,}}{\delta x_0} + y_0'\frac{\delta V_{,}}{\delta y_0} + z_0'\frac{\delta V_{,}}{\delta z_0}\right) - m\int_0^t R\,dt;$$

and similarly with the analogous symbol R', we have

$$(586)\ldots\quad mm'W = mm'\left(x\frac{\delta V_{,}'}{\delta x'} + y\frac{\delta V_{,}'}{\delta y'} + z\frac{\delta V_{,}'}{\delta z'} - x_0\frac{\delta V_{,}'}{\delta x_0'} - y_0\frac{\delta V_{,}'}{\delta y_0'} + z_0\frac{\delta V_{,}'}{\delta z_0'}\right) - m'\int_0^t R'\,dt.$$

We may therefore put, in a symmetrical form, the following expression for the characteristic function V:

$$(587)\ldots\quad V = mV_{,} + m'V_{,}' + \frac{mm'}{2}\left(x'\frac{\delta V_{,}}{\delta x} + y'\frac{\delta V_{,}}{\delta y} + z'\frac{\delta V_{,}}{\delta z} + x_0'\frac{\delta V_{,}}{\delta x_0} + y_0'\frac{\delta V_{,}}{\delta y_0} + z_0'\frac{\delta V_{,}}{\delta z_0}\right.$$
$$\left. + x\frac{\delta V_{,}'}{\delta x'} + y\frac{\delta V_{,}'}{\delta y'} + z\frac{\delta V_{,}'}{\delta z'} + x_0\frac{\delta V_{,}'}{\delta x_0'} + y_0\frac{\delta V_{,}'}{\delta y_0'} + z_0\frac{\delta V_{,}'}{\delta z_0'}\right) - \frac{m}{2}\int_0^t R\,dt - \frac{m'}{2}\int_0^t R'\,dt:$$

it may however be more convenient occasionally to employ the less symmetric forms (585) (586) for the part $mm'W$. And if we denote by $\overline{x}, \overline{y}, \overline{z}$ the heliocentric rectangular coordinates of m and use a similar notation for the other heliocentric coordinates, we easily find by [5.],

$$(588)\ldots\quad \frac{d\overline{x}}{dt} = \frac{1}{m}\frac{\delta V}{\delta x} = \frac{\delta V_{,}}{\delta x} + m'\frac{\delta W}{\delta x}; \qquad (589)\ldots\quad \frac{d\overline{y}}{dt} = \frac{\delta V_{,}}{\delta y} + m'\frac{\delta W}{\delta y};$$

$$(590)\ldots\quad \frac{d\overline{z}}{dt} = \frac{\delta V_{,}}{\delta z} + m'\frac{\delta W}{\delta z}; \qquad (591)\ldots\quad \frac{d\overline{x}'}{dt} = \frac{1}{m'}\frac{\delta V}{\delta x'} = \frac{\delta V_{,}'}{\delta x'} + m\frac{\delta W}{\delta x'};$$

(592)... ; (593)...

In like manner we have the 6 initial expressions

$$(594)\ldots\quad -\left(\frac{d\overline{x}}{dt}\right)_0 = \frac{1}{m}\frac{\delta V}{\delta x_0} = \frac{\delta V_{,}}{\delta x_0} + m'\frac{\delta W}{\delta x_0}; \qquad (595)\ldots\quad ; \qquad (596)\ldots$$

$$(597)\ldots\quad -\left(\frac{d\overline{x}'}{dt}\right)_0 = \frac{1}{m'}\frac{\delta V}{\delta x_0'} = \frac{\delta V_{,}'}{\delta x_0'} + m\frac{\delta W}{\delta x_0'}; \qquad (598)\ldots\quad ; \qquad (599)\ldots$$

Hence

$$(600)\ldots\quad \delta x\frac{d\overline{x}}{dt} + \delta y\frac{d\overline{y}}{dt} + \delta z\frac{d\overline{z}}{dt} = \delta V_{,} + m'\delta W,$$

---

* [*Mécanique Céleste*, Book II, Chap. VI.]

δ here referring only to x, y, z; & similarly

(601)... $$\delta x' \frac{d\bar{x}'}{dt} + \delta y' \frac{d\bar{y}'}{dt} + \delta z' \frac{d\bar{z}'}{dt} = \delta' V'_{,} + m\delta' W,$$

δ' referring to x', y', z'. In like manner

(602)... $$\delta x_0 \left(\frac{d\bar{x}}{dt}\right)_0 + \delta y_0 \left(\frac{d\bar{y}}{dt}\right)_0 + \delta z_0 \left(\frac{d\bar{z}}{dt}\right)_0 = -\delta_0 V_{,} - m'\delta_0 W,$$

(603)... $$\delta x'_0 \left(\frac{d\bar{x}'}{dt}\right)_0 + \delta y'_0 \left(\frac{d\bar{y}'}{dt}\right)_0 + \delta z'_0 \left(\frac{d\bar{z}'}{dt}\right)_0 = -\delta'_0 V'_{,} - m\delta'_0 W.$$

a, a' are here treated as constant in $V_{,}, V'_{,}$. Hence

(604)... $$0 = \frac{\delta V_{,}}{\delta r_0} + m' \frac{\delta W}{\delta r_0} + \frac{x_0}{r_0}\left(\frac{d\bar{x}}{dt}\right)_0 + \frac{y_0}{r_0}\left(\frac{d\bar{y}}{dt}\right)_0 + \frac{z_0}{r_0}\left(\frac{d\bar{z}}{dt}\right)_0;$$

(605)... $$0 = \frac{\delta V'_{,}}{\delta r'_0} + m \frac{\delta W}{\delta r'_0} + \frac{x'_0}{r'_0}\left(\frac{d\bar{x}'}{dt}\right)_0 + \frac{y'_0}{r'_0}\left(\frac{d\bar{y}'}{dt}\right)_0 + \frac{z'_0}{r'_0}\left(\frac{d\bar{z}'}{dt}\right)_0.$$

And if we suppose as before that θ, θ' are centrobaric longitudes in one common plane, neglecting inclinations, then

(606)... $$0 = \frac{\delta V_{,}}{\delta \theta_0} + m' \frac{\delta W}{\delta \theta_0} + x_0 \left(\frac{d\bar{y}}{dt}\right)_0 - y_0 \left(\frac{d\bar{x}}{dt}\right)_0;$$

(607)... $$0 = \frac{\delta V'_{,}}{\delta \theta'_0} + m \frac{\delta W}{\delta \theta'_0} + x'_0 \left(\frac{d\bar{y}'}{dt}\right)_0 - y'_0 \left(\frac{d\bar{x}'}{dt}\right)_0.$$

In all these 1st differentiations of $V_{,}, V'_{,}$ we consider a, a' as constant; but in passing to the second differentiations of $V_{,}, V'_{,}$ we should be obliged to consider a, a' as variable and as homogeneous functions of the 1st dimension of $r + r_0$, $r' + r'_0$, χ, χ', $\frac{mh^{-1}}{1+m}$, $\frac{m'h^{-1}}{1+m'}$, involving also but not homogeneously the quotient $\frac{1+m'}{1+m}$; because they are determined by the equation (300), & by a relation between this latter quotient & the 5 ratios of $r + r_0$, $r' + r'_0$, χ, χ', a, a', obtained by equating the two values of the elliptic time T.

And with respect to the disturbed time t, it is in general $= \frac{\delta V}{\delta h}$, as I conjectured some pages ago. For by the idea of my characteristic function V, we have

(608)... $$V = \Sigma \int mv\,ds;$$

$\because$ (609)... $$\delta V = \Sigma \int m(\delta v\,ds + v\,\delta ds) = \Sigma \int mv\,\delta v\,dt + \Sigma \int_0^t m\left(\frac{dx}{dt}\frac{d\delta x}{dt} + \frac{dy}{dt}\frac{d\delta y}{dt} + \frac{dz}{dt}\frac{d\delta z}{dt}\right) dt;$$

also by the law of living force

(610)... $$\tfrac{1}{2}\Sigma mv^2 = h + F,$$

F being a function of the coordinates & masses, independent of h, & such that

(611)... $$\delta F = \Sigma m \left(\delta x \frac{d}{dt}\frac{dx}{dt} + \delta y \frac{d}{dt}\frac{dy}{dt} + \delta z \frac{d}{dt}\frac{dz}{dt} \right);$$

∵ (612)... $$\Sigma m v\, \delta v\, dt = \delta h\, dt + \delta F\, dt,$$

and

(613)... $$\delta V = t\delta h + \Sigma m \left(\frac{dx}{dt}\delta x - \frac{dx_0}{dt}\delta x_0 + \frac{dy}{dt}\,\delta y - \frac{dy_0}{dt}\delta y_0 + \frac{dz}{dt}\,\delta z - \frac{dz_0}{dt}\,\delta z_0 \right) .$$

This is the general dynamical Law of Varying Action, including the variation δh, and we see that it gives in general

(614)... $$t = \frac{\delta V}{\delta h}\cdot$$

[Various investigations regarding the disturbing function and heliocentric coordinates. Comparison with Lagrange.]

[**31**.] Imagine then that we have expressed the Characteristic Function V, to the accuracy of the product of the masses, under the form

(615)... $$V = V^{(1)} + V^{(2)},$$

in which

(616)... $$V^{(1)} = mV_{,} + m'V'_{,},$$

and

(617)... $$V^{(2)} = mm'W,$$

& that the parts $V^{(1)}$ and $V^{(2)}$ are each expressed as functions of the initial & final coordinates & masses of the planets, & of the constant h; we shall have the following equations

(618)... $$0 = m\left(\frac{d\overline{x}}{dt}\right)_0 + \frac{\delta V^{(1)}}{\delta x_0} + \frac{\delta V^{(2)}}{\delta x_0};$$

(619)... $$0 = m\left(\frac{d\overline{y}}{dt}\right)_0 + \frac{\delta V^{(1)}}{\delta y_0} + \frac{\delta V^{(2)}}{\delta y_0},$$ (620)... $$0 = m\left(\frac{d\overline{z}}{dt}\right)_0 + \frac{\delta V^{(1)}}{\delta z_0} + \frac{\delta V^{(2)}}{\delta z_0};$$

(621)... $$0 = m'\left(\frac{d\overline{x}'}{dt}\right)_0 + \frac{\delta V^{(1)}}{\delta x'_0} + \frac{\delta V^{(2)}}{\delta x'_0};$$ (622)... (623)...

and

(624)... $$0 = -t + \frac{\delta V^{(1)}}{\delta h} + \frac{\delta V^{(2)}}{\delta h}:$$

by which we can determine x, y, z, x', y', z', h as functions of

$$x_0, y_0, z_0, x'_0, y'_0, z'_0, \left(\frac{d\overline{x}}{dt}\right)_0, \left(\frac{d\overline{y}}{dt}\right)_0, \left(\frac{d\overline{z}}{dt}\right)_0, \left(\frac{d\overline{x}'}{dt}\right)_0, \left(\frac{d\overline{y}'}{dt}\right)_0, \left(\frac{d\overline{z}'}{dt}\right)_0, m, m', t.$$

The elliptic parts of these functions are had by suppressing $V^{(2)}$; & if the perturbations of the coordinates be represented, as in [6], by $m'\xi$, $m'\eta$, $m'\zeta$, $m\xi'$, $m\eta'$, $m\zeta'$ (ζ not having the same meaning here which it had in [29]) we shall have

$$(625)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta x_0}+m'\left(\xi\frac{\delta^2V^{(1)}}{\delta x_0\,\delta x}+\eta\,\frac{\delta^2V^{(1)}}{\delta x_0\,\delta y}+\zeta\frac{\delta^2V^{(1)}}{\delta x_0\,\delta z}\right)+m\left(\xi'\frac{\delta^2V^{(1)}}{\delta x_0\,\delta x'}+\eta'\,\frac{\delta^2V^{(1)}}{\delta x_0\,\delta y'}+\zeta'\,\frac{\delta^2V^{(1)}}{\delta x_0\,\delta z'}\right);$$

$$(626)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta y_0}+$$

$$(627)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta z_0}+$$

$$(628)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta x'_0}+$$

$$(629)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta y'_0}+$$

$$(630)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta z'_0}+$$

$$(631)\ldots\quad 0=\frac{\delta V^{(2)}}{\delta h}+m'\left(\xi\,\frac{\delta^2V^{(1)}}{\delta x_0\,\delta h}+\right.$$

Now, in differentiating $V^{(1)}$, we may consider it as an explicit function of the quantities $r+r_0$, $r'+r'_0$, χ, χ', a, a'; & if we put for abbreviation

$$(632)\ldots\quad r+r_0=\rho,\qquad\qquad (633)\ldots\quad r'+r'_0=\rho',$$

we have

$$(634)\ldots\quad \delta V^{(1)}=\frac{\delta V^{(1)}}{\delta\rho}\,\delta\rho+\frac{\delta V^{(1)}}{\delta\rho'}\,\delta\rho'+\frac{\delta V^{(1)}}{\delta\chi}\,\delta\chi+\frac{\delta V^{(1)}}{\delta\chi'}\,\delta\chi'+\frac{\delta V^{(1)}}{\delta a}\,\delta a+\frac{\delta V^{(1)}}{\delta a'}\,\delta a',$$

in which

$$(635)\ldots\quad \frac{\delta V^{(1)}}{\delta a}\,\delta a+\frac{\delta V^{(1)}}{\delta a'}\,\delta a'=0;$$

it seems evident that we may differentiate this last equation & ∵ that we have

$$(636)\ldots\quad \delta^2V^{(1)}=\delta^2_{\rho,\rho',\chi,\chi'}V^{(1)}+\delta a\,\delta_{\rho,\rho',\chi,\chi'}\frac{\delta V^{(1)}}{\delta a}+\delta a'\delta_{\rho,\rho',\chi,\chi'}\frac{\delta V^{(1)}}{\delta a'}$$
$$=\delta^2_{\rho,\rho',\chi,\chi'}V^{(1)}+\delta a\,\delta_{\rho,\chi}\frac{\delta V^{(1)}}{\delta a}+\delta a'\,\delta_{\rho',\chi'}\frac{\delta V^{(1)}}{\delta a'}.$$

However it may be more simple to proceed in the following manner. In (616), $V_{,}$ is an explicit function of the 3 quantities a, ρ, χ; and $V'_{,}$ is a similar explicit function of the 3 quantities a', ρ', χ'; and each of these 2 similar functions is homogeneous of the dimension $\frac{1}{2}$, so that

$$(637)\ldots\quad \frac{\delta V_{,}}{\delta a}=\frac{1}{a}\left(\frac{V_{,}}{2}-\chi\frac{\delta V_{,}}{\delta\chi}-\rho\frac{\delta V_{,}}{\delta\rho}\right);\qquad (638)\ldots\quad \frac{\delta V'_{,}}{\delta a'}=\frac{1}{a'}\left(\frac{V'_{,}}{2}-\chi'\frac{\delta V'_{,}}{\delta\chi'}-\rho'\frac{\delta V'_{,}}{\delta\rho'}\right).$$

We have also, with a sufficient approximation for the purpose of calculating the perturbations, the two following relations to determine a, a', as functions of m, m', ρ, ρ', χ, χ', h,

$$(639)\ldots\quad \frac{1}{2}\frac{\delta V^{(1)}}{\delta h}=a^2\frac{\delta V_{,}}{\delta a}=a'^2\frac{\delta V'_{,}}{\delta a'}\left(=\frac{T}{2}\right);\qquad (640)\ldots\quad \frac{m}{a}+\frac{m'}{a'}+2h=0:$$

& by differentiating these we can find the partial differential coefficients of a, a', & $\because$ of $V^{(1)}$, with respect to ρ, ρ', χ, χ', h. Thus if we put

(641)... $\frac{m}{h}=\mu;$ (642)... $\frac{m'}{h}=\mu';$

(643)... $$\frac{\mu}{a}+\frac{\mu'}{a'}=-2,$$

we shall have

(644)... $V^{(1)}=h\,(\mu V_{,}+\mu' V'_{,})=h\times$ a homogeneous function, 1st dimension, of $\mu, \mu', \rho, \rho', \chi, \chi'$; & if we put, instead of (634), the following expression

(645)... $$\delta V^{(1)}=\frac{\delta V^{(1)}}{\delta\rho}\,\delta\rho+\frac{\delta V^{(1)}}{\delta\rho'}\,\delta\rho'+\frac{\delta V^{(1)}}{\delta\chi}\,\delta\chi+\frac{\delta V^{(1)}}{\delta\chi'}\,\delta\chi'+\frac{\delta V^{(1)}}{\delta h}\,\delta h,$$

we shall have

(646)... $\frac{\delta V^{(1)}}{\delta\rho}=m\,\frac{\delta V_{,}}{\delta\rho};$ (647)... $\frac{\delta V^{(1)}}{\delta\chi}=m\,\frac{\delta V_{,}}{\delta\chi};$

(648)... $\frac{\delta V'}{\delta\rho'}=m'\,\frac{\delta V'_{,}}{\delta\rho'};$ (649)... $\frac{\delta V'}{\delta\chi'}=m'\,\frac{\delta V'_{,}}{\delta\chi'};$

observing that by (639), (640)

(650)... $$m\,\frac{\delta V_{,}}{\delta a}\,\delta a+m'\,\frac{\delta V'_{,}}{\delta a'}\,\delta a'=\frac{\delta V^{(1)}}{\delta h}.$$

Also

(651)... $$\frac{\delta^2 V^{(1)}}{\delta\rho^2}=m\,\frac{\delta^2 V_{,}}{\delta\rho^2}+m\,\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}\,\frac{\delta a}{\delta\rho};$$

(652)... $$\frac{\delta^2 V^{(1)}}{\delta\rho\,\delta\rho'}=m\,\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}\,\frac{\delta a}{\delta\rho'}=m'\,\frac{\delta^2 V'_{,}}{\delta\rho'\,\delta a'}\,\frac{\delta a'}{\delta\rho};$$

as a verification of which last we have, by (640),

(653)... $$\frac{m}{a^2}\,\frac{\delta a}{\delta\rho'}+\frac{m'}{a'^2}\,\frac{\delta a'}{\delta\rho'}=0;$$

we ought therefore to have

(654)... $$0=a^2\,\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}\,\frac{\delta a'}{\delta\rho'}+a'^2\,\frac{\delta^2 V'_{,}}{\delta\rho'\,\delta a'}\,\frac{\delta a'}{\delta\rho};$$

but by (639)

(655)... $$a^2\,\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}=\left(a'^2\,\frac{\delta^2 V'_{,}}{\delta a'^2}+2a'\,\frac{\delta V'_{,}}{\delta a'}\right)\frac{\delta a'}{\delta\rho}-\frac{2a'^2}{a}\,\frac{\delta V'_{,}}{\delta a'}\,\frac{\delta a}{\delta\rho};$$

however it is clear that the equation of the two last members of (652) ought to be a consequence of the 2 equations

(656)... $m\,\frac{\delta V_{,}}{\delta a}\,\frac{\delta a}{\delta\rho}+m'\,\frac{\delta V'_{,}}{\delta a'}\,\frac{\delta a'}{\delta\rho}=0,$ (657)... $m\,\frac{\delta V_{,}}{\delta a}\,\frac{\delta a}{\delta\rho'}+m'\,\frac{\delta V'_{,}}{\delta a'}\,\frac{\delta a'}{\delta\rho'}=0;$

and, in fact, if we differentiate the 1st with respect to ρ' and the 2nd with respect to ρ & compare, we find the equation which was to be proved. We have thus the three ratios of $\frac{\delta a}{\delta\rho}, \frac{\delta a}{\delta\rho'}, \frac{\delta a'}{\delta\rho}, \frac{\delta a'}{\delta\rho'}$; to find these 4 coefficients themselves & more generally to find δa, $\delta a'$, we have, by (639),

$$(658)\ldots\quad 0=\left(a^2\frac{\delta^2 V_{,}}{\delta a^2}+2a\frac{\delta V_{,}}{\delta a}\right)\delta a-\left(a'^2\frac{\delta^2 V'_{,}}{\delta a'^2}+2a'\frac{\delta V'_{,}}{\delta a'}\right)\delta a'+a^2\delta_{\rho,\chi}\frac{\delta V_{,}}{\delta a}-a'^2\delta_{\rho',\chi'}\frac{\delta V'_{,}}{\delta a'};$$

∵ if we put

$$(659)\ldots\quad A=m\frac{\delta V_{,}}{\delta a}\left(a'^2\frac{\delta^2 V'_{,}}{\delta a'^2}+2a'\frac{\delta V'_{,}}{\delta a'}\right)+m'\frac{\delta V'_{,}}{\delta a'}\left(a^2\frac{\delta^2 V_{,}}{\delta a^2}+2a\frac{\delta V_{,}}{\delta a}\right),$$

we shall have

$$(660)\ldots\quad \delta a=\frac{m'}{A}\frac{\delta V'_{,}}{\delta a'}\left(a'^2\delta_{\rho',\chi'}\frac{\delta V'_{,}}{\delta a'}-a^2\delta_{\rho,\chi}\frac{\delta V_{,}}{\delta a}\right)+\frac{\delta h}{A}\frac{\delta V^{(1)}}{\delta h}\left(a'^2\frac{\delta^2 V'_{,}}{\delta a'^2}+2a'\frac{\delta V'_{,}}{\delta a'}\right);$$

$$\&\quad(661)\ldots\quad \delta a'=\frac{m}{A}\frac{\delta V_{,}}{\delta a}\left(a^2\delta_{\rho,\chi}\frac{\delta V_{,}}{\delta a}-a'^2\delta_{\rho',\chi'}\frac{\delta V'_{,}}{\delta a'}\right)+\frac{\delta h}{A}\frac{\delta V^{(1)}}{\delta h}\left(a^2\frac{\delta^2 V_{,}}{\delta a^2}+2a\frac{\delta V_{,}}{\delta a}\right).$$

Hence

$$(662)\ldots\quad \frac{\delta a}{\delta\rho}=\quad\frac{m'a^2}{A}\frac{\delta V'_{,}}{\delta a'}\frac{\delta^2 V_{,}}{\delta\rho\,\delta a};\qquad (663)\ldots\quad \frac{\delta a}{\delta\rho'}=\frac{m'a'^2}{A}\frac{\delta V'_{,}}{\delta a'}\frac{\delta^2 V'_{,}}{\delta\rho'\delta a'};$$

$$(664)\ldots\quad \frac{\delta a'}{\delta\rho'}=-\frac{ma'^2}{A}\frac{\delta V_{,}}{\delta a}\frac{\delta^2 V'_{,}}{\delta\rho'\delta a'};\qquad (665)\ldots\quad \frac{\delta a'}{\delta\rho}=\frac{ma^2}{A}\frac{\delta V_{,}}{\delta a}\frac{\delta^2 V_{,}}{\delta\rho\,\delta a};$$

$$(666)\ldots\quad \frac{\delta^2 V^{(1)}}{\delta\rho^2}=m\frac{\delta^2 V_{,}}{\delta\rho^2}\quad\frac{mm'a^2}{A}\frac{\delta V'_{,}}{\delta a'}\left(\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}\right)^2;\qquad (667)\ldots\quad \frac{\delta^2 V^{(1)}}{\delta\rho\,\delta\rho'}=\frac{mm'T}{2a}\frac{\delta^2 V_{,}}{\delta\rho\,\delta a}\frac{\delta^2 V'_{,}}{\delta\rho'\delta a'}.$$

In the same manner we might calculate $\frac{\delta^2 V^{(1)}}{\delta\rho'^2}, \frac{\delta^2 V^{(1)}}{\delta\rho\,\delta\chi}, \frac{\delta^2 V^{(1)}}{\delta\rho'\delta\chi'}, \frac{\delta^2 V^{(1)}}{\delta\chi^2}, \frac{\delta^2 V^{(1)}}{\delta\chi\,\delta\chi'}, \frac{\delta^2 V^{(1)}}{\delta\chi'^2}$. and then we should be able to deduce the coefficients $\frac{\delta^2 V^{(1)}}{\delta x_0^2}$, &c., in (625)...(631). But the following method appears preferable.

[**32**.] Our disturbing function W may be considered as giving certain small alterations in the initial direction and velocity and elliptic time of each planet, such that by applying these we can calculate the place of each planet by the laws of elliptic motion. Thus

$$(668)\ldots\quad \frac{2}{r_0}-\frac{1}{a}=\left(\frac{\delta V_{,}}{\delta x_0}\right)^2+\left(\frac{\delta V_{,}}{\delta y_0}\right)^2+\left(\frac{\delta V_{,}}{\delta z_0}\right)^2$$
$$=\left(\frac{d\overline{x}}{dt}\right)_0^2+\left(\frac{d\overline{y}}{dt}\right)_0^2+\left(\frac{d\overline{z}}{dt}\right)_0^2+2m'\left\{\left(\frac{d\overline{x}}{dt}\right)_0\frac{\delta W}{\delta x_0}+\left(\frac{d\overline{y}}{dt}\right)_0\frac{\delta W}{\delta y_0}+\left(\frac{d\overline{z}}{dt}\right)_0\frac{\delta W}{\delta z_0}\right\};$$

and if there had been no perturbation of m by m', that is, if m' were $=0$, we should have had $a=$ a value $a_{,}$ depending on the circumstances of projection of m and such that, by (503),

$$(669)\ldots\quad \frac{2}{r}-\frac{1}{a_{,}}=(1+m)^2\left\{\left(\frac{d\overline{x}}{dt}\right)_0^2+\left(\frac{d\overline{y}}{dt}\right)_0^2+\left(\frac{d\overline{z}}{dt}\right)_0^2\right\};$$

if then we change a to $a_{,}+m'a_{,,}$, as in (326), but with new meanings of $a_{,}$, $a_{,,}$ different from the former, we shall have

$$(670)\ldots\quad \frac{a_{,,}}{2a^2}=\frac{dx_0}{dt}\frac{dx'_0}{dt}+\frac{dy_0}{dt}\frac{dy'_0}{dt}+\frac{dz_0}{dt}\frac{dz'_0}{dt}+\frac{\delta W}{\delta x_0}\frac{dx_0}{dt}+\frac{\delta W}{\delta y_0}\frac{dy_0}{dt}+\frac{\delta W}{\delta z_0}\frac{dz_0}{dt}.$$

Now to calculate the perturbation of the centrobaric radius vector $r_{,} = r_{,} + m'r_{,,}$, we can determine the undisturbed part* $r_{,}$ as a function of $r_0, \frac{dr_0}{dt}, a_{,}, m, T$; for which purpose we may use the formula

$$(671)\ldots \qquad r_{,} = a_{,}(1 - e_{,}\cos u),$$

considering $e_{,}$, u as functions of $r_0, \frac{dr_0}{dt}, a_{,}, m, T$, such that

$$(672)\ldots \qquad T = (1+m)\, a_{,}^{\frac{3}{2}} \{u - u_0 - e_{,}(\sin u - \sin u_0)\};$$

$$(673)\ldots \quad r_0 = a_{,}(1 - e_{,}\cos u_0); \qquad (674)\ldots \quad r_0 \frac{dr_0}{dt} = \frac{\sqrt{a_{,}}\,.\,e_{,}\sin u_0}{1+m},$$

& whatever relation is thus obtained, by the theory of elliptic motion, between $r_{,}, r_0, \frac{dr_0}{dt}(1+m)$, $a_{,}, \frac{T}{1+m}$, the same relation must hold identically between $r, r_0, -\frac{\delta V_{,}}{\delta r_0}, a, 2a^2\frac{\delta V_{,}}{\delta a}$: we may therefore differentiate this relation with reference to the disturbing mass m', making

$$(675)\ldots \quad \frac{\delta r}{\delta m'} = r_{,,}; \qquad (676)\ldots \quad \frac{\delta a}{\delta m'} = a_{,,};$$

and

$$(677)\ldots \qquad \frac{\delta}{\delta m'} \,.\, -\frac{\delta V_{,}}{\delta r_0} = \frac{\delta W}{\delta r_0} + \frac{x_0}{r_0}\frac{dx_0'}{dt} + \frac{y_0}{r_0}\frac{dy_0'}{dt} + \frac{z_0}{r_0}\frac{dz_0'}{dt};$$

$$(678)\ldots \qquad m'\frac{\delta}{\delta m'} \,.\, 2a^2\frac{\delta V_{,}}{\delta a} = -mm'\frac{\delta W}{\delta h}.$$

We have ∵

$$(679)\ldots \qquad r_{,,} = a_{,,}\frac{r}{a} + ae\sin u \,.\, u_{,,} - ae_{,,}\cos u;$$

$$(680)\ldots \qquad 0 = a_{,,}\frac{r_0}{a} + ae\sin u_0 \,.\, u_{0,,} - ae_{,,}\cos u_0;$$

$$(681)\ldots \quad r_0\frac{\delta W}{\delta r_0} + x_0\frac{dx_0'}{dt} + y_0\frac{dy_0'}{dt} + z_0\frac{dz_0'}{dt} = \frac{a_{,,}e\sin u_0}{2\sqrt{a}} + \sqrt{a}\,.\,e_{,,}\sin u_0 + \sqrt{a}\,.\,e\cos u_0\,.\,u_{0,,};$$

$$(682)\ldots \qquad 0 = \frac{m}{\sqrt{a}}\frac{\delta W}{\delta h} + \tfrac{3}{2}a_{,,}Ta^{-\frac{3}{2}} + ru_{,,} - r_0u_{0,,} - ae_{,,}(\sin u - \sin u_0);$$

besides (670) for $a_{,,}$.

And if we neglect excentricities and inclinations in perturbations, we may put

$$(683)\ldots \qquad r_{,,} = a_{,,} - ae_{,,}\cos u,$$

$$(684)\ldots \qquad e_{,,} = a^{-\frac{1}{2}}\operatorname{cosec} u_0 \left(r_0\frac{\delta W}{\delta r_0} + x_0\frac{dx_0'}{dt} + y_0\frac{dy_0'}{dt}\right),$$

$$(685)\ldots \qquad a_{,,} = 2a^2\left\{\frac{dx_0}{dt}\left(\frac{dx_0'}{dt} + \frac{\delta W}{\delta x_0}\right) + \frac{dy_0}{dt}\left(\frac{dy_0'}{dt} + \frac{\delta W}{\delta y_0}\right)\right\}.$$

---

* [The notation here is inconsistent. $r_{,}, \theta_{,}$ are the coordinates at time t of the point which moves along the *fixed ellipse* of semi-major axis $a_{,}$. The ellipse of semi-major axis a passes through the points (r_0, θ_0) and (r, θ), T being the time in this ellipse. In this method of calculating perturbations if we put m' equal to zero in r, θ, a, T we get respectively $r_{,}, \theta_{,}, a_{,}, t$ so that $T - m'\frac{\delta T}{\delta m'} = t$, whilst by equation (624) $t = T + mm'\frac{\delta W}{\delta h}$.]

The difficulty is ∴ reduced to calculating these partial differential coefficients of W, $\frac{\delta W}{\delta x_0}$, $\frac{\delta W}{\delta y_0}$, since we may put

$$(686)\ldots \qquad e'' = a^{-\frac{1}{2}} \operatorname{cosec} u_0 \left\{ x_0 \left(\frac{dx_0'}{dt} + \frac{\delta W}{\delta x_0} \right) + y_0 \left(\frac{dy_0'}{dt} + \frac{\delta W}{\delta y_0} \right) \right\},$$

& ∴

$$(687)\ldots \quad r_{,,} = \left(2a^2 \frac{dx_0}{dt} - \sqrt{a} \frac{\cos u}{\sin u_0} x_0 \right) \left(\frac{\delta W}{\delta x_0} + \frac{dx_0'}{dt} \right) + \left(2a^2 \frac{dy_0}{dt} - \sqrt{a} \frac{\cos u}{\sin u_0} y_0 \right) \left(\frac{\delta W}{\delta y_0} + \frac{dy_0'}{dt} \right).$$

Yet I have some doubt whether this last process of approximation is legitimate, & whether u_0 may not vanish or become indeterminate when e is neglected.

It seems safer to employ the approximate expression (178)

$$r_{,} = a + (r_0 - a) \cos nt + a^{\frac{3}{2}} \left(\frac{dr_0}{dt} \right) \sin nt,$$

& ∴ to put

$$(688)\ldots \qquad r_{,,} = a_{,,} (1 - \cos nt) + \sqrt{a} \sin nt \left(r_0 \frac{\delta W}{\delta r_0} + x_0 \frac{dx_0'}{dt} + y_0 \frac{dy_0'}{dt} \right),$$

that is,

$$(689)\ldots \quad r_{,,} = \left(2a^2 \frac{dx_0}{dt} (1 - \cos nt) + \sqrt{a}\, x_0 \sin nt \right) \left(\frac{\delta W}{\delta x_0} + \frac{dx_0'}{dt} \right)$$
$$+ \left(2a^2 \frac{dy_0}{dt} (1 - \cos nt) + \sqrt{a}\, y_0 \sin nt \right) \left(\frac{\delta W}{\delta y_0} + \frac{dy_0'}{dt} \right).$$

But how to calculate $\frac{\delta W}{\delta x_0}$, $\frac{\delta W}{\delta y_0}$ or $\frac{\delta W}{\delta r_0}$, $\frac{\delta W}{\delta \theta_0}$? Since we may put, neglecting excentricities,

$$(690)\ldots \qquad a_{,,} = 2a^2 \left\{ \frac{dx_0}{dt} \frac{dx_0'}{dt} + \frac{dy_0}{dt} \frac{dy_0'}{dt} + a^{-\frac{3}{2}} \frac{\delta W}{\delta \theta_0} \right.$$

[**33**.] W may be developed as an explicit function of a, a', ϵ, ϵ', k, k', l, l', t; & may be put under the form

$$(691)\ldots \qquad W = (W) + \Delta W = (W) + k \frac{\delta \Delta W}{\delta k} + k' \frac{\delta \Delta W}{\delta k'} + l \frac{\delta \Delta W}{\delta l} + l' \frac{\delta \Delta W}{\delta l'},$$

(W) being the circular part. Then, by (22), (W) being a function of a, a', ϵ, ϵ', t,

$$(692)\ldots \quad \delta W = \frac{\delta (W)}{\delta t} \delta t + \frac{\delta (W)}{\delta a} \delta a + \frac{\delta (W)}{\delta a'} \delta a' + \frac{\delta (W)}{\delta \epsilon} \delta \theta_0 + \frac{\delta (W)}{\delta \epsilon'} \delta \theta_0' + \left(\frac{\delta \Delta W}{\delta l} - 2 \frac{\delta (W)}{\delta \epsilon} \right) \delta l$$
$$\left. \frac{\delta \Delta W}{\delta l'} - 2 \frac{\delta (W)}{\delta \epsilon'} \right) \delta l' - \frac{\delta \Delta W}{\delta k} \delta \left(\frac{r_0}{a} \right) - \frac{\delta \Delta W}{\delta k'} \delta \left(\frac{r_0'}{a'} \right);$$

also since

$$(693)\ldots \qquad t = a^{\frac{3}{2}} \{ \theta - \theta_0 - 2k \sin nt + 2l (1 - \cos nt) \},$$

we have

$$(694)\ldots \quad \delta t = a^{\frac{3}{2}} (\delta \theta - \delta \theta_0) + \tfrac{3}{2} \sqrt{a} (\theta - \theta_0) \delta a + 2a^{\frac{3}{2}} \sin nt \,.\, \delta \frac{r_0}{a} + 2a^{\frac{3}{2}} (1 - \cos nt) \delta l;$$

and again

$$(695)\ldots \qquad l = \sqrt{a} \frac{dr_0}{dt} = - \sqrt{a} \frac{\delta V_{,}}{\delta r_0}, \qquad\qquad (696)\ldots \quad l' = - \sqrt{a'} \frac{\delta V_{,}'}{\delta r_0'},$$

∵ (neglecting excentricities in results)

$$(697)\ldots\quad \delta l = -\sqrt{a}\,\delta\frac{\delta V_{,}}{\delta r_0}, \qquad (698)\ldots\quad \delta l' = -\sqrt{a'}\,\delta\frac{\delta V'_{,}}{\delta r'_0};$$

$$(699)\ldots\quad \frac{\delta W}{\delta r_0} = \frac{\delta a}{\delta r_0}\Bigg[\left\{\sqrt{a}\left(\frac{3}{2}nt + 2\sin nt\right) - 2a^2(1-\cos nt)\frac{\delta^2 V_{,}}{\delta r_0\,\delta a}\right\}\frac{\delta(W}{\delta t}$$

$$\frac{\delta(W)}{\delta a} - \sqrt{a}\left(\frac{\delta\Delta W}{\delta l} - 2\frac{\delta(W)}{\delta\epsilon}\right)\frac{\delta^2 V_{,}}{\delta r_0\,\delta a} + \frac{\delta\Delta W}{a\,\delta k}\Bigg]$$

$$-2\left\{\sqrt{a}\sin nt + a^2(1-\cos nt)\frac{\delta^2 V_{,}}{\delta r_0^2}\right\}\frac{\delta(W)}{\delta t} - \sqrt{a}\left(\frac{\delta\Delta W}{\delta l} - 2\frac{\delta(W)}{\delta\epsilon}\right)\frac{\delta^2 V_{,}}{\delta r_0^2} - \frac{\delta\Delta W}{a\,\delta k}$$

$$+\frac{\delta a'}{\delta r_0}\left\{\frac{\delta(W)}{\delta a'} + \frac{\delta\Delta W}{a'\,\delta k'} - \sqrt{a'}\left(\frac{\delta\Delta W}{\delta l'} - 2\frac{\delta(W)}{\delta\epsilon'}\right)\frac{\delta^2 V'_{,}}{\delta r'_0\,\delta a'}\right\};$$

$$(700)\ldots\quad \frac{\delta W}{\delta\theta_0} =$$

and it seems hardly possible that the ratio of the masses m, m' should disappear from the perturbation coefficient $r_{,,}$, unless, upon reducing the expression (692) for δW to an explicit (and, if we please, in a certain sense symmetrical) function of δr, $\delta\theta$, δr_0, $\delta\theta_0$, δa and of $\delta r'$, $\delta\theta'$, $\delta r'_0$, $\delta\theta'_0$, $\delta a'$ without introducing the masses (by the formulae of elliptic motion considering the two ellipses as independent) we can then eliminate simultaneously the variations δa, $\delta a'$ by the relation (658), or by some equivalent relation. Changing ∵ t to $\frac{T-T'}{2}$ for the sake of symmetry and employing other symmetrical expressions wherever it is possible to do so, we are led to try whether the following condition is satisfied,

$$(701)\ldots\quad 0 = \frac{\delta T'}{\delta a'}\left[\frac{\delta W}{\delta a}\right] + \frac{\delta T}{\delta a}\left[\frac{\delta W}{\delta a'}\right],$$

in which

$$(702)\ldots\quad \left[\frac{\delta W}{\delta a}\right] = \frac{1}{2}\frac{\delta(W)}{\delta t}\frac{\delta T}{\delta a} + \frac{\delta(W)}{\delta a} + \frac{\delta\Delta W}{a\,\delta k} + \left(\frac{\delta\Delta W}{\delta l} - 2\frac{\delta(W)}{\delta\epsilon}\right)\frac{\delta l}{\delta a};$$

$$(703)\ldots\quad \left[\frac{\delta W}{\delta a'}\right] = \frac{1}{2}\frac{\delta(W)}{\delta t}\frac{\delta T'}{\delta a'} + \frac{\delta(W)}{\delta a'} + \frac{\delta\Delta W}{a'\,\delta k'} + \left(\frac{\delta\Delta W}{\delta l'} - 2\frac{\delta(W)}{\delta\epsilon'}\right)\frac{\delta l'}{\delta a'}.$$

Also, by (363) (367),

$$(704)\ldots\quad \frac{\delta l}{\delta a} = -\frac{1}{a}\tan\frac{nt}{2}, \qquad \& \qquad (705)\ldots\quad \frac{\delta l'}{\delta a'} = -\frac{1}{a'}\tan\frac{n't}{2};$$

& by (304) &c.,

$$(706)\ldots\quad \frac{\delta T}{\delta a} = \sqrt{a}\left(\frac{3}{2}nt - 4\tan\frac{nt}{2}\right); \qquad \frac{\delta T'}{\delta a'} = \sqrt{a'}\left(\frac{3}{2}n't - 4\tan\frac{n't}{2}\right);$$

it remains ∵ to calculate $\frac{\delta(W)}{\delta t}$, $\frac{\delta(W)}{\delta a}$, $\frac{\delta(W)}{\delta a'}$, $\frac{\delta(W)}{\delta\epsilon}$, $\frac{\delta(W)}{\delta\epsilon'}$, $\frac{\delta\Delta W}{\delta k}$, $\frac{\delta\Delta W}{\delta k'}$, $\frac{\delta\Delta W}{\delta l}$, $\frac{\delta\Delta W}{\delta l'}$. Now since $W = \int_0^t Q\,dt$, (571), we have

(707)... $\frac{\delta(W)}{\delta t}=Q;$ (708)... $\frac{\delta(W)}{\delta a}=\int_0^t \frac{\delta Q}{\delta a}\,dt;$ (709)... $\frac{\delta(W)}{\delta a'}=\int_0^t \frac{\delta Q}{\delta a'}\,dt;$

(710)... $\frac{\delta(W)}{\delta \epsilon}=\int_0^t \frac{\delta Q}{\delta \epsilon}\,dt;$ (711)... $\frac{\delta(W)}{\delta \epsilon'}=\int_0^t \frac{\delta Q}{\delta \epsilon'}\,dt;$ (712)... $\frac{\delta\Delta W}{\delta k}=\int_0^t \frac{\delta Q}{\delta k}\,dt;$

(713)... $\frac{\delta\Delta W}{\delta k'}=\int_0^t \frac{\delta Q}{\delta k'}\,dt;$ (714)... $\frac{\delta\Delta W}{\delta l}=\int_0^t \frac{\delta Q}{\delta l}\,dt;$ (715)... $\frac{\delta\Delta W}{\delta l'}=\int_0^t \frac{\delta Q}{\delta l'}\,dt.$

If we put equation (701) under the form

(716)... $$0=\frac{\left[\frac{\delta W}{\delta a}\right]}{\frac{\delta T}{\delta a}}+\frac{\left[\frac{\delta W}{\delta a'}\right]}{\frac{\delta T'}{\delta a'}},$$

it is easy to see that it gives, when $t=0$,

(717)... $$0=a\frac{d}{dt}\left[\frac{\delta W}{\delta a}\right]+a'\frac{d}{dt}\left[\frac{\delta W}{\delta a'}\right];$$

in which (t being $=0$) we have

(718)... $$a\frac{d}{dt}\left[\frac{\delta W}{\delta a}\right]=-\tfrac{1}{4}Q+a\frac{\delta Q}{\delta a}+\frac{\delta Q}{\delta k};$$

(719)... $$a'\frac{d}{dt}\left[\frac{\delta W}{\delta a'}\right]=-\tfrac{1}{4}Q+a'\frac{\delta Q}{\delta a'}+\frac{\delta Q}{\delta k'};$$

we ought ∵ to have, when $t=0$,

(720)... $$\tfrac{1}{2}Q=a\frac{\delta Q}{\delta a}+a'\frac{\delta Q}{\delta a'}+\frac{\delta Q}{\delta k}+\frac{\delta Q}{\delta k'}.$$

Is this relation true? By (572)

$$Q=\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\{(x-x')^2+(y-y')^2\}^{-\frac{1}{2}}-(xx'+yy')(r^{-3}+r'^{-3});$$

and since

(721)... $$\frac{dx}{dt}=\frac{d(r\cos\theta)}{dt}=\cos\theta\frac{dr}{dt}-r\sin\theta\frac{d\theta}{dt},$$

& (722)... $$\frac{dy}{dt}=\sin\theta\frac{dr}{dt}+r\cos\theta\frac{d\theta}{dt},$$

we have

(723)... $$\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}=rr'\cos(\theta-\theta')\frac{d\theta}{dt}\frac{d\theta'}{dt}$$
$$+\sin(\theta-\theta')\left(r'\frac{dr}{dt}\frac{d\theta'}{dt}-r\frac{dr'}{dt}\frac{d\theta}{dt}\right)+\cos(\theta-\theta')\frac{dr}{dt}\frac{dr'}{dt};$$

therefore, neglecting no powers of the excentricities, we have, in polar coordinates,

(724)... $$Q=\cos(\theta-\theta')\left(\frac{dr}{dt}\frac{dr'}{dt}+rr'\frac{d\theta}{dt}\frac{d\theta'}{dt}-\frac{r'}{r^2}-\frac{r}{r'^2}\right)$$
$$+\sin(\theta-\theta')\left(\frac{dr}{dt}\cdot r'\frac{d\theta'}{dt}-\frac{dr'}{dt}\cdot r\frac{d\theta}{dt}\right)+\{r^2+r'^2-2rr'\cos(\theta-\theta')\}^{-\frac{1}{2}}$$

& this is the symmetrical function, relatively to the two elliptic orbits, which we are to develope and integrate with regard to t in order to find my disturbing function W. In the immediate question we neglect squares and products of excentricities; and we have ∵ by [22],

(363)... $r = a - ak\cos nt + al\sin nt$; (364)... $\theta = nt + \epsilon + 2k\sin nt + 2l\cos nt$;

& (725)... $\dfrac{dr}{dt} = an\,(k\sin nt + l\cos nt)$; (726)... $\dfrac{d\theta}{dt} = n\{1 + 2k\cos nt - 2l\sin nt\}$;

thus

(727)... $\dfrac{d\theta}{dt} = \dfrac{\sqrt{a}}{r^2}$, (728)... $\dfrac{d\theta'}{dt} = \dfrac{\sqrt{a'}}{r'^2}$;

& if we put for abbreviation $\Delta\theta$, $\Delta\theta'$ to denote the parts $2k\sin nt + 2l\cos nt$, $2k'\sin n't + 2l'\cos n't$ of θ, θ', & similarly use Δr, $\Delta r'$, we shall have

(729)... $\dfrac{dr}{dt} = \dfrac{an}{2}\Delta\theta$, (730)... $\dfrac{dr'}{dt} = \dfrac{a'n'}{2}\Delta\theta'$;

& finally, neglecting squares and products of excentricities,

(731)... $$Q = \cos(\theta - \theta')\left\{\frac{\sqrt{aa'}}{rr'} - \frac{r'}{r^2} - \frac{r}{r'^2}\right\} + \frac{\sin(\theta - \theta')}{2\sqrt{aa'}}(\Delta\theta - \Delta\theta') + \{r^2 + r'^2 - 2rr'\cos(\theta - \theta')\}^{-\frac{1}{2}}:$$

that is, if we put

(732)... $$(Q) = \cos(nt - n't + \epsilon - \epsilon')\left\{\frac{1}{\sqrt{aa'}} - \frac{a'}{a^2} - \frac{a}{a'^2}\right\} + \{a^2 + a'^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon')\}^{-\frac{1}{2}},$$

(733)... $$\Delta Q = \left\{\frac{a'}{a^2} + \frac{a}{a'^2} - \frac{1}{2\sqrt{aa'}} - aa'\{a^2 + a'^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon')\}^{-\frac{3}{2}}\right\}$$
$$\times \sin(nt - n't + \epsilon - \epsilon')(\Delta\theta - \Delta\theta') + \frac{\delta\Delta Q}{\delta\Delta r}\Delta r + \frac{\delta\Delta Q}{\delta\Delta r'}\Delta r',$$

in which

(734)... $$\frac{\delta\Delta Q}{\delta\Delta r} = \left(\frac{2a'}{a^3} - \frac{1}{a'^2} - \frac{1}{\sqrt{a'a^3}}\right)\cos(nt - n't + \epsilon - \epsilon')$$
$$+ \frac{a'\cos(nt - n't + \epsilon - \epsilon') - a}{\{a^2 + a'^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon')\}^{\frac{3}{2}}},$$

(735)... $$\frac{\delta\Delta Q}{\delta\Delta r'} = \left(\frac{2a}{a'^3} - \frac{1}{a^2} - \frac{1}{\sqrt{aa'^3}}\right)\cos(nt - n't + \epsilon - \epsilon') + \frac{a\cos(nt - n't + \epsilon - \epsilon') - a'}{\{a^2 + a'^2 - 2aa'\cos(nt - n't + \epsilon - \epsilon')\}^{\frac{3}{2}}},$$

we shall have

(736)... $$Q = (Q) + \Delta Q.$$

Hence when $t = 0$,

(737)... $$\frac{\delta Q}{\delta k} + \frac{\delta Q}{\delta k'} = -a\frac{\delta\Delta Q}{\delta\Delta r} - a'\frac{\delta\Delta Q}{\delta\Delta r'} = \left(\frac{2}{\sqrt{aa'}} - \frac{a}{a'^2} - \frac{a'}{a^2}\right)\cos(\epsilon - \epsilon')$$
$$+ \{a^2 + a'^2 - 2aa'\cos(\epsilon - \epsilon')\}^{-\frac{1}{2}};$$

also

(738)... $$a\frac{\delta Q}{\delta a} + a'\frac{\delta Q}{\delta a'} = -(Q) = \left(\frac{a'}{a^2} + \frac{a}{a'^2} - \frac{1}{\sqrt{aa'}}\right)\cos(\epsilon - \epsilon') - \{a^2 + a'^2 - 2aa'\cos(\epsilon - \epsilon')\}^{-\frac{1}{2}};$$

∵ (739)... $$a\frac{\delta Q}{\delta a} + a'\frac{\delta Q}{\delta a'} + \frac{\delta Q}{\delta k} + \frac{\delta Q}{\delta k'} = \frac{1}{\sqrt{aa'}}\cos(\epsilon - \epsilon');$$

but this is not $=\frac{1}{2}Q$. My conjecture $\because$ is not confirmed respecting the possibility of eliminating δa, $\delta a'$ from δW by the relation

$$(740)\ldots\qquad \delta T = \delta T'$$

without the variation of the relation $\frac{m}{a}+\frac{m'}{a'}+2h=0$. And, indeed, I ought to have perceived that this latter relation was necessary, for without it we should have no introduction of δh into δW, that is, we should have $\frac{\delta W}{\delta h}=0$, & $\because$ the elliptic time would not differ from the true time.

[34.] Since then the part $\left[\frac{\delta W}{\delta a}\right]\delta a+\left[\frac{\delta W}{\delta a'}\right]\delta a'$ of δW, when reduced by the relation (740) to the form

$$\left[\frac{\delta W}{\delta a}\right]+\frac{\frac{\delta T}{\delta a}\left[\frac{\delta W}{\delta a'}\right]}{\frac{\delta T'}{\delta a'}}\Bigg\}\,\delta a+\frac{\delta_{\rho,\chi}T-\delta_{\rho',\chi'}T'}{\frac{\delta T'}{\delta a'}}$$

involves still δa, it seems impossible that the expression (688) for $r_{\prime\prime}$ should not involve the ratio $\frac{m'}{m}$, except by our having

$$(741)\ldots\qquad 2\sqrt{a}\,\frac{\delta a}{\delta\theta_0}(1-\cos nt)+a^{\frac{3}{2}}\sin nt\,.\,\frac{\delta a}{\delta r_0}=0;$$

that is, by (740),

$$(742)\ldots\qquad \frac{\delta T}{\delta r_0}=-\frac{2}{a}\tan\frac{nt}{2}\cdot\frac{\delta T}{\delta\theta_0}.$$

And, in fact, by (304),

$$(743)\ldots\qquad \frac{\delta T}{\delta\theta_0}=-a^{\frac{3}{2}};\qquad\qquad (744)\ldots\qquad \frac{\delta T}{\delta r_0}=2\sqrt{a}\tan\frac{nt}{2};$$

$\because$ the condition (742) is satisfied.

If then we put the total variation of W under the form

$$(745)\ldots\qquad \delta W=\left[\frac{\delta W}{\delta a}\right]\delta a+\left[\frac{\delta W}{\delta a'}\right]\delta a'+\left[\frac{\delta W}{\delta r_0}\right]\delta r_0+\left[\frac{\delta W}{\delta\theta_0}\right]\delta\theta_0+\left[\frac{\delta W}{\delta r}\right]\delta r+\left[\frac{\delta W}{\delta\theta}\right]\delta\theta,$$

we shall have for the coefficient $r_{\prime\prime}$ in the perturbation of r the expression

$$(746)\ldots\qquad r_{\prime\prime}=2\sqrt{a}\left[\frac{\delta W}{\delta\theta_0}\right](1-\cos nt)+\left[\frac{\delta W}{\delta r_0}\right]\frac{\sin nt}{n}+\frac{2a'n'}{n}\cos(\epsilon-\epsilon')(1-\cos nt)$$
$$+\frac{a'n'}{n}\sin(\epsilon-\epsilon')\sin nt;$$

$$(747)\ldots\qquad \left[\frac{\delta W}{\delta r_0}\right]=-\frac{\delta\Delta W}{a\,\delta k}+\frac{\cot\!\text{an}\,nt}{a}\left(2\frac{\delta(W)}{\delta\epsilon}-\frac{\delta\Delta W}{\delta l}\right);\qquad\qquad (748)\ldots\qquad \left[\frac{\delta W}{\delta\theta_0}\right]=\frac{\delta(W)}{\delta\epsilon}.$$

Hence, when $t=0$, we have

$$(749)\ldots\qquad \frac{dr_{\prime\prime}}{dt}=a'n'\sin(\epsilon-\epsilon')+\left[\frac{\delta W}{\delta r_0}\right],$$

in which, at present,

(750)... $$\left[\frac{\delta W}{\delta r_0}\right]=2\sqrt{a}\left\{\frac{\delta(Q)}{\delta\epsilon}-\frac{\delta\Delta Q}{\delta\Delta\theta}\right\}=-a'n'\sin(\epsilon-\epsilon');$$

∵ not only $r_{\prime\prime}$ but $\frac{dr_{\prime\prime}}{dt}$ vanishes with t, which is, so far, a verification of our result (746) because we know that the perturbation must have this property. Whatever t may be, we have

(751)... $$\frac{\delta(W)}{\delta\epsilon}-\frac{1}{2}\frac{\delta\Delta W}{\delta l}=\int_0^t\left(\frac{\delta(Q)}{\delta\epsilon}-\frac{1}{2}\frac{\delta(\Delta Q)}{\delta l}\right)dt;$$

and

(752)... $$r_{\prime\prime}=2\sqrt{a}\int_0^t\frac{\delta(Q)}{\delta\epsilon}dt-\sqrt{a}\sin nt\int_0^t\frac{\delta\Delta Q}{\delta k}dt-\sqrt{a}\cos nt\int_0^t\frac{\delta\Delta Q}{\delta l}dt$$
$$+2\sqrt{\frac{a^2}{a'}}\cos(\epsilon-\epsilon')(1-\cos nt)+\sqrt{\frac{a^3}{a'}}\sin(\epsilon-\epsilon')\sin nt.$$

This expression may be put under the form

(753)... $$r_{\prime\prime}=f(t)-f(0),$$

in which

(754)... $$f(t)=\frac{2\sqrt{a}(Q)}{n-n'}-\frac{2\cos(\epsilon-\epsilon')\cos nt}{n\sqrt{a'}}+\frac{\sin(\epsilon-\epsilon')\sin nt}{n\sqrt{a'}}$$
$$-2\sqrt{a}\left\{\sin nt\int_0^t\sin nt\frac{\delta\Delta Q}{\delta\Delta\theta}dt+\cos nt\int_0^t\cos nt\frac{\delta\Delta Q}{\delta\Delta\theta}dt\right\}$$
$$+\frac{1}{n}\left\{\sin nt\int_0^t\cos nt\frac{\delta\Delta Q}{\delta\Delta r}dt-\cos nt\int_0^t\sin nt\frac{\delta\Delta Q}{\delta\Delta r}dt\right\},$$

so that we have

(755)... $$r_{\prime\prime}=\frac{2\sqrt{a}(Q)-2\sqrt{a}(Q_0)}{n-n'}+\frac{2\cos(\epsilon-\epsilon')(1-\cos nt)+\sin(\epsilon-\epsilon')\sin nt}{n\sqrt{a}}$$
$$-2\sqrt{a}\int_0^t\frac{\delta\Delta Q}{\delta\Delta\theta}dt+\frac{2}{a}\left\{\sin nt\int_0^t\cos nt\left(\int_0\frac{\delta\Delta Q}{\delta\Delta\theta}dt\right)dt-\cos nt\int_0^t\sin nt\left(\int_0^t\frac{\delta\Delta Q}{\delta\Delta\theta}dt\right)dt\right\}$$
$$+\frac{1}{n}\left\{\sin nt\int_0^t\cos nt\frac{\delta\Delta Q}{\delta\Delta r}dt-\cos nt\int_0^t\sin nt\frac{\delta\Delta Q}{\delta\Delta r}dt\right\}.$$

Besides

(756)... $$\int_0^t\frac{\delta\Delta Q}{\delta\Delta\theta}dt=\frac{(Q)-(Q_0)}{n-n'}+\frac{\cos(\epsilon-\epsilon')-\cos(nt-n't+\epsilon-\epsilon')}{2\sqrt{aa'}(n-n')};$$

∵ (757)... $$r_{\prime\prime}=\frac{\cos(nt-n't+\epsilon-\epsilon')-\cos(\epsilon-\epsilon')}{(n-n')\sqrt{a'}}+\frac{2\cos(\epsilon-\epsilon')(1-\cos nt)+\sin(\epsilon-\epsilon')\sin nt}{n\sqrt{a'}}$$
$$+\sin nt\int_0^t\cos nt\,.\,S\,.\,dt+\cos nt\int_0^t\sin nt\,.\,S\,.\,dt,$$
$$\left(\left(\frac{2a'}{a^3}-\frac{1}{a'^2}-\frac{1}{\sqrt{a'a^3}}\right)=a'(n-n')(2n+n')\right),$$

if we put for abbreviation

(758)... $$S=\frac{2}{a}\int_0^t\frac{\delta\Delta Q}{\delta\Delta\theta}dt+\frac{1}{n}\frac{\delta\Delta Q}{\delta\Delta r}.$$

To develope this function, let us employ the former expression, [**10**],

$$(a^2 - 2aa'\cos\omega + a'^2)^{-\frac{1}{2}} = \frac{1}{2a'} \sum_{(i)-\infty}^{\infty} b_{\frac{1}{2}}^{(i)} \cos i\omega;$$

we find

(759)... $$\frac{1}{n}\frac{\delta\Delta Q}{\delta\Delta r} = \frac{a'}{n}(n-n')(2n+n')\cos(nt-n't+\epsilon-\epsilon') + \frac{1}{2a'n}\sum_{(i)-\infty}^{\infty}\frac{\delta b_{\frac{1}{2}}^{(i)}}{\delta a}\cos(int-in't+i\epsilon-i\epsilon'),$$

& (760)... $$\frac{2}{a}\int_0^t \frac{\delta\Delta Q}{\delta\Delta\theta}\,dt = \frac{a'}{n-n'}(nn'-2n^2-2n'^2)\{\cos(nt-n't+\epsilon-\epsilon')-\cos(\epsilon-\epsilon')\} + \frac{a^{-1}a'^{-1}}{n-n'}\sum_{(i)-\infty}^{\infty} b_{\frac{1}{2}}^{(i)}\{\cos(int-in't+i\epsilon-i\epsilon')-\cos(i\epsilon-i\epsilon')\};$$

∵ (761)... $$S = \tfrac{1}{2}\sum_{(i)-\infty}^{\infty} S_i\cos i(nt+n't+\epsilon-\epsilon') = \tfrac{1}{2}S_0 + S_1\cos(nt-n't+\epsilon-\epsilon') + \sum_{(i)\,2}^{\infty} S_i\cos i(nt-n't+\epsilon-\epsilon'),$$

in which, if $i>1$,

(762)... $$S_i = \frac{2b_{\frac{1}{2}}^{(i)}}{aa'(n-n')} + \frac{1}{a'n}\frac{\delta b_{\frac{1}{2}}^{(i)}}{\delta a};$$

when $i=1$, then

(763)... $$S_i = S_1 = \frac{2b_{\frac{1}{2}}^{(1)}}{aa'(n-n')} + \frac{1}{a'n}\frac{\delta b_{\frac{1}{2}}^{(1)}}{\delta a} + \frac{a'(nn'-2n^2-2n'^2)}{n-n'} + \frac{a'(n-n')(2n+n')}{n};$$

and

(764)... $$\tfrac{1}{2}S_0 = -\frac{a}{n-n'}(nn'-2n^2-2n'^2)\cos(\epsilon-\epsilon') - \frac{1}{aa'(n-n')}\sum_{(i)-\infty}^{\infty} b_{\frac{1}{2}}^{(i)}\cos i(\epsilon-\epsilon') + \frac{1}{2a'n}\frac{\delta b_{\frac{1}{2}}^{(0)}}{\delta a} + \frac{b_{\frac{1}{2}}^{(0)}}{aa'(n-n')}.$$

Now

(765)... $$\sin nt\int_0^t \cos nt\,.\,S\,.\,dt - \cos nt\int_0^t \sin nt\,.\,S\,.\,dt$$
$$\frac{n}{2}\sum_{(i)-\infty}^{\infty}\frac{S_i\{\cos i(nt-n't+\epsilon-\epsilon')-\cos nt\cos i(\epsilon-\epsilon')\}}{n^2-i^2(n-n')^2} + \frac{n-n'}{2}\sum_{(i)-\infty}^{\infty}\frac{S_i\sin i(\epsilon-\epsilon')\sin nt}{n^2-i^2(n-n')^2},$$

because

$$2\sin nt\int_0^t \cos nt\cos(\mu t+\nu)\,dt - 2\cos nt\int_0^t \sin nt\sin(\mu t+\nu)\,dt$$
$$= \sin nt\int_0^t \cos(\overline{\mu+n}\,.\,t+\nu)\,dt + \sin nt\int_0^t \cos(\overline{\mu-n}\,.\,t+\nu)\,dt$$
$$- \cos nt\int_0^t \sin(\overline{\mu+n}\,.\,t+\nu)\,dt + \cos nt\int_0^t \sin(\overline{\mu-n}\,.\,t+\nu)\,dt$$
$$= \frac{2n}{n^2-\mu^2}\{\cos(\mu t+\nu)-\cos nt\cos\nu\} + \frac{2\mu\sin nt\sin\nu}{n^2-\mu^2}.$$

(As a verification, this function & its 1st differential coefficient both vanish with t.)

Thus $r_{\prime\prime}$ may be developed in a series of the form (90), namely

(766)... $$r_{\prime\prime} = \tfrac{1}{2}\sum_{(i-\infty}^{\infty}\frac{R_i\cos(int-in't+i\epsilon-i\epsilon')}{i^2(n-n')^2-n^2} + [R_{\prime}]\cos nt + [R_{\prime\prime}]\sin nt,$$

in which, when $i > 1$,

(767)... $$R_i = -nS_i = -\frac{2nb^{(i)}_{\frac{1}{2}}}{aa'(n-n')} - \frac{1}{a'}\frac{\delta b^{(i)}_{\frac{1}{2}}}{\delta a};$$

when $i = 1$,

(768)... $$R_1 = -nS_1 + \frac{2a'n'}{n-n'};$$

that is,

(769)... $$R_1 + \frac{2nb^{(1)}_{\frac{1}{2}}}{aa'(n-n')} + \frac{1}{a'}\frac{\delta b^{(1)}_{\frac{1}{2}}}{\delta a}$$
$$= \frac{a'n'\{(n-n')^2 - n^2\}}{n-n'} - a'(n-n')(2n+n') - \frac{a'n(nn' - 2n^2 - 2n'^2)}{n-n'}$$
$$\frac{a'n'}{n-n'}\left\{n'(n'-2n) - \left(1 + \frac{2n}{n'}\right)(n-n')^2 - n^2 + 2nn' + \frac{2n^3}{n'}\right\} = \frac{2a'n'n^2}{n-n'};$$

& (770)... $$\tfrac{1}{2}R_0 = -\frac{n}{2}S_0 + a'n'n\cos(\epsilon-\epsilon')\left\{\frac{n}{n-n'} - 2\right\}$$
$$\frac{2a'n^3}{n-n'}\cos(\epsilon-\epsilon') + \frac{n}{aa'(n-n')}\left(\sum_{(i)-\infty}^{\infty} b^{(i)}_{\frac{1}{2}}\cos i(\epsilon-\epsilon') - b^{(0)}_{\frac{1}{2}}\right) - \frac{1}{2a'}\frac{\delta b^{(0)}_{\frac{1}{2}}}{\delta a};$$

while the coefficients $[R_,]$ and $[R_{,,}]$ may be determined by the condition that $r_{,,}$ and $\frac{dr_{,,}}{dt}$ shall vanish with t. All these coefficients agree perfectly with those obtained in [**10**], & ∵ the present method of calculating the perturbation of the radius vector agrees perfectly in results with the former. That former method was the usual one of integrating ordinary differential equations of the 2nd order; the present method consisted in a previous integration of a partial differential equation of the 1st order & in a subsequent differentiation of the result, namely, of my characteristic function V.

[**35.**] It is remarkable that in this last differentiation, so far as was required for determining the perturbation of the radius vector of a nearly circular orbit, the semiaxes a, a', though considered as variable, have been permitted to be treated as constant, that is, their variations disappear as does also the variation of the time: so that on regarding the disturbing function W to contain explicitly only the coordinates, the semiaxes, and the time (without the masses or their ratio), we were at liberty to vary this function with respect to the coordinates only. I suspect that this will be found to extend to all powers of excentricities and inclinations, at least in the perturbation of the radius vector and perhaps in that of longitude also.

Without neglecting any powers of excentricities or inclinations & without introducing masses or h we may put W under the form of a function of the initial and final coordinates & of the time; namely by expressing the semiaxes a, a' & the other elliptic elements as functions of the coordinates & of the time by the theory of elliptic motion. In the variation of W, it will ∵ be sufficient (in order to determine the influence of the masses) to calculate the variation δt, or its equivalent δT, or $\delta T'$, by the solution of the following equations:

(771)... $$\delta t = \frac{\delta T}{\delta a}\delta a + \delta_{\rho,\chi}T;$$

(772)... $$\delta t = \frac{\delta T'}{\delta a'}\delta a' + \delta_{\rho',\chi'}T';$$

(773)... $$m\frac{\delta a}{a^2} + m'\frac{\delta a'}{a'^2} = 2\delta h:$$

which give

(774)...
$$\delta t = \frac{\frac{m}{a^2}\frac{\delta T'}{\delta a'}\,\delta_{\rho,\chi}T + \frac{m'}{a'^2}\frac{\delta T}{\delta a}\,\delta_{\rho',\chi'}T' + 2\frac{\delta T}{\delta a}\frac{\delta T'}{\delta a'}\,\delta h}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}}.$$

Since the part of the perturbation of each coordinate (rectangular or polar) of m, which involves partial differential coefficients of W, may be put under the form

$$m'\left(L\frac{\delta W}{\delta x_0} + M\frac{\delta W}{\delta y_0} + N\frac{\delta W}{\delta z_0} - mP\frac{\delta W}{\delta h}\right),$$

while δW may be put under the form

(775)...
$$\delta W = \frac{\delta W}{\delta x}\;\delta x + \left\{\frac{\delta W}{\delta x_0}\right\}\delta x_0 + \ldots + \left\{\frac{\delta W}{\delta t}\right\}\delta t,$$

we see that the only part of each perturbation of m, involving the disturbed mass, is of the form

$$\frac{\frac{mm'}{a^2}\frac{\delta T'}{\delta a'}\left\{\frac{\delta W}{\delta t}\right.}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}}\quad L\frac{\delta T}{\delta x_0} + M\frac{\delta T}{\delta y_0} + N\frac{\delta T}{\delta z_0} - 2a^2P\frac{\delta T}{\delta a}$$

& since $\left\{\frac{\delta W}{\delta t}\right\}$ does not in general vanish, it is necessary & sufficient, for the independence of the perturbations on the disturbed mass, that we should have identically

(776)...
$$0 = L\frac{\delta T}{\delta x_0} + M\frac{\delta T}{\delta y_0} + N\frac{\delta T}{\delta z_0} - 2a^2P\frac{\delta T}{\delta a}.$$

It will be useful, in every particular case, to verify this latter condition; but we know *a priori* that it must be satisfied. We may ∵ in calculating the perturbations treat the time t as constant in W, not only where it expressly enters that function, but in each elliptic element in W, which becomes thus a function of the coordinates and of this constant time, its form being determined by the theory of elliptic motion.

For example, in calculating the perturbation of the radius vector, r, we may use the elliptic equation

(777)...
$$\int_{r_0}^{r}\frac{\pm r\,dr}{\sqrt{2r - \frac{r^2}{a} - a(1-e^2)}}\qquad {}_{r_0}\frac{\pm r\,dr}{\sqrt{2(r-r_0) - \frac{r^2-r_0^2}{a} + r_0^2\left(\frac{dr_0}{dt}\right)^2}},$$

the upper or lower sign being taken according as r is increasing or diminishing, that is, according as $\sin u$ is positive or negative, u being the excentric anomaly, so that (489)... $r = a - ae\cos u$, and

(778)...
$$\frac{du}{dt} > 0,$$

& ∵

(779)...
$$\pm\sqrt{2r - \frac{r^2}{a} - a(1-e^2)} = \sqrt{a}\,.\,e\,.\sin u.$$

Hence, supposing r, t, a, $\frac{dr_0}{dt}$ to vary, while r_0 remains constant, we have

$$(780)\ldots \quad \delta t = \frac{r\,\delta r}{\sqrt{a}\,.\,e\,.\sin u} - \frac{\sqrt{a}}{e}\,\delta a\,.\int_{u_0}(1-e\cos u)\left(1-e\,\frac{\cos u+\cos u_0}{2}\right)(\cos u_0-\cos u)\,\frac{du}{\sin u^2}$$

$$-\frac{ar_0\sin u_0}{e}\,\delta\,\frac{dr_0}{dt}\,.\int_{u_0}^{u}(1-e\cos u)\,\frac{du}{\sin u^2};$$

$$\because\ (781)\ldots \quad \delta r = \frac{\sqrt{a}\,.\,e\,.\sin u}{r}\,.\,\delta t + \frac{a^{\frac{3}{2}}r_0}{r}\{\sin(u-u_0)-e(\sin u-\sin u_0)\}\,\delta\,\frac{dr_0}{dt}$$

$$+a\,\frac{\delta a}{r}\left\{\text{vers}\,(u-u_0)-\frac{e}{2}\{2\cos u-2\cos u_0+3(u-u_0)\sin u-\sin u_0\sin(u-u_0)\}\right\}$$

observing that

$$(782)\ldots \quad \int_{u_0}^{u}(\cos u_0-\cos u)\,\frac{du}{\sin u^2} = \Delta\,\text{cosec}\,u-\cos u_0\,\Delta\,\text{cotan}\,u$$

$$= \frac{1-\cos u_0\cos u}{\sin u} - \frac{1-\cos u_0^2}{\sin u_0} = \frac{\text{vers}\,(u-u_0)}{\sin u};$$

$$(783)\ldots \quad \int_{u_0}^{u}(\cos u_0+3\cos u)(\cos u_0-\cos u)\,\frac{du}{\sin u^2}$$

$$= \int_{u_0}^{u}(\cos u_0^2+2\cos u_0\cos u-3\cos u^2)\,\frac{du}{\sin u^2}$$

$$= 3\Delta u+(3-\cos u_0^2)\,\Delta\,\text{cotan}\,u-2\cos u_0\,\Delta\,\text{cosec}\,u$$

$$= 3\Delta u+\frac{2\Delta\cos u}{\sin u}-\frac{\sin u_0}{\sin u}\sin\Delta u;$$

$$\&\ (784)\ldots \quad \int_{u_0}^{u}(\cos u_0^2-\cos u^2)\cos u\,\frac{du}{\sin u^2} = \int_{u_0}^{u}(\sin u^2-\sin u_0^2)\,\frac{\cos u\,du}{\sin u^2}$$

$$= \Delta\sin u+\sin u_0^2\,\Delta\,\text{cosec}\,u = \frac{(\Delta\sin u)^2}{\sin u}.$$

We have therefore for the perturbation $m'r_{\prime\prime}$ of the centrobaric radius vector, neglecting no powers of excentricities or inclinations, the formula, by [32.],

$$(785)\ldots \quad r_{\prime\prime} = \frac{a^{\frac{3}{2}}}{r}\{\sin(u-u_0)-e(\sin u-\sin u_0)\}\left(r_0\frac{\delta W}{\delta r_0}+x_0\frac{dx_0'}{dt}+y_0\frac{dy_0'}{dt}+z_0\frac{dz_0'}{dt}\right)$$

$$-\frac{a^3}{r}\,.\left\{\left(\frac{\delta W}{\delta x_0}+\frac{dx_0'}{dt}\right)\frac{dx_0}{dt}+\left(\frac{\delta W}{\delta y_0}+\frac{dy_0'}{dt}\right)\frac{dy_0}{dt}+\left(\frac{\delta W}{\delta z_0}+\frac{dz_0'}{dt}\right)\frac{dz_0}{dt}\right\}$$

$$\times\left[2\,\text{vers}\,(u-u_0)-e\{2\cos u-2\cos u_0+3(u-u_0)\sin u-\sin u_0\sin(u-u_0)\}+e^2(\sin u-\sin u_0)^2\right],$$

the time being treated as constant in these last differentiations of W.

Now, it is to be observed that the equations (771)...(773) give not only (774), but

(786)... $$\delta a = \frac{\frac{m'}{a'^2}(\delta_{\rho',\chi'}T' - \delta_{\rho,\chi}T) + 2\frac{\delta T'}{\delta a'}\delta h}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}}$$

(787)... $$\delta a' = \frac{\frac{m}{a^2}(\delta_{\rho,\chi}T - \delta_{\rho',\chi'}T') + 2\frac{\delta T}{\delta a}\delta h}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}}.$$

If then we consider Q as a function of the 6 elliptic elements of each planet & of the time, & put its variation under the form

(788)... $$\delta Q = \frac{\delta Q}{\delta t}\delta t + \frac{\delta Q}{\delta a}\delta a + \ldots,$$

so that we shall have

(789)... $$\delta W = \delta\int_0^t Q\,dt = Q\,\delta t + \delta a\int_0^t \frac{\delta Q}{\delta a}dt + \ldots;$$

we may put, in full,

(790)... $$\begin{aligned}\delta W = {}& \left(\int_0^t \frac{\delta Q}{\delta a}dt + \frac{\delta\epsilon}{\delta a}\int_0^t \frac{\delta Q}{\delta\epsilon}dt + \ldots\right)\frac{\frac{m'}{a'^2}(\delta_{\rho',\chi'}T' - \delta_{\rho,\chi}T) + 2\frac{\delta T'}{\delta a'}\delta h}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}} \\ & + \left(\int_0^t \frac{\delta Q}{\delta a'}dt + \frac{\delta\epsilon'}{\delta a'}\int_0^t \frac{\delta Q}{\delta\epsilon'}dt + \ldots\right)\frac{\frac{m}{a^2}(\delta_{\rho,\chi}T - \delta_{\rho',\chi'}T') + 2\frac{\delta T}{\delta a}\delta h}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}} \\ & + \left(\frac{\delta\epsilon}{\delta x_0}\delta x_0 + \frac{\delta\epsilon}{\delta y_0}\delta y_0 + \frac{\delta\epsilon}{\delta z_0}\delta z_0 + \frac{\delta\epsilon}{\delta x}\delta x + \frac{\delta\epsilon}{\delta y}\delta y + \frac{\delta\epsilon}{\delta z}\delta z\right)\int_0^t \frac{\delta Q}{\delta\epsilon}dt + \ldots \\ & + \left(\frac{\delta\epsilon'}{\delta x'_0}\delta x'_0 + \frac{\delta\epsilon'}{\delta y'_0}\delta y'_0 + \frac{\delta\epsilon'}{\delta z'_0}\delta z'_0 + \frac{\delta\epsilon'}{\delta x'}\delta x' + \frac{\delta\epsilon'}{\delta y'}\delta y' + \frac{\delta\epsilon'}{\delta z'}\delta z'\right)\int_0^t \frac{\delta Q}{\delta\epsilon'}dt + \ldots \\ & + \frac{Q\left(\frac{m}{a^2}\frac{\delta T'}{\delta a'}\delta_{\rho,\chi}T + \frac{m'}{a'^2}\frac{\delta T}{\delta a}\delta_{\rho',\chi'}T' + 2\frac{\delta T}{\delta a}\frac{\delta T'}{\delta a'}\delta h\right)}{\frac{m}{a^2}\frac{\delta T'}{\delta a'} + \frac{m'}{a'^2}\frac{\delta T}{\delta a}}.\end{aligned}$$

in which $\frac{\delta\epsilon}{\delta a}$, $\frac{\delta\epsilon}{\delta x_0}$, &c. may be calculated from $V_,$ and $\frac{\delta\epsilon'}{\delta a'}$, $\frac{\delta\epsilon'}{\delta x'_0}$, &c. from $V'_,$. And if we have obtained, by the theory of elliptic motion, a relation between any variable coordinate such as x and the quantities $x_0, y_0, z_0, \frac{dx_0}{dt}(1+m), \frac{dy_0}{dt}(1+m), \frac{dz_0}{dt}(1+m), \frac{t}{1+m}$, which is identically true when $m'=0$, the same relation must hold good rigorously & identically between the same variable coordinate and x_0, y_0, z_0, $-\frac{\delta V_,}{\delta x_0}$, $-\frac{\delta V_,}{\delta y_0}$, $-\frac{\delta V_,}{\delta z_0}$, $2a^2\frac{\delta V_,}{\delta a}$; $V_,$ being treated here as a function of the independent variables x, y, z, x_0, y_0, z_0, a; in fact $V_,$ is a function of ρ, χ, a, depending on no other quantities except so far as they may be involved in these, (& being with respect to these homogeneous of dimension $\frac{1}{2}$); & although this function is subject to the partial differential equation connecting the 3 coefficients $-\frac{\delta V_{,}}{\delta x_0}, -\frac{\delta V_{,}}{\delta y_0}, -\frac{\delta V_{,}}{\delta z_0}$, namely that the sum of their squares is $=\frac{2}{r_0}-\frac{1}{a}$, yet when along with these 3 coefficients we have also $\frac{\delta V_{,}}{\delta a}$, (or $V_{,}$), we can in general determine x, y, z; we may $\because$ regard x, y, z as rigorously and identically functions of $x_0, y_0, z_0, -\frac{\delta V_{,}}{\delta x_0}, -\frac{\delta V_{,}}{\delta y_0}, -\frac{\delta V_{,}}{\delta z_0}, 2a^2\frac{\delta V_{,}}{\delta a}$, not involving the masses nor h, & determined merely by the form of the function $V_{,}$; & we know by the theory of elliptic motion, as connected with this latter function $V_{,}$, that if m' were $=0$ we should have $-\frac{\delta V_{,}}{\delta x_0}, \ldots, 2a^2\frac{\delta V_{,}}{\delta a}$ equal respectively to $(1+m)\frac{dx_0}{dt}, \ldots, \frac{t}{1+m}$; if $\because$ without thinking of $V_{,}$ we can in any way deduce, from the known theory of elliptic motion, the dependence of x, y, z on $x_0, y_0, z_0, (1+m)\frac{dx_0}{dt}, \ldots, \frac{t}{1+m}$, we shall thereby discover the rigorous & identical dependence of x, y, z on $x_0, y_0, z_0, -\frac{\delta V_{,}}{\delta x_0}, -\frac{\delta V_{,}}{\delta y_0}, -\frac{\delta V_{,}}{\delta z_0}, 2a^2\frac{\delta V_{,}}{\delta a}$. In this last identical dependence (if we only neglect squares and products of masses in perturbations or even rigorously if we suppose W so determined as to satisfy the equation

$$V=mV_{,}+m'V'_{,}+mm'W)$$

we may change $-\frac{\delta V_{,}}{\delta x_0}, -\frac{\delta V_{,}}{\delta y_0}, -\frac{\delta V_{,}}{\delta z_0}$ to $(1+m)\frac{dx_0}{dt}+m'\left(\frac{\delta W}{\delta x_0}+\frac{dx'_0}{dt}\right)$, &c.; and $2a^2\frac{\delta V_{,}}{\delta a}$ to $\frac{t}{1+m}-\frac{mm'}{1+m}\frac{\delta W}{\delta h}$: if $\because$ we neglect squares and products of masses in perturbations & suppose that the ordinary theory of elliptic motion of an evanescent mass m about a fixed unit mass has given an expression of the form

$$\text{(791)}\ldots\qquad \delta x=\frac{\delta x}{\delta x_0}\delta x_0+\ldots+\frac{dx}{dt}\delta t+L\delta\frac{dx_0}{dt}+M\delta\frac{dy_0}{dt}+N\delta\frac{dz_0}{dt},$$

we shall have for the perturbation of x

$$\text{(792)}\ldots\qquad m'x_{\prime\prime}=m'\left\{L\left(\frac{\delta W}{\delta x_0}+\frac{dx'_0}{dt}\right)+M\left(\frac{\delta W}{\delta y_0}+\frac{dy'_0}{dt}\right)+\ldots-m\frac{dx}{dt}\frac{\delta W}{\delta h}\right\};$$

and because L, M, N must be such, by (776), that

$$\text{(793)}\ldots\qquad L\frac{\delta T}{\delta x_0}+M\frac{\delta T}{\delta y_0}+N\frac{\delta T}{\delta z_0}=2a^2\frac{dx}{dt}\frac{\delta T}{\delta a}\left(=t\frac{dx}{dt}\right);$$

$$\text{(794)}\ldots\qquad x_{\prime\prime}=-2a^2\frac{dx}{dt}\left(\int_0^t\frac{\delta Q}{\delta a}dt+\frac{\delta\epsilon}{\delta a}\int_0^t\frac{\delta Q}{\delta\epsilon}dt++++\right)+L\frac{dx'_0}{dt}+M\frac{dy'_0}{dt}+N\frac{dz'_0}{dt}$$
$$+\left(L\frac{\delta\epsilon}{\delta x_0}+M\frac{\delta\epsilon}{\delta y_0}+N\frac{\delta\epsilon}{\delta z_0}\right)\int_0^t\frac{\delta Q}{\delta\epsilon}dt++++;\text{*}$$

the four *pluses* referring to the four other elliptic elements besides a & ϵ.

* [From (786) and (787), $L\frac{\delta a}{\delta x_0}+M\frac{\delta a}{\delta y_0}+N\frac{\delta a}{\delta z_0}-2a^2\frac{dx}{dt}\frac{\delta a}{\delta h}=-2a^2\frac{dx}{dt}$,
$L\frac{\delta a'}{\delta x_0}+M\frac{\delta a'}{\delta y_0}+N\frac{\delta a'}{\delta z_0}-2a^2\frac{dx}{dt}\frac{\delta a'}{\delta h}=0.$]

[**36.**] And if we wish to refer the coordinates to the centre of the Sun and not to the centre of gravity of the system, and denote as before the heliocentric coordinates by $\bar{x}$, &c., we have

(795)... $\bar{x} = (1+m)\,x + m'x'$, (796)... $\bar{x}' = (1+m')\,x' + mx$,

$\because$ (797)... $x = \frac{(1+m')\,\bar{x} - m'\bar{x}'}{1+m+m'}$ (798)... $x' = \frac{(1+m)\,\bar{x}' - m\bar{x}}{1+m+m'}$,

so that the equation

(799)... $$\delta V = m\left(\frac{d\bar{x}}{dt}\,\delta x - \frac{d\bar{x}_0}{dt}\,\delta x_0 + - + -\right) + m'\left(\frac{d\bar{x}'}{dt}\,\delta x' - \frac{d\bar{x}'_0}{dt}\,\delta x'_0 + - + -\right)$$

becomes

(800)... $$(1+m+m')\,\delta V = m\left(\frac{d\bar{x}}{dt}\,\delta\bar{x} - \frac{d\bar{x}_0}{dt}\,\delta\bar{x}_0 + - + -\right) + m'\left(\frac{d\bar{x}'}{dt}\,\delta\bar{x}' - \frac{d\bar{x}'_0}{dt}\,\delta\bar{x}'_0 + - + -\right.$$
$$\left. + mm'\left\{\left(\frac{d\bar{x}}{dt} - \frac{d\bar{x}'}{dt}\right)(\delta\bar{x} - \delta\bar{x}') - \left(\frac{d\bar{x}_0}{dt} - \frac{d\bar{x}'_0}{dt}\right)(\delta\bar{x}_0 - \delta\bar{x}'_0) + - + -\right\}\right.;$$

this last equation being *rigorous*. It gives approximately

(801)... $\frac{d\bar{x}_0}{dt} = m'\frac{d\bar{x}'_0}{dt} - \frac{1+m}{m}\frac{\delta V}{\delta\bar{x}_0}$; (802)... $\frac{uy_0}{dt} = \text{\&c.}$; (803)... $\frac{d\bar{z}_0}{dt} = \text{\&c.}$:

& if we represent by $\bar{V}_{,}$, $\bar{V}'_{,}$ what $V_{,}$ $V'_{,}$ become when we change ρ, χ, a & ρ', χ', a' to $\bar{\rho}$, $\bar{\chi}$, $\bar{a}$ & $\bar{\rho}'$, $\bar{\chi}'$, $\bar{a}'$, that is, to $(1+m)\,\rho$, &c., we may put

(804)... $$V = \frac{m\bar{V}_{,}}{\sqrt{1+m}} + \frac{m'\bar{V}'_{,}}{\sqrt{1+m'}} + mm'\,\bar{\bar{W}},$$

in which the new heliocentric disturbing function is

(805)... $$\bar{\bar{W}} = \bar{W} - \left(\bar{x}'\,\frac{\delta\bar{V}_{,}}{\delta\bar{x}} + \bar{y}'\,\frac{\delta\bar{V}_{,}}{\delta\bar{y}} + \bar{z}'\,\frac{\delta\bar{V}_{,}}{\delta\bar{z}} + \bar{x}'_0\,\frac{\delta\bar{V}_{,}}{\delta\bar{x}_0} + + + \bar{x}\,\frac{\delta V'_{,}}{\delta\bar{x}'} + + + \bar{x}_0\,\frac{\delta V'_{,}}{\delta\bar{x}'_0} + +\right),$$

or, omitting the upper bars on 2nd side,*

(806)... $$\bar{\bar{W}} = -\int_0^t \frac{R\,dt}{m'} - \left(x\,\frac{\delta V'_{,}}{\delta x'} + x_0\,\frac{\delta V'_{,}}{\delta x'_0} + + + +\right).$$

Hence

(807)... $\frac{d\bar{x}_0}{dt} = -\sqrt{1+m}\,\frac{\delta\bar{V}_{,}}{\delta\bar{x}_0} + \frac{\delta}{\delta x_0}\int_0^t R\,dt$; (808)... $\frac{wy_0}{dt} = \text{\&c.}$; (809)... $\frac{d\bar{z}_0}{dt} = \text{\&c.}$

When $\because$ we shall have formed from the elliptic theory, or from the properties of $\bar{V}_{,}$, an equation of the form

(810)... $$\delta\bar{x} = \frac{\delta\bar{x}}{\delta\bar{x}_0}\,\delta\bar{x}_0 + \frac{\delta\bar{x}}{\delta\bar{y}_0}\,\delta\bar{y}_0 + \frac{\delta\bar{x}}{\delta\bar{z}_0}\,\delta\bar{z}_0 + 2\,\frac{d\bar{x}}{dt}\,\delta\,.\,\bar{a}^2\,\frac{\delta\bar{V}_{,}}{\delta\bar{a}} + \frac{\delta\bar{x}}{\delta\frac{\delta\bar{V}_{,}}{\delta\bar{x}_0}}\,\delta\,\frac{\delta\bar{V}_{,}}{\delta x_0} + +,$$

we may at once express the *perturbation of the heliocentric coordinate* $\bar{x}$ under the form

(811)... $$\delta\bar{x} = m'\bar{x}_{\prime\prime} = \frac{\delta x}{\delta\frac{\delta V_{,}}{\delta x_0}}\cdot\frac{\delta}{\delta x_0}\int_0^t R\,dt + \frac{\delta x}{\delta\frac{\delta V_{,}}{\ }}\cdot\frac{\delta}{\delta y_0}\int_0^t R\,dt + \frac{\delta x}{\delta\frac{\delta V_{,}}{\delta z_0}}\cdot\frac{\delta}{\delta z_0}\int_0^t R\,dt;$$

[Cf. equation (585), p. 51.]

R having the same meaning as in the *Méc. Cél.*, & $\int_0^t R\,dt$ being considered as a function of the initial and final coordinates and of the time, which is to be differentiated as if the time were constant. We may ∵ put for the *perturbation of the heliocentric radius vector* $\bar{r}$

$$(812)\ldots \qquad \delta\bar{r} = m'\bar{r}_{\prime\prime} = \frac{\delta r}{\delta \frac{\delta V_{\prime}}{\delta x_0}} \cdot \frac{\delta}{\delta x_0} \int_0^t R\,dt + \frac{\delta r}{\delta \frac{\delta V_{\prime}}{\delta y_0}} \cdot \frac{\delta}{\delta y_0} \int_0^t R\,dt + \frac{\delta r}{\delta \frac{\delta V_{\prime}}{\delta z_0}} \cdot \frac{\delta}{\delta z_0} \int_0^t R\,dt;$$

r being considered as a function of x_0, y_0, z_0, $\frac{\delta V_{\prime}}{\delta x_0}$, $\frac{\delta V_{\prime}}{\delta y_0}$, $\frac{\delta V_{\prime}}{\delta z_0}$, $2a^2\frac{\delta V_{\prime}}{\delta a}$. The variation of this function may be put by (781) under the form

$$(813)\ldots \qquad \delta r = \frac{\delta r}{\delta r_0}\delta r_0 + \frac{dr}{dt}\delta t + \frac{\delta r}{\delta a}\delta a + \frac{\delta r}{\delta \frac{dr_0}{dt}}\delta\frac{dr_0}{dt};$$

$$\because\ (814)\ldots \qquad \frac{\delta r}{\delta \frac{\delta V_{\prime}}{\delta x_0}} \quad -\frac{x_0}{r_0}\frac{\delta}{\delta \frac{dr_0}{dt}} - 2a^2\frac{\delta r}{\delta a}\frac{dx_0}{dt};\ \&\text{c.};^*$$

& consequently

$$(815)\ldots \qquad -m'\bar{r}_{\prime\prime} = \frac{\delta r}{\delta \frac{dr_0}{dt}} \cdot \frac{\delta}{\delta r_0}\int_0^t R\,dt + 2a^2\frac{\delta r}{\delta a}\left\{\frac{dx_0}{dt}\cdot\frac{\delta}{\delta x_0}\int_0^t R\,dt + \frac{dy_0}{dt}\cdot\frac{\delta}{\delta y_0}\int_0^t R\,dt + \frac{dz_0}{dt}\cdot\frac{\delta}{\delta z_0}\int_0^t R\,dt\right\}.$$

In short, the perturbations of the heliocentric coordinates $\bar{x}, \bar{y}, \bar{z}$ may be regarded as arising from small perturbations of the initial components of velocity $\frac{d\bar{x}_0}{dt}, \frac{d\bar{y}_0}{dt}, \frac{d\bar{z}_0}{dt}$ without any change in the initial coordinates, or of the time; and the perturbations of $\frac{d\bar{x}_0}{dt}$, &c. are $-\frac{\delta}{\delta x_0}\int_0^t R\,dt$, &c., the time being treated as constant.

[**37.**] We may put for abbreviation

$$(816)\ldots \quad \frac{d\bar{x}_0}{dt} = \bar{X}_0, \qquad (817)\ldots \quad \frac{d\bar{y}_0}{dt} = \bar{Y}_0, \qquad (818)\ldots \quad \frac{d\bar{z}_0}{dt} = \bar{Z}_0;$$

and may suppose that R has been developed as an explicit function (omitting the bars) of t, x_0, y_0, z_0, X_0, Y_0, Z_0, x'_0, y'_0, z'_0, X'_0, Y'_0, Z'_0. Then

$$(819)\ldots \qquad \frac{\delta}{\delta x_0}\int_0^t R\,dt = \int_0^t \frac{\delta R}{\delta x_0}dt + \frac{\delta X_0}{\delta x_0}\int_0^t \frac{\delta R}{\delta X_0}dt + \frac{\delta Y_0}{\delta x_0}\int_0^t \frac{\delta R}{\delta Y_0}dt + \frac{\delta Z_0}{\delta x_0}\int_0^t \frac{\delta R}{\delta Z_0}dt,$$

together with similar expressions for $\frac{\delta}{\delta y_0}\int_0^t R\,dt$ & $\frac{\delta}{\delta z_0}\int_0^t R\,dt$. Hence by (811)

* $\left[\Sigma\frac{x_0}{r_0}\frac{\delta V_{\prime}}{\delta x_0} = -\frac{dr_0}{dt} \text{ and } 2\frac{dx_0}{dt}\delta\frac{\delta V_{\prime}}{\delta x_0} = -\frac{\delta a}{a^2}.\right]$

$$(820)\ldots\quad \delta\bar{x}=m'\bar{x}_{\prime\prime}=-\left(\frac{\delta x}{\delta X_0}\int_0^t\frac{\delta R}{\delta x_0}dt+\frac{\delta x}{\delta Y_0}\int_0^t\frac{\delta R}{\delta y_0}dt+\frac{\delta x}{\delta Z_0}\int_0^t\frac{\delta R}{\delta z_0}dt\right)$$
$$-\frac{\delta x}{\delta X_0}\left(\frac{\delta X_0}{\delta x_0}\int_0^t\frac{\delta R}{\delta X_0}dt+\frac{\delta Y_0}{\delta x_0}\int_0^t\frac{\delta R}{\delta Y_0}dt+\frac{\delta Z_0}{\delta x_0}\int_0^t\frac{\delta R}{\delta Z_0}dt\right)$$
$$-\frac{\delta x}{\delta Y_0}\left(\frac{\delta X_0}{\delta y_0}\int_0^t\frac{\delta R}{\delta X_0}dt+\frac{\delta Y_0}{\delta y_0}\int_0^t\frac{\delta R}{\delta Y_0}dt+\frac{\delta Z_0}{\delta y_0}\int_0^t\frac{\delta R}{\delta Z_0}dt\right)$$
$$-\frac{\delta x}{\delta Z_0}\left(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\right);$$

x being considered as a function x_0, y_0, z_0, X_0, Y_0, Z_0, t, and X_0, Y_0, Z_0 being functions of x, y, z, x_0, y_0, z_0, t. We have also, neglecting masses,

$$(821)\ldots\quad X_0=-\frac{\delta V_{\prime}}{\delta x_0};\qquad\qquad (822)\ldots\quad Y_0=-\frac{\delta V_{\prime}}{\delta y_0};$$

$$(823)\ldots\quad Z_0=-\frac{\delta V_{\prime}}{\delta z_0};\qquad\qquad (824)\ldots\quad t=T=2a^2\frac{\delta V_{\prime}}{\delta a};$$

$$\because\ (825)\ldots\qquad \delta X_0=\frac{1}{2a^2}\frac{\delta T}{\delta x_0}\frac{\delta_{\rho,\chi}T-\delta t}{\frac{\delta T}{\delta a}}-\delta_{x_0,y_0,z_0}\frac{\delta V_{\prime}}{\delta x_0}-\delta_{x,y,z}\frac{\delta V_{\prime}}{\delta x_0};$$

so that*

$$(826)\ldots\quad \frac{\delta X_0}{\delta x_0}=\frac{\left(\frac{\delta T}{\delta x_0}\right)^2}{2a^2\frac{\delta T}{\delta a}}-\frac{\delta^2 V_{\prime}}{\delta x_0^2},\qquad\qquad (827)\ldots\quad \frac{\delta X_0}{\delta y_0}=\frac{\frac{\delta T}{\delta x_0}\frac{\delta T}{\delta y_0}}{2a^2\frac{\delta T}{\delta a}}-\frac{\delta^2 V_{\prime}}{\delta x_0\,\delta y_0}=\frac{\delta Y_0}{\delta x_0};\ \&c.;$$

$$(828)\ldots\quad \frac{\delta x}{\delta X_0}\frac{\delta X_0}{\delta x_0}+\frac{\delta x}{\delta Y_0}\frac{\delta X_0}{\delta y_0}+\frac{\delta x}{\delta Z_0}\frac{\delta X_0}{\delta z_0}=\frac{\delta x}{\delta X_0}\frac{\delta X_0}{\delta x_0}+\frac{\delta x}{\delta Y_0}\frac{\delta Y_0}{\delta x_0}+\frac{\delta x}{\delta Z_0}\frac{\delta Z_0}{\delta x_0}=-\frac{\delta x}{\delta x_0};$$

observing that the equation

$$(829)\ldots\quad \delta x=\frac{\delta x}{\delta x_0}\delta x_0+\frac{\delta x}{\delta y_0}\delta y_0+\frac{\delta x}{\delta z_0}\delta z_0+\frac{\delta x}{\delta X_0}\delta X_0+\frac{\delta x}{\delta Y_0}\delta Y_0+\frac{\delta x}{\delta Z_0}\delta Z_0+\frac{\delta x}{\delta t}\delta t$$

must hold good independent of any relation between δx, δy, δz, δx_0, δy_0, δz_0, δt, when δX_0, δY_0, δZ_0 are changed to their values

$$(830)\ldots\quad \delta X_0=\frac{\delta X_0}{\delta x}\delta x+\frac{\delta X_0}{\delta y}\delta y+\frac{\delta X_0}{\delta z}\delta z+\frac{\delta X_0}{\delta x_0}\delta x_0+\frac{\delta X_0}{\delta y_0}\delta y_0+\frac{\delta X_0}{\delta z_0}\delta z_0+\frac{\delta X_0}{\delta t}\delta t,\ \&c.$$

Hence the formula (820) for the perturbation of the heliocentric coordinate $\bar{x}$ becomes

$$(831)\ldots\quad \delta\bar{x}=m'\bar{x}_{\prime\prime}=\frac{\delta x}{\delta x_0}\int_0^t\frac{\delta R}{\delta X_0}dt-\frac{\delta x}{\delta X_0}\int_0^t\frac{\delta R}{\delta x_0}dt+\frac{\delta x}{\delta y_0}\int_0^t\frac{\delta R}{\delta Y_0}dt-\frac{\delta x}{\delta Y_0}\int\frac{\delta R}{\delta y_0}dt+-.$$

* $\left[\delta X_0=-\delta\frac{\delta S}{\delta x_0}\right.$, where S is the Principal Function.$\left.\right]$

And this expression agrees perfectly with a result of Lagrange,* who conceives the six elements x_0, y_0, z_0, X_0, Y_0, Z_0 to vary (whereas I only conceive the 3 latter to be disturbed) & who gives for the perturbations of these 6 elements expressions equivalent to

$$+\int_0^t \frac{\delta R}{\delta X_0}\,dt, \quad -\int_0^t \frac{\delta R}{\delta x_0}\,dt, \quad \&c.$$

My former expression (811) may be thus abridged:

$$(832)\ldots \qquad \delta\overline{x} = -\left(\frac{\delta x}{\delta X_0}\frac{\delta}{\delta x_0} + \frac{\delta x}{\delta Y_0}\frac{\delta}{\delta y_0} + \frac{\delta x}{\delta Z_0}\frac{\delta}{\delta z_0}\right) R\,dt.$$

But as an enunciation of my theorem of perturbation, including this and other analogous expressions for $\delta\overline{y}$, $\delta\overline{z}$, I see none simpler than the statement at the end of section [**36.**], namely that the perturbations of the final coordinates may be considered to arise from perturbations of the initial components of relative heliocentric velocities X_0, Y_0, Z_0 such that

$$(833)\ldots \quad \delta\overline{X}_0 = -\frac{\delta}{\delta x_0}\int_0^t R\,dt; \qquad (834)\ldots \quad \delta\overline{Y}_0 = -\frac{\delta}{\delta y_0}\int_0^t R\,dt;$$

$$(835)\ldots \quad \delta\overline{Z}_0 = -\frac{\delta}{\delta z_0}\int_0^t R\,dt;$$

a result which seems to me to constitute a new theorem in dynamics. The theorem may be thus presented:

$$(836)\ldots \quad \delta\frac{\delta V_{,}}{\delta x_0} = \frac{\delta}{\delta x_0}\int_0^t R\,dt; \qquad (837)\ldots \quad \delta\frac{\delta V_{,}}{\delta y_0} = \frac{\delta}{\delta y_0}\int_0 R\,dt; \qquad (838)\ldots \quad \&c.;$$

$$(839)\ldots \quad \delta\,.\,a^2\frac{\delta V_{,}}{\delta a} = 0; \qquad (840)\ldots \quad \delta x_0 = 0.;$$

or thus

$$(841)\ldots \qquad \frac{\delta}{\delta x_0}\int_0^t R\,dt = \delta_{x,y,z}\frac{\delta V_{,}}{\delta x_0} - \frac{\dfrac{\delta^2 V_{,}}{\delta a\,\delta x_0}\,\delta_{x,y,z}\dfrac{\delta V_{,}}{\delta a}}{\dfrac{\delta^2 V_{,}}{\delta a^2} + \dfrac{2}{a}\dfrac{\delta V_{,}}{\delta a}} \quad \&c.$$

Hence

$$(842)\ldots \qquad \frac{\delta}{\delta r_0}\int_0^t R\,dt = \delta_{x,y,z}\frac{\delta V_{,}}{\delta r_0} - \frac{\dfrac{\delta}{\delta r_0}\dfrac{\delta V_{,}}{\delta a}\,\delta_{x,y,z}\dfrac{\delta V_{,}}{\delta a}}{\dfrac{\delta^2 V_{,}}{\delta a^2} + \dfrac{2}{a}\dfrac{\delta V_{,}}{\delta a}}$$

[**38.**] We have

$$(843)\ldots \qquad X_0 = -\frac{x_0}{r_0}\frac{\delta V_{,}}{\delta \rho} + \frac{x - x_0}{\chi}\frac{\delta V_{,}}{\delta \chi},$$

$$\because\ (844)\ldots \qquad = x_0 F + X_0 G,$$

& similarly

$$(845)\ldots \quad y = y_0 F + Y_0 G, \quad \& \quad (846)\ldots \quad z = z_0 F + Z_0 G,$$

if we put, for abbreviation,

$$(847)\ldots \quad F = 1 + \frac{\chi}{r_0}\frac{\delta V_{,}}{\delta \rho}\left(\frac{\delta V_{,}}{\delta \chi}\right)^{-1}; \qquad (848)\ldots \quad G = \chi\left(\frac{\delta V_{,}}{\delta \chi}\right)^{-1}.$$

---

* [*Mécanique Analytique*, Tome I, 2^e Partie, Sect. V.]

These coefficients F, G are by Lagrange called sometimes T & V; see 7th section of *Méc. Anal.*, § I & § II. Lagrange does not appear to have found any finite expressions for these coefficients; but he gives the 1st terms of their expansions according to the powers of the time, namely, neglecting t^4 & using my present notation,

$$(849)\ldots \quad F = 1 - \frac{t^2}{2r_0^3} + \frac{t^3}{2r_0^4}\frac{dr_0}{dt}; \qquad (850)\ldots \quad G = t - \frac{t^3}{6r_0^3}.$$

By my methods we have by [28.]

$$(851)\ldots \qquad G = \chi\left(\frac{\delta V_{,}}{\delta\chi}\right)^{-1} = a^{\frac{3}{2}}\left\{\sin(u-u_0) - 2\zeta\sin\frac{u-u_0}{2}\right.$$

in which $\zeta = e\cos\frac{u+u_0}{2}$; also

$$(852)\ldots \qquad t = a^{\frac{3}{2}}\left\{u - u_0 - 2\zeta\sin\frac{u-u_0}{2}\right\};$$

& again

$$(853)\ldots \quad F = 1 + \frac{G}{r_0}\frac{\delta V_{,}}{\delta\rho} = 1 - \frac{2a}{r_0}\left(\sin\frac{u-u_0}{2}\right)^2; \qquad (854)\ldots \quad \rho = r + r_0 = 2a\left(1 - \zeta\cos\frac{u-u_0}{2}\right);$$

$$\&\quad (533)\ldots \quad \chi = 2a\sin\frac{u-u_0}{2}\,.\,\sqrt{1-\zeta^2}.$$

Hence $t-G$ and $r_0(1-F)$ are explicit functions of a, $u-u_0$, namely

$$(855)\ldots \qquad (t-G) = a^{\frac{3}{2}}\{u-u_0-\sin(u-u_0)\},$$

$$(856)\ldots \qquad r_0(1-F) = 2a\left(\sin\frac{u-u_0}{2}\right)^2 = a\{1-\cos(u-u_0)\};$$

and these functions are such that

$$(857)\ldots \quad \frac{(t-G)^2}{r_0^3(1-F)^3} = \frac{\{u-u_0-\sin(u-u_0)\}^2}{\{1-\cos(u-u_0)\}^3} = \frac{\left(\frac{\Delta u^3}{2\,.\,3} - \frac{\Delta u^5}{2\,.\,3\,.\,4\,.\,5} + \&\text{c.}\right)^2}{\left(\frac{\Delta u^2}{2} - \frac{\Delta u^4}{2\,.\,3\,.\,4} + \&\text{c.}\right)^3}$$

$$2\frac{\left(1 - \frac{\Delta u^2}{4\,.\,5} + \frac{\Delta u^4}{4\,.\,5\,.\,6\,.\,7} - \&\text{c.}\right)^3}{\left(1 - \frac{\Delta u^2}{3\,.\,4} + \frac{\Delta u^4}{3\,.\,4\,.\,5\,.\,6} - \&\text{c.}\right)^2} = \frac{2}{9}\left\{1 + \Delta u^2\left(\frac{1}{6} - \frac{3}{20}\right) + \&\text{c.}\right.$$

$$= \frac{2}{9}\left\{1 + \frac{\Delta u^2}{60} + \&\text{c.}\right\} = \text{a function of } \Delta u.$$

We have ∵ finite expressions for 5 relations between the 8 quantities F, G, a, t, ρ, χ, ζ, Δu; & if we eliminate ζ we get 4 finite relations between the 4 quantities F, G, a, Δu & the 3 others t, ρ, χ, namely

$$(858)\ldots \quad \frac{t}{\sqrt{a}} = a\Delta u + (\rho - 2a)\tan\frac{\Delta u}{2}; \qquad (859)\ldots \quad \chi^2 = 4a^2\left(\sin\frac{\Delta u}{2}\right)^2 - (\rho-2a)^2\left(\tan\frac{\Delta u}{2}\right)^2;$$

---

* [(533) and (540).]

together with (855) & (856). Instead of (859) we may use the following:

(860)... $$\chi^2 = 4a^2\left(\sin\frac{\Delta u}{2}\right)^2 - \left(\frac{t}{\sqrt{a}} - a\,\Delta u\right)^2 = 2a^2 \operatorname{vers}\Delta u - \left(a\,\Delta u - \frac{t}{\sqrt{a}}\right)^2.$$

When the orbit tends to become parabolic, this expression tends to the following:

(861)... $$2t\,(\sqrt{a}\,.\,\Delta u) - \frac{(\sqrt{a}\,.\,\Delta u)^4}{12} = \chi^2;$$

whilst (858) tends to the form

(862)... $$t = \frac{\rho}{2}\,(\sqrt{a}\,.\,\Delta u) - \frac{1}{12}\,(\sqrt{a}\,.\,\Delta u)^3$$

Put for a moment

(863)... $$U = \sqrt{a}\,.\,\Delta u,$$

then

(864)... $2tU = \chi^2 + \dfrac{U^4}{12}$; & (865)... $2t = \rho U - \dfrac{U^3}{6}$; ∵ (866)... $6tU = \rho U^2 + 2\chi^2$;

∵ (867)... $\rho U = 3t \pm \sqrt{9t^2 - 2\rho\chi^2}$; (868)... $\dfrac{U^3}{6} = t \pm \sqrt{9t^2 - 2\rho\chi^2}$;

(869)... $$\rho^3 U^3 = 108t^3 - 18t\rho\chi^2 \pm (36t^2 - 2\rho\chi^2)\sqrt{9t^2 - 2\rho\chi^2} = 6t\rho^3 \pm 6\rho^3\sqrt{9t^2 - 2\rho\chi^2}:$$

∵ (870)... $$3t\,(18t^2 - 3\rho\chi^2 - \rho^3) = \pm\,(18t^2 - \rho\chi^2 - 3\rho^3)\sqrt{9t^2 - 2\rho\chi^2};$$

∵ putting

(871)... $$9t^2 - 2\rho\chi^2 = \tau,$$

we have

(872)... $$(\tau + 2\rho\chi^2)\,(2\tau + \rho\chi^2 - \rho^3)^2 = \tau\,(2\tau + 3\rho\chi^2 - 3\rho^3)^2.$$

∵ (873)... $$\rho^3\chi^2\,(\chi^2 - \rho^2)^2 = 4\tau\rho\,(\chi^2 - \rho^2)\,(\tau + \rho\chi^2 - \rho^3) - 4\tau\rho\chi^2\,(\tau + \rho\chi - \rho^3) = -\,4\tau\rho^3\,(\tau + \rho\chi^2 - \rho^3);$$

∵, since neither ρ nor $\chi^2 - \rho^2$ can vanish,

(874)... $4\tau^2 - 4\tau\rho\,(\rho^2 - \chi^2) + \chi^2\,(\rho^2 - \chi^2)^2 = 0$; ∵ (875)... $\{2\tau - \rho\,(\rho^2 - \chi^2)\}^2 = (\rho^2 - \chi^2)^3$;

∵ (876)... $$18t^2 = \rho\,(\rho^2 + 3\chi^2) \mp (\rho^2 - \chi^2)^{\frac{3}{2}};$$

∵ (877)... $$(6t)^2 = (\rho + \chi)^3 + (\rho - \chi)^3 \mp 2\,(\rho^2 - \chi^2)^{\frac{3}{2}};$$

∵ (878)... $$t = \tfrac{1}{6}\,\{(\rho + \chi)^{\frac{3}{2}} \mp (\rho - \chi)^{\frac{3}{2}}\},$$

which agrees with the known expression for time in a parabola.

When the orbit is of moderate excentricity, (860) gives nearly for small arcs

(879)... $$2\sqrt{a}\,\Delta u = 2U = \frac{t}{a} + \frac{\chi^2}{t};$$

&, for such arcs, (858) gives

(880)... $t = \dfrac{\rho U}{2}$; ∵ (881)... $\dfrac{t}{a} + \dfrac{\chi^2}{t} = \dfrac{4t}{\rho}$; ∵ (882)... $t = \dfrac{\chi}{\sqrt{\dfrac{4}{\rho} - \dfrac{1}{a}}}.$

In fact the radical in the denominator is, for small arcs, the velocity. Hence, for small arcs,

(883)... $$\frac{1}{a}=\frac{4}{\rho}-\frac{\chi^2}{t^2}.$$

In general the relation between a, t, ρ, χ may be had by eliminating Δu between (858) & (859) or (860): that is, between the following equations

(884)... $$\frac{t}{a^{\frac{3}{2}}}-\Delta u=\left(\frac{\rho}{a}-2\right)\tan\frac{\Delta u}{2}=\mp\sqrt{2\operatorname{vers}\Delta u-\frac{\chi^2}{a^2}}\left(=-2\zeta\sin\frac{\Delta u}{2}=-e\,.\,\Delta\sin u\right).$$

For small arcs and moderate excentricities, we have, along with (883), the expression

(885)... $$\Delta u=\frac{2t}{\rho\sqrt{a}}=\frac{2}{\rho}\sqrt{\frac{4t^2}{\rho}-\chi^2}\cdot$$

∵ (886)... $F=1-\frac{2t^2}{\rho^2 r_0}$; (887)... $G=t-\frac{4t^3}{3\rho^3}$;

which evidently agree, so far, with (849) & (850).

For parabolic arcs*

(888)... $G=\mp\sqrt{9t^2-2\rho\chi^2}$; ∵ (889)... $(2G)^2=(\rho^2-\chi^2)\{\sqrt{\rho+\chi}\mp\sqrt{\rho-\chi}\}^2$;

∵ (890)... $G=\frac{1}{2}\sqrt{\rho^2-\chi^2}\{\sqrt{\rho+\chi}\mp\sqrt{\rho-\chi}\}$;

(891)... $U=\sqrt{\rho+\chi}\mp\sqrt{\rho-\chi}$; (892)... $F=1-\frac{U^2}{2r_0}$.

In general we may consider F & G as functions of t & may accordingly denote them by F_t, G_t. Suppose now that $\bar{z}_0$ and $\frac{d\bar{z}_0}{dt}$ vanish, so that if it were not for perturbation the planet m would have moved about the Sun in the plane of xy; & let us calculate the perturbation of $\bar{z}$ by the formula (835), $\delta\bar{Z}_0=-\frac{\delta}{\delta z_0}\int_0^t R\,dt$, which gives here

(893)... $$m'\bar{z}_{\prime\prime}=\delta\bar{z}=-G_t\frac{\delta}{\delta z_0}\int_0^t R\,dT.$$

In calculating this last partial differential coefficient we may consider in $\int_0^t R\,dT$ not only the whole time t as given but x, y, x', y', z' as given functions of the variable intermediate time T; z or z_T alone varies with z_0 and is the varying vertical ordinate (varying with T) of the elliptic path by which m would go from x_0, y_0, z_0 to x, y, z (if we put z or $z_T=m'\bar{z}_{\prime\prime}$ we should introduce squares of m') in the given time t; & since in this elliptic path there would be an initial vertical velocity Z_0, such that

(894)... $$0=z_t=F_t z_0+G_t Z_0,$$

we shall have

(895)... $Z_0=-\frac{F_t}{G_t}z_0$, & ∵ (896)... $z_T=F_T z_0+G_T Z_0=\frac{F_T G_t-F_t G_T}{G_t}z_0$,

& (897)... $$\frac{\delta}{\delta z_0}\int_0^t R\,dT=\int_0^t F_T\frac{\delta R}{\delta z}dT-\frac{F_t}{G_t}\int_0^t G_T\frac{\delta R}{\delta z}dT.$$

* [From (855) $t-G=\frac{U^3}{6}$, ∵ $G=\mp\sqrt{9t^2-2\rho\chi^2}=\pm\sqrt{\tau}$. From (875) we get (889). (891) follows from (867).]

We have ∵, for the sought perturbation of the coordinate perpendicular to the primitive heliocentric plane in any planet's or comet's orbit, the expression

$$(898)\ldots \qquad m'\bar{z}_{\prime\prime} = \delta\bar{z} = F_t \int_0^t G_T \frac{\delta R}{\delta z}\, dT - G_t \int_0^t F_T \frac{\delta R}{\delta z}\, dT;$$

in which z_0, z_T are to be made $= 0$ in G_T, F_T and z_t, z_0 in F_t, G_t, while (z being $= 0$)

$$(899)\ldots \qquad \frac{\delta R}{\delta z} = \frac{m'z'}{r'^3} - m'z' \{(x-x')^2 + (y-y')^2 + z'^2\}^{-\frac{3}{2}}.$$

We see also that

$$(900)\ldots \qquad m'\frac{d\bar{z}_{\prime\prime}}{dt} = \frac{dF_t}{dt}\int_0^t G_T \frac{\delta R}{\delta z}\, dT - \frac{dG_t}{dt}\int_0^t F_T \frac{\delta R}{\delta z}\, dT:$$

from which it follows that not only the perturbation $m'\bar{z}_{\prime\prime}$ itself but also the differential of that perturbation at the end of any time t is the same as if the planet or comet were moving in a momentary ellipse corresponding to an initial z_0 and Z_0 represented by

$$\int_0^t G_T \frac{\delta R}{\delta z}\, dT \ \& \ -\int_0^t F_T \frac{\delta R}{\delta z}\, dT.$$

This last result is interesting from its analogy to the results of Lagrange, & it seems likely to be useful in calculating the perturbations of planets and comets. It agrees completely with Lagrange's theory; $G_T \frac{\delta R}{\delta z}$ & $F_T \frac{\delta R}{\delta z}$ corresponding to $\frac{\delta\Gamma}{\delta Z_0}$ & $\frac{\delta\Pi}{\delta z_0}$. Perhaps however my finite expressions for F_T, G_T may be found useful in the applications besides the curiosity of deducing the same results by considerations and methods so different.

We have here supposed xy to be the plane of the primitive orbit; but leaving this plane arbitrary and employing the expression (831) under the clearer form

$$(901)\ldots \qquad \delta\bar{x} = m'\bar{x}_{\prime\prime} = \frac{\delta x_t}{\delta x_0}\int_0^t \frac{\delta R_T}{\delta X_0}\, dT - \frac{\delta x_t}{\delta X_0}\int_0^t \frac{\delta R_T}{\delta x_0}\, dT + - + -,$$

we find by differentiation

$$(902)\ldots \qquad \frac{d\delta\bar{x}}{dt} = m'\frac{d\bar{x}_{\prime\prime}}{dt} = \frac{d}{dt}\left(\frac{\delta x_t}{\delta x_0}\right)\int_0^t \frac{\delta R_T}{\delta X_0}\, dT - \frac{d}{dt}\left(\frac{\delta x}{\delta X_0}\right)\int_0^t \frac{\delta R_T}{\delta x_0}\, dT + - + -,$$

observing that

$$(903)\ldots \qquad \frac{\delta R_t}{\delta X_0}\frac{\delta x_t}{\delta x_0} - \frac{\delta R_t}{\delta x_0}\frac{\delta x_t}{\delta X_0} + - + - = \left(\frac{\delta R_t}{\delta x_t}\frac{\delta x_t}{\delta X_0} + \frac{\delta R_t}{\delta y_t}\frac{\delta y_t}{\delta X_0} + \frac{\delta R_t}{\delta z_t}\frac{\delta z_t}{\delta X_0}\right)\frac{\delta x_t}{\delta x_0} - + - + - = 0.$$

We thus ∵ not only verify Lagrange's result by finding one, which, so far as it goes, agrees with it, & shews that the disturbed planet will have the plane which Lagrange's theory assigns at the end of any time t; but also we rededuce Lagrange's whole result (respecting the variations of the 6 elements x_0, X_0, &c.) by finding, by our own method, that the velocity and direction of velocity of the disturbed planet at the same time t are such as Lagrange assigns. That is, the development of my method conducts to the same conception of 6 varying elements and of an ellipse which represents not only the momentary plane but the momentary motion of a planet or comet or other disturbed body.

[*Independence of perturbations on disturbed mass when squares and products of masses are neglected.*]

[39.] We have as yet only assumed, & not deduced in a general manner from the development of our own method, the independence of the perturbations on the disturbed mass, when the squares and products of the masses are neglected. It is however not difficult to establish this essential theorem. We reduced it, before, to the following

$$(776)\ldots \qquad L\frac{\delta T}{\delta x_0}+M\frac{\delta T}{\delta y_0}+N\frac{\delta T}{\delta z_0}=2a^2P\frac{\delta T}{\delta a},$$

in which L, M, N, P are coefficients such that, when x_0, y_0, z_0 are given but X_0, Y_0, Z_0, t vary, we have $L\delta X_0+M\delta Y_0+N\delta Z_0+P\delta t$ for the variation of an elliptic coordinate. Now using for simplicity the rectangular coordinates x, y, z we have the four equations

$$(904)\ldots \qquad -\delta X_0=\frac{\delta^2 V_{,}}{\delta x\,\delta x_0}\delta x+\frac{\delta^2 V_{,}}{\delta y\,\delta x_0}\delta y+\frac{\delta^2 V_{,}}{\delta z\,\delta x_0}\delta z+\frac{\delta^2 V_{,}}{\delta a\,\delta x_0}\delta a;$$

$$(905)\ldots \qquad -\delta Y_0=\delta_{x,y,z}\frac{\delta V_{,}}{\delta y_0}+\frac{\delta^2 V_{,}}{\delta a\,\delta y_0}\delta a; \qquad (906)\ldots \qquad -\delta Z_0=\&\text{c.}:$$

$$(907)\ldots \qquad \delta t=\delta\,.\,2a^2\frac{\delta V_{,}}{\delta a}=\frac{\delta T}{\delta x}\delta x+\frac{\delta T}{\delta y}\delta y+\frac{\delta T}{\delta z}\delta z+\frac{\delta T}{\delta a}\delta a;$$

which may be put under the form

$$(908)\ldots \qquad -2a^2\delta X_0=\frac{\delta T}{\delta x_0}\delta a+2a^2\delta_{x,y,z}\frac{\delta V_{,}}{\delta x_0};$$

$$(909)\ldots \qquad -2a^2\delta Y_0=\frac{\delta T}{\delta y_0}\delta a+2a^2\delta_{x,y,z}\frac{\delta V_{,}}{\delta y_0}; \qquad (910)\ldots \quad \&\text{c.} \qquad (911)\ldots \quad \delta t=\frac{\delta T}{\delta a}\delta a+\delta_{x,y,z}T.$$

They give, therefore, eliminating δa,

$$(912)\ldots \qquad 2a^2\frac{\delta T}{\delta a}\delta X_0+\frac{\delta T}{\delta x_0}\delta t=\frac{\delta T}{\delta x_0}\delta_{x,y,z}T-2a^2\frac{\delta T}{\delta a}\delta_{x,y,z}\frac{\delta V_{,}}{\delta x_0},$$

$$(913)\ldots \qquad 2a^2\frac{\delta T}{\delta a}\delta Y_0+\frac{\delta T}{\delta y_0}\delta t=\frac{\delta T}{\delta y_0}\delta_{x,y,z}T-2a^2\frac{\delta T}{\delta a}\delta_{x,y,z}\frac{\delta V_{,}}{\delta y_0}, \qquad (914)\ldots \quad \&\text{c.};$$

& consequently for δx, δy, δz expressions of the form*

$$(915)\ldots \qquad \delta x=L\left(\delta X_0+\frac{\overline{\delta x_0}}{2a^2\frac{\delta T}{\delta a},}\right|+M(\qquad)+N(\qquad),$$

so that if we represent this variation by

$$(916)\ldots \qquad \delta x=L\delta x_0+M\delta y_0+N\delta z_0+P\delta t,$$

we have evidently the required condition

$$(776)\ldots \qquad L\frac{\delta T}{\delta x_0}+M\frac{\delta T}{\delta y_0}+N\frac{\delta T}{\delta z_0}=2a^2P\frac{\delta T}{\delta a}.$$

---

* [It is assumed that we *can* solve this equation for δx, δy, δz. This is true except for Kinetic Foci.]

[*Proof that the two partial differential equations have a common solution.** *The expansion of V for various laws of force.*]

[**40**.] We have not, in the present series of investigations, attempted to integrate directly the system of the 2 partial differential equations for elliptic motion which may for abbreviation be written thus (omitting the lower index in $V_{\prime}$):

$$(917)\ldots \qquad \left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2=\frac{2}{r}-\frac{1}{a}=2F(r),$$

$$(918)\ldots \qquad \left(\frac{\delta V}{\delta x_0}\right)^2+\left(\frac{\delta V}{\delta y_0}\right)^2+\left(\frac{\delta V}{\delta z_0}\right)^2=\frac{2}{r_0}-\frac{1}{a}=2F(r_0),$$

F being such that its derived function $F'(r)$ represents the law of repulsive force.

Since V is evidently a function of r, r_0, χ, a, and

$$(919)\ldots \quad \left(\frac{\delta r}{\delta x}\right)^2+\left(\frac{\delta r}{\delta y}\right)^2+\left(\frac{\delta r}{\delta z}\right)^2=1; \qquad (920)\ldots \quad \left(\frac{\delta r_0}{\delta x_0}\right)^2+\left(\frac{\delta r_0}{\delta y_0}\right)^2+\left(\frac{\delta r_0}{\delta z_0}\right)^2=1;$$

$$(921)\ldots \quad \left(\frac{\delta \chi}{\delta x}\right)^2+\left(\frac{\delta \chi}{\delta y}\right)^2+\left(\frac{\delta \chi}{\delta z}\right)^2=1; \qquad (922)\ldots \quad \left(\frac{\delta \chi}{\delta x_0}\right)^2+\left(\frac{\delta \chi}{\delta y_0}\right)^2+\left(\frac{\delta \chi}{\delta z_0}\right)^2=1;$$

$$(923). \qquad 2\left(\frac{\delta\chi}{\delta x}\frac{\delta r}{\delta x}+\frac{\delta\chi}{\delta y}\frac{\delta r}{\delta y}+\frac{\delta\chi}{\delta z}\frac{\delta r}{\delta z}\right)=\frac{2x(x-x_0)+\ldots}{r\chi}=\frac{r^2+\chi^2-r_0^2}{r\chi};$$

$$(924). \qquad 2\left(\frac{\delta\chi}{\delta x_0}\frac{\delta r_0}{\delta x_0}+\frac{\delta\chi}{\delta y_0}\frac{\delta r_0}{\delta y_0}+\frac{\delta\chi}{\delta z_0}\frac{\delta r_0}{\delta z_0}\right)=\frac{2x_0(x_0-x)+\ldots}{r_0\chi}=\frac{r_0^2+\chi^2-r^2}{r_0\chi};$$

while r is independent of x_0, y_0, z_0 and r_0 of x, y, z: we have the following transformed equations

$$(925)\ldots \qquad \left(\frac{\delta V}{\delta r}\right)^2+\left(\frac{\delta V}{\delta \chi}\right)^2+\frac{r^2+\chi^2-r_0^2}{r\chi}\frac{\delta V}{\delta r}\frac{\delta V}{\delta \chi}=2F(r);$$

$$(926)\ldots \qquad \left(\frac{\delta V}{\delta r_0}\right)^2+\left(\frac{\delta V}{\delta \chi}\right)^2+\frac{r_0^2+\chi^2-r^2}{r_0\chi}\frac{\delta V}{\delta r_0}\frac{\delta V}{\delta \chi}=2F(r_0).$$

Now in general if t be a function of 3 independent variables x, y, z and if we represent its total variation by the formula†

$$(927)\ldots \qquad \delta t=p\,\delta x+q\,\delta y+r\,\delta z,$$

& suppose we are required to integrate a system of 2 partial differential equations of the 1st order, not containing t itself, which may ∵ be put under the form

$$(928)\ldots \quad p=\phi(r,x,y,z), \qquad (929)\ldots \quad q=\psi(r,x,y,z),$$

ϕ & ψ being known functions. We have the 3 conditions

$$(930)\ldots \qquad \left(\frac{\delta^2 t}{\delta x\,\delta y}=\right)\frac{\delta\phi}{\delta r}\frac{\delta r}{\delta y}+\frac{\delta\phi}{\delta y}=\frac{\delta\psi}{\delta r}\frac{\delta r}{\delta x}+\frac{\delta\psi}{\delta x};$$

$$(931)\ldots \qquad \left(\frac{\delta^2 t}{\delta x\,\delta z}=\right)\frac{\delta\phi}{\delta r}\frac{\delta r}{\delta z}+\frac{\delta\phi}{\delta z}=\frac{\delta r}{\delta x};$$

$$(932)\ldots \qquad \left(\frac{\delta^2 t}{\delta y\,\delta z}=\right)\frac{\delta\psi}{\delta r}\frac{\delta r}{\delta z}+\frac{\delta\psi}{\delta z}=\frac{\delta r}{\delta y};$$

$\frac{\delta t}{\delta z}=r$ being considered as an unknown function of x, y, z.

* [See Appendix, note 2, p. 614.]

† [x, y, z, t are now any variables and are not used in the foregoing sense.]

Eliminating $\frac{\delta r}{\delta x}, \frac{\delta r}{\delta y}$ between these 3 conditions we get

$$(933)\ldots \qquad \frac{\delta\phi}{\delta r}\frac{\delta\psi}{\delta z}+\frac{\delta\phi}{\delta y}=\frac{\delta\psi}{\delta r}\frac{\delta\phi}{\delta z}+\frac{\delta\psi}{\delta x};$$

unless ∵ this condition is satisfied the proposed system of partial differential equations cannot be satisfied by any function t of 3 *independent* variables x, y, z: though it may perhaps be satisfied in other ways, like the total equations called unintegrable between x, y, z. When the condition (933) is satisfied, the integration of the proposed system is reduced to the integration of the system (931), (932), which is in some respects simpler than the proposed, as being only of the 1st degree: it has however the disadvantage of containing the sought variable r itself as well as its partial differential coefficients. And if we can thus determine r as a function of x, y, z we shall have p, q by the proposed system, & thus shall only have to integrate the total differential equation now necessarily integrable, (927).

To apply these general remarks to the present question, let the proposed system be rewritten as follows:

$$(934)\ldots \quad p^2+r^2+\frac{x^2-y^2+z^2}{xz}pr=2F(x). \qquad (935)\ldots \quad q^2+r^2+\frac{y^2-x^2+z^2}{yz}qr=2F(y).$$

These give, if we put

$$(936)\ldots \quad 2p+\frac{r}{xz}(x^2-y^2+z^2)=\lambda \quad \text{and} \quad (937)\ldots \quad 2q+\frac{r}{yz}(y^2-x^2+z^2)=\mu,$$

$$(938)\ldots \quad \frac{\delta\phi}{\delta r}=\frac{\delta p}{\delta r}=-\frac{2r}{\lambda}-\frac{x^2-y^2+z^2}{xz\lambda}p; \qquad (939)\ldots \quad \frac{\delta\psi}{\delta r}=-\frac{2r}{\mu}-\frac{y^2-x^2+z^2}{yz\mu}q;$$

$$(940)\ldots \quad \frac{\delta\phi}{\delta y}=\frac{2ypr}{xz\lambda}; \qquad (941)\ldots \quad \frac{\delta\psi}{\delta x}=\frac{2xqr}{yz\mu};$$

$$(942)\ldots \quad \frac{\delta\phi}{\delta z}=\frac{x^2-y^2-z^2}{xz^2\lambda}pr; \qquad (943)\ldots \quad \frac{\delta\psi}{\delta z}=\frac{y^2-x^2-z^2}{yz^2\mu}qr;$$

∵ the condition (933), when multiplied by $\frac{xyz^3\lambda\mu}{r}$, becomes

$$(944)\ldots \quad (x^2+z^2-y^2)\,q\,\{(x^2+z^2-y^2)\,p+2xzr\}+2yzp\,\{2qyz+r\,(y^2+z^2-x^2)\}$$
$$=(y^2+z^2-x^2)\,p\,\{(y^2+z^2-x^2)\,q+2yzr\}+2xzq\,\{2pxz+r\,(x^2+z^2-y^2)\};$$

which is, in fact, identical.

It is ∵ possible in the present case to satisfy the system of the 2 proposed partial differential equations by a function of 3 independent variables, though it may be hard to discover its form. To assist in this discovery, we have by (931), (932) the expression

$$(945)\ldots \quad \delta r=\frac{\delta\phi}{\delta z}\,\delta x+\frac{\delta\psi}{\delta z}\,\delta y+\frac{\delta r}{\delta z}\left(\delta z+\frac{\delta\phi}{\delta r}\,\delta x+\frac{\delta\psi}{\delta r}\,\delta y\right);$$

we are led to try whether we can satisfy simultaneously the 2 differential equations

$$(946)\ldots \quad \delta z+\frac{\delta\phi}{\delta r}\,\delta x+\frac{\delta\psi}{\delta r}\,\delta y=0$$

$$\&\ (947)\ldots \quad \delta r=\frac{\delta\phi}{\delta z}\,\delta x+\frac{\delta\psi}{\delta z}\,\delta y.$$

Or we may employ the equation (927), observing that

$$(948)\ldots\qquad \lambda^2-8F(x)+\frac{r^2}{x^2z^2}(x+y+z)(x+y-z)(x-y+z)(-x+y+z)=0;$$

$$(949)\ldots\qquad \mu^2-8F(y)+\frac{r^2}{y^2z^2}(\qquad\qquad\qquad\qquad\qquad\qquad)=0;$$

& ∵ that

$$(950)\ldots\quad \left(\frac{\lambda}{2}\right)^2=2F(x)-r^2(\sin\widehat{xz})^2;\qquad (951)\ldots\quad \left(\frac{\mu}{2}\right)^2=2F(y)-r^2(\sin\widehat{yz})^2.$$

For thus we find

$$(952)\ldots\qquad p=-r\cos\widehat{xz}\pm\sqrt{2F(x)-r^2(\sin\widehat{xz})^2};$$

$$(953)\ldots\qquad q=-r\cos\widehat{yz}\pm\sqrt{2F(y)-r^2(\sin\widehat{yz})^2};$$

& ∵

$$(954)\ldots\quad \delta t=r(\delta z-\cos\widehat{xz}\,\delta x-\cos\widehat{yz}\,\delta y)\pm\sqrt{2F(x)-r^2(\sin\widehat{xz})^2}\,\delta x\pm\sqrt{2F(y)-r^2(\sin\widehat{yz})^2}\,\delta y.$$

Can we integrate the equation

$$(955)\ldots\qquad \delta z=\cos\widehat{xz}\,\delta x+\cos\widehat{yz}\,\delta y=\frac{x^2+z^2-y^2}{2xz}\,\delta x+\frac{y^2+z^2-x^2}{2yz}\,\delta y?$$

We can, putting

$$(956)\ldots\qquad \widehat{xy}=\text{const.},$$

that is,

$$(957)\ldots\qquad \frac{x^2+y^2-z^2}{xy}=\text{const.}=2\cos\widehat{xy};$$

& thus we find that we have

$$(958)\ldots\qquad \delta z-\cos\widehat{xz}\,\delta x-\cos\widehat{yz}\,\delta y=\frac{xy\sin\widehat{xy}}{z}\,\delta\widehat{xy}=\frac{xy\sin\widehat{xy}\,\delta\widehat{xy}}{\sqrt{x^2+y^2-2xy\cos\widehat{xy}}};$$

& thus we might transform (954) into an equation between $t, x, y, \widehat{xy}$. But it does not seem that this would have any utility.

When will t be a function of $z, x+y$? In this case $r=\frac{\delta t}{\delta z}$ will also be such a function; and we shall have

$$(959)\ldots\qquad \frac{\delta\phi}{\delta z}+\frac{\delta\phi}{\delta r}\frac{\delta r}{\delta z}=\frac{\delta\psi}{\delta z}+\frac{\delta\psi}{\delta r}\frac{\delta r}{\delta z},$$

that is, multiplying by $-xyz^3\lambda\mu$,

$$(960)\ldots\quad \left\{(y^2+z^2-x^2)pr+(x^2+z^2-y^2)pz\frac{\delta r}{\delta z}+2xz^2r\frac{\delta r}{\delta z}\right\}\{(y^2+z^2-x^2)r+2yzq\}$$
$$=\{(x^2+z^2-y^2)qr+\qquad\qquad\qquad\qquad\qquad\qquad\}\{(x^2+z^2-y^2)r+2xzp\};$$

in which the part independent of $\frac{\delta r}{\delta z}$ gives

$$(961)\ldots\quad \{(y^2+z^2-x^2)^2p-(x^2+z^2-y^2)^2q\}r=2pqz\{x(x^2+z^2-y^2)-y(y^2+z^2-x^2)\}.$$

Make

$$(962)\ldots\quad \delta t=p'\delta x'+q'\delta y'+r'\delta z';\qquad (963)\ldots\quad x'=x+y+z;$$

$$(964)\ldots\quad y'=x+y-z;\qquad (965)\ldots\quad z'=x-y.$$

Then

(966)... $p = p' + q' + r'$; (967)... $q = p' + q' - r'$.

(968)... $r = p' - q'$; (969)... $x = \dfrac{x' + y' + 2z'}{4}$;

(970)... $y = \dfrac{x' + y' - 2z'}{4}$; (971)... $z = \dfrac{x' - y'}{2}$; (972)... $x^2 - y^2 = \dfrac{z'(x' + y')}{2}$;

$$\because \quad (973)\ldots \quad p'^2 + q'^2 + r'(p' + q') + \frac{r'^2}{2} + \frac{(x' - y')^2 + 2z'(x' + y')}{(x' - y')^2 + 2z'(x' - y')}\{p'^2 - q'^2 + r'(p' - q')\} = F\left(\frac{x' + y' + 2z'}{4}\right);$$

$$(974)\ldots \quad p'^2 + q'^2 - r'(p' + q') + \frac{r'^2}{2} + \frac{(x' - y')^2 - 2z'(x' + y')}{(x' - y')^2 - 2z'(x' - y')}\{p'^2 - q'^2 - r'(p' - q')\} = F\left(\frac{x' + y' - 2z'}{4}\right);$$

$$(975)\ldots \quad \left(p'^2 + q'^2 + \frac{r'^2}{2}\right)\{(x' - y')^2 \pm 2z'(x' - y')\} \pm r'(p' + q')(x' - y')^2 + 2r'z'(p' + q')(x' - y') + \{(x' - y')^2 \pm 2z'(x' + y')\}\{p'^2 - q'^2 \pm r'(p' - q')\} = \{(x' - y')^2 \pm 2z'(x' - y')\} F\left(\frac{x' + y' \pm 2z'}{4}\right);$$

$$\because \quad (976)\ldots \quad z'(x' - y')\left\{2p'^2 + 2q'^2 + r'^2 - F\left(\frac{x' + y' + 2z'}{4}\right) - F\left(\frac{x' + y' - 2z'}{4}\right)\right\} + 2r'p'(x' - y')^2 + 2z'(x' + y')(p'^2 - q'^2) = 0.$$

Let

$$(977)\ldots \quad \frac{x^2 + y^2 - z^2}{xy} = u = 2\cos\widehat{xy};$$

then

(978)... $\dfrac{\delta u}{\delta z} = -\dfrac{2z}{xy}$, (979)... $\dfrac{\delta u}{\delta x} = \dfrac{x^2 + z^2 - y^2}{x^2 y}$, (980)... $\dfrac{\delta u}{\delta y} = \dfrac{y^2 + z^2 - x^2}{y^2 x}$;

∵ the system (934), (935) becomes

(981)... $p^2 + r^2 - \dfrac{2pr\dfrac{\delta u}{\delta x}}{\dfrac{\delta u}{\delta z}} = 2F(x)$; (982)... $q^2 + r^2 - \dfrac{2qr\dfrac{\delta u}{\delta y}}{\dfrac{\delta u}{\delta z}} = 2F(y)$.

Hence

(983)... $\dfrac{\delta\phi}{\delta r} = \dfrac{\delta p}{\delta r} = \dfrac{p\dfrac{\delta u}{\delta x} - r\dfrac{\delta u}{\delta z}}{p\dfrac{\delta u}{\delta z} - r\dfrac{\delta u}{\delta x}}$; (984)... $\dfrac{\delta\psi}{\delta r} = \dfrac{\delta q}{\delta r} = -\dfrac{q\dfrac{\delta u}{\delta y} - r\dfrac{\delta u}{\delta z}}{q\dfrac{\delta u}{\delta z} - r\dfrac{\delta u}{\delta y}}$;

& the equation (946) becomes

$$(985)\ldots \quad \left(p\frac{\delta u}{\delta x} - r\frac{\delta u}{\delta z}\right)\left(q\frac{\delta u}{\delta z} - r\frac{\delta u}{\delta y}\right)\delta x + \left(q\frac{\delta u}{\delta y} - r\frac{\delta u}{\delta z}\right)\left(p\frac{\delta u}{\delta z} - r\frac{\delta u}{\delta x}\right)\delta y + \left(p\frac{\delta u}{\delta z} - r\frac{\delta u}{\delta x}\right)\left(q\frac{\delta u}{\delta z} - r\frac{\delta u}{\delta y}\right)\delta z = 0.$$

Let t be supposed developable in a converging series according to positive integral powers of z,

(986)... $$t = t_1 z + t_2 z^2 + t_3 z^3 + \&c.,$$

that is,

(987)... $$V = t_1 \chi + t_2 \chi^2 + t_3 \chi^3 + \&c.$$

Then

(988)... $$r = \frac{\delta t}{\delta z} = t_1 + 2t_2 z + 3t_3 z^2 + \&c.;$$ (989)... $$p = \frac{\delta t}{\delta x} = z\frac{\delta t_1}{\delta x} + z^2 \frac{\delta t_2}{\delta x} + \&c.;$$

(990)... $$q = \frac{\delta t}{\delta y} = z\frac{\delta t_1}{\delta y} + z^2 \frac{\delta t_2}{\delta y} + \&c.;$$

and neglecting z^3, (934) becomes

(991)... $$t_1^2 + 4t_1 t_2 z + (4t_2^2 + 6t_1 t_3) z^2 + \frac{x^2 - y^2 + z^2}{x}\left(\frac{\delta t_1}{\delta x} + z\frac{\delta t_2}{\delta x} + z^2 \frac{\delta t_3}{\delta x}\right)(t_1 + 2t_2 z + 3t_3 z^2) + z^2\left(\frac{\delta t_1}{\delta x}\right)^2 = 2F(x);$$

or, neglecting z,

(992)... $$t_1^2 + (x^2 - y^2)\frac{t_1 \delta t_1}{x \delta x} = 2F(x);$$

& similarly

(993)... $$t_1^2 + (y^2 - x^2)\frac{t_1 \delta t_1}{y \delta y} = 2F(y);$$

∵ (994)... $$\delta . (x^2 - y^2) t_1^2 = 4F(x)\, x\delta x - 4F(y)\, y\delta y,$$ ∵ (995)... $$t_1^2 = \frac{4\int_y^x F(x)\, x dx}{x^2 - y^2}$$

that is, nearly,

(996)... $$V^2 = \frac{4\chi^2 \int_{r_0}^r F(r)\, r dr}{r^2 - r_0^2}$$

For example, in the solar system,

(997)... $$F(r) = \frac{1}{r} - \frac{1}{2a},$$ ∵ (998)... $$\int_{r_0}^r F(r)\, r dr = (r - r_0) - \frac{r^2 - r_0^2}{4a};$$

∵ (999)... $$\frac{\int_{r_0}^r F(r)\, r dr}{r^2 - r_0^2} = \frac{1}{r + r_0} - \frac{1}{4a},$$

&, nearly,

(1000)... $$V^2 = \frac{4\chi^2}{r + r_0} - \frac{\chi^2}{a}.$$

If the force be directly as the distance,

(1001)... $$F(r) = h - \frac{r^2}{2};$$ ∵ (1002)... $$\int_{r_0}^r F(r)\, r dr = h(r^2 - r_0^2) - \frac{r^4 - r_0^4}{8},$$

∵ (1003)... $$V^2 = 2h\chi^2 - \frac{\chi^2}{2}(r^2 + r_0^2), \text{ nearly.}$$

Proceeding to the coefficient of z in (991), we have

$$(1004)\ldots \qquad 0 = 4t_1^2 t_2 + (x^2 - y^2)\frac{t_1^2 \delta t_2}{x\,\delta x} + 2t_2 (x^2 - y^2)\frac{t_1\,\delta t_1}{x\,\delta x};$$

$$\because \ (1005)\ldots \qquad 0 = 4\frac{\delta t_2}{x\,\delta x}\int_y^x F(x)\,x\,dx + t_2\{2t_1^2 + 4F(x)\};$$

in like manner, interchanging x & y,

$$(1006)\ldots \qquad 0 = -4t_1^2 t_2 + (x^2 - y^2)\frac{t_1^2\,\delta t_2}{y\,\delta y} + 2t_2 (x^2 - y^2)\frac{t_1\,\delta t_1}{y\,\delta y};$$

$\because$ multiplying (1004) (1006) by $x\,\delta x$, $y\,\delta y$ & adding,

$$(1007)\ldots \qquad 0 = (x^2 - y^2)\,t_1^2\,\delta t_2 + 2(x^2 - y^2)\,t_2 t_1\,\delta t_1 + 4t_1^2 t_2 (x\,\delta x - y\,\delta y),$$

that is, integrating,

$$(1008)\ldots \qquad (x^2 - y^2)^2 t_1^2 t_2 = \text{const.} = 0,$$

since we suppose that t_1, t_2 do not become infinite when the chord z or χ vanishes, & $\because$ when $x^2 = y^2$. Hence

$$(1009)\ldots \qquad t_2 = 0,$$

& indeed it might have been assumed as evident that the development of t or V contains only odd powers of the chord.

Making this last assumption, we are to try whether we can satisfy the 2 partial differential equations (934), (935) by a development

$$(1010)\ldots \qquad t = t_1 z + t_3 z^3 + t_5 z^5 + \&c.;$$

that is

$$(1011)\ldots \qquad V = t_1 \chi + t_3 \chi^3 + t_5 \chi^5 + \&c.$$

Putting for abbreviation

$$(1012)\ldots \quad z^2 = z_,; \qquad (1013)\ldots \quad x^2 = x_,; \qquad (1014)\ldots \quad y^2 = y_,; \qquad (1015)\ldots \quad \frac{V}{\chi} = \frac{t}{z} = f;$$

we have

$$(1016)\ldots \quad t = zf; \qquad\qquad (1017)\ldots \quad r = \frac{\delta t}{\delta z} = f + 2z_,\frac{\delta f}{\delta z_,};$$

$$(1018)\ldots \quad p = \frac{\delta t}{\delta x} = 2xz\frac{\delta f}{\delta x_,}; \qquad\qquad (1019)\ldots \quad q = 2yz\frac{\delta t}{\delta y_,};$$

and the partial differential equations become

$$(1020)\ldots \quad \left(f + 2z_,\frac{\delta f}{\delta z_,}\right)^2 + 2\left(f + 2z_,\frac{\delta f}{\delta z_,}\right)\frac{\delta f}{\delta x_,}(x_, - y_, + z_,) + 4x_, z_,\left(\frac{\delta f}{\delta x_,}\right)^2 = 2F(\sqrt{x_,});$$

$$(1021)\ldots \quad \left(\qquad\qquad\right)^2 + 2\left(\qquad\qquad\right)\frac{\delta f}{\delta y_,}(y_, - x_, + z_,) + 4y_, z_,\left(\frac{\delta f}{\delta y_,}\right)^2 = 2F(\sqrt{y_,}).$$

We have also the assumed development with which we are to try to satisfy these equations, namely

(1022)... $$f = t_1 + t_3 z_{,} + t_5 z_{,}^2 + \&c.,$$

or more simply

(1023)... $$f = f_0 + f_1 z_{,} + f_2 z_{,}^2 + \&c.$$

Making $z_{,} = 0$, we find

(1024)... $f_0^2 + 2f_0 \frac{\delta f_0}{\delta x_{,}}(x_{,} - y_{,}) = 2F(\sqrt{x_{,}})$; (1025)... $f_0^2 + 2f_0 \frac{\delta f_0}{\delta y_{,}}(y_{,} - x_{,}) = 2F(\sqrt{y_{,}})$;

(1026)... $$(x_{,} - y_{,})\,\delta f_0^2 + f_0^2 \,.\, \delta(x_{,} - y_{,}) = 2F(\sqrt{x_{,}})\,\delta x_{,} - 2F(\sqrt{y_{,}})\,\delta y_{,},$$

∵ since f_0 is finite when $x_{,} = y_{,}$,

(1027)... $$f_0^2 = \frac{2\int_{y_{,}}^{x_{,}} F(\sqrt{x_{,}})\,dx_{,}}{x_{,} - y_{,}}$$

agreeing with (995).

Proceeding to $z_{,}$ but continuing to neglect $z_{,}^2$, we may put

(1028)... $f + 2z_{,} \frac{\delta f}{\delta z_{,}} = f_0 + 3f_1 z_{,}$; (1029)... $\frac{\delta f}{\delta x_{,}} = \frac{\delta f_0}{\delta x_{,}} + z_{,} \frac{\delta f_1}{\delta x_{,}}$;

∵ (1030)... $$0 = 3f_0 f_1 + 3f_1 \frac{\delta f_0}{\delta x_{,}}(x_{,} - y_{,}) + f_0 \frac{\delta f_1}{\delta x_{,}}(x_{,} - y_{,}) + f_0 \frac{\delta f_0}{\delta x_{,}} + 2x_{,}\left(\frac{\delta f_0}{\delta x_{,}}\right)^2;$$

& similarly

(1031)... $$0 = 3f_0 f_1 + 3f_1 \frac{\delta f_0}{\delta y_{,}}(y_{,} - x_{,}) + f_0 \frac{\delta f_1}{\delta y_{,}}(y_{,} - x_{,}) + f_0 \frac{\delta f_0}{\delta y_{,}} + 2y_{,}\left(\frac{\delta f_0}{\delta y_{,}}\right)^2;$$

∵ (1032)... $$\begin{aligned}0 = {} & f_0(x_{,} - y_{,})\,\delta f_1 + 3f_1(x_{,} - y_{,})\,\delta f_0 + 3f_0 f_1\,\delta(x_{,} - y_{,}) \\ & + f_0\left(\frac{\delta f_0}{\delta x_{,}}\delta x_{,} - \frac{\delta f_0}{\delta y_{,}}\delta y_{,}\right) + 2\left(\frac{\delta f_0}{\delta x_{,}}\right)^2 x_{,}\,\delta x_{,} - 2\left(\frac{\delta f_0}{\delta y_{,}}\right)^2 y_{,}\,\delta y_{,};\end{aligned}$$

that is,

(1033)... $$\begin{aligned}\delta\,.\,f_0^3(x_{,} - y_{,})^3 f_1 = {} & -f_0^2(x_{,} - y_{,})^2\left(f_0 \frac{\delta f_0}{\delta x_{,}}\delta x_{,} - f_0 \frac{\delta f_0}{\delta y_{,}}\delta y_{,}\right) \\ & -2(x_{,} - y_{,})^2\left\{\left(f_0 \frac{\delta f_0}{\delta x_{,}}\right)^2 x_{,}\,\delta x_{,} - \left(f_0 \frac{\delta f_0}{\delta y_{,}}\right)^2 y_{,}\,\delta y_{,}\right\} \\ = {} & \Big[\int_{y_{,}}^{x_{,}} F(\sqrt{x_{,}})\,dx_{,}\,\{f_0^2(\delta x_{,} + \delta y_{,}) - 2F(\sqrt{x_{,}})\,\delta x_{,} - 2F(\sqrt{y_{,}})\,\delta y_{,}\} \\ & - \{\tfrac{1}{2}f_0^2 - F(\sqrt{x_{,}})\}^2\,\delta\,.\,x_{,}^2 + \{\tfrac{1}{2}f_0^2 - F(\sqrt{y_{,}})\}^2\,\delta\,.\,y_{,}^2\Big].\end{aligned}$$

To simplify this expression we have by (1024)

(1034)... $$-2(x_{,} - y_{,})^2\left(f_0 \frac{\delta f_0}{\delta x_{,}}\right)^2 x_{,}\,\delta x_{,} = (x_{,} - y_{,})\,f_0 \frac{\delta f_0}{\delta x_{,}}(x_{,} - y_{,} + y_{,})\{f_0^2 - 2F(\sqrt{x_{,}})\}\,\delta x_{,};$$

∵ (1035)... $$-2(x_{,}-y_{,})^2\left(f_0\frac{\delta f_0}{\delta x_{,}}\right)^2 x_{,}\,\delta x_{,} - f_0^2(x_{,}-y_{,})^2 f_0\frac{\delta f_0}{\delta x_{,}}\,\delta x_{,}$$

$$= y_{,}(x_{,}-y_{,})f_0^3\frac{\delta f_0}{\delta x_{,}}\,\delta x_{,} - 2(x_{,}-y_{,})f_0\frac{\delta f_0}{\delta x_{,}}F(\sqrt{x_{,}})\,x_{,}\,\delta x_{,}$$

$$=\{F(\sqrt{x_{,}})-\tfrac{1}{2}f_0^2\}\{y_{,}f_0^2\,\delta x_{,} - F(\sqrt{x_{,}})\,\delta\,.\,x_{,}^2\}$$

$$=-2\{\tfrac{1}{2}f_0^2-F(\sqrt{x_{,}})\}^2 x_{,}\,\delta x_{,} + 2\{\tfrac{1}{2}f_0^2-F(\sqrt{x_{,}})\}\left(\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}\right)\delta x_{,}$$

$$=-\left\{\frac{\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}}{x_{,}-y_{,}}-F(\sqrt{x_{,}})\right\}^2\delta\,.\,x_{,}^2 + 2\left\{\frac{\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}}{x_{,}-y_{,}}-F(\sqrt{x_{,}})\right\}.\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}\,.\,\delta x_{,}$$

$$=-\frac{x_{,}\,\delta x_{,}}{2(x_{,}-y_{,})^2}[I'-2y_{,}\{F(\sqrt{x_{,}})-F(\sqrt{y_{,}})\}]^2-\frac{I'-2y_{,}\{F(\sqrt{x_{,}})-F(\sqrt{y_{,}})\}}{x_{,}-y_{,}}\left(\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}\right)\delta x_{,};$$

observing that

(1036)... $$\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}=x_{,}F(\sqrt{x_{,}})-y_{,}F(\sqrt{y_{,}})-\tfrac{1}{2}I',$$

if we put for abbreviation

(1037)... $$I'=\int_{y_{,}}^{x_{,}}F'(\sqrt{x_{,}})\sqrt{x_{,}}\,dx_{,};$$

∵ changing $x_{,}$, $y_{,}$ to $y_{,}$, $x_{,}$ & subtracting the result from the expression (1035), we find

(1038)... $$\delta\,.\,f_0^3(x_{,}-y_{,})^3 f_1 = A_{x_{,},y_{,}} - A_{y_{,},x_{,}},$$

in which

(1039)... $$A_{x_{,},y_{,}}=-\frac{2I^2y_{,}\,\delta x_{,}}{(x_{,}-y_{,})^2}+\frac{2I\,.\,F(\sqrt{x_{,}})\,.\,(x_{,}+y_{,})\,\delta x_{,}}{x_{,}-y_{,}}-2\{F(\sqrt{x_{,}})\}^2x_{,}\,\delta x_{,};$$

(1040)... $$A_{y_{,},x_{,}}=-\frac{2I^2x_{,}\,\delta y_{,}}{(x_{,}-y_{,})^2}+\frac{2I\,.\,F(\sqrt{y_{,}})\,.\,(x_{,}+y_{,})\,\delta y_{,}}{x_{,}-y_{,}}-2\{F(\sqrt{y_{,}})\}^2y_{,}\,\delta y_{,};$$

(1041)... $$I=\int_{y_{,}}^{x_{,}}F(\sqrt{x_{,}})\,dx_{,}\,.$$

Hence

(1042)... $$\tfrac{1}{2}\delta\,.\,f_0^3(x_{,}-y_{,})^3f_1=\frac{I^2(x_{,}\,\delta y_{,}-y_{,}\,\delta x_{,})}{(x_{,}-y_{,})^2}$$
$$+\frac{I(x_{,}+y_{,})}{x_{,}-y_{,}}\{F(\sqrt{x_{,}})\,\delta x_{,}-F(\sqrt{y_{,}})\,\delta y_{,}\}+\{F(\sqrt{y_{,}})\}^2y_{,}\,\delta y_{,}-\{F(\sqrt{x_{,}})\}^2x_{,}\,\delta x_{,};$$

that is,

(1043)... $$\delta\left\{\tfrac{1}{2}f_0^3(x_{,}-y_{,})^3f_1+\int_{y_{,}}^{x_{,}}\{F(\sqrt{x_{,}})\}^2x_{,}\,dx_{,}\right\}$$
$$=\frac{I(x_{,}+y_{,})}{x_{,}-y_{,}}\left\{\frac{I(x_{,}\,\delta y_{,}-y_{,}\,\delta x_{,})}{x_{,}^2-y_{,}^2}+F(\sqrt{x_{,}})\,\delta x_{,}-F(\sqrt{y_{,}})\,\delta y_{,}\right\}.$$

In order that this should be an exact variation of a function of two independent variables $x_{,}$, $y_{,}$, we must have

$$(1044)\ldots\quad \frac{\delta}{\delta y_{,}}\left\{\frac{I\frac{\delta I}{\delta x_{,}}(x_{,}+y_{,})}{x_{,}-y_{,}}-\frac{I^2 y_{,}}{(x_{,}-y_{,})^2}\right\}=\frac{\delta}{\delta x}\left\{\frac{I\frac{\delta I}{\delta y_{,}}(x_{,}+y_{,})}{x_{,}-y_{,}}+\frac{I^2 x_{,}}{(x_{,}-y_{,})^2}\right\},$$

observing that

$$(1045)\ldots\quad \frac{\delta I}{\delta x_{,}}=F(\sqrt{x_{,}});\qquad (1046)\ldots\quad \frac{\delta I}{\delta y_{,}}=-F(\sqrt{y_{,}}).$$

Now

$$(1047)\ldots\quad -\frac{y_{,}}{(x_{,}-y_{,})^2}=-\frac{x_{,}}{(x_{,}-y_{,})^2}+\frac{1}{x_{,}-y_{,}};\qquad (1048)\ldots\quad \frac{x_{,}}{(x_{,}-y_{,})^2}=\frac{y_{,}}{(x_{,}-y_{,})^2}+\frac{1}{x_{,}-y_{,}};$$

$$\because\ (1049)\ldots\quad \frac{\delta}{\delta y_{,}}\left(-\frac{y_{,}}{(x_{,}-y_{,})^2}\right)=-\frac{2x_{,}}{(x_{,}-y_{,})^3}+\frac{1}{(x_{,}-y_{,})^2}\qquad -\frac{x_{,}+y_{,}}{(x_{,}-y_{,})^3}$$

$$=-\frac{2y_{,}}{(x_{,}-y_{,})^3}\qquad \frac{1}{(x_{,}-y_{,})^2}=\frac{\delta}{\delta x_{,}}\,\frac{x_{,}}{(x_{,}-y_{,})^2}$$

also

$$(1050)\ldots\quad \frac{\delta}{\delta y_{,}}\,\frac{x_{,}+y_{,}}{x_{,}-y_{,}}=\frac{2x_{,}}{(x_{,}-y_{,})^2};\qquad (1051)\ldots\quad \frac{\delta}{\delta x_{,}}\,\frac{x_{,}+y_{,}}{x_{,}-y_{,}}=-\frac{2y_{,}}{(x_{,}-y_{,})^2}.$$

$\therefore$ (1044) is evidently true. And since

$$(1052)\ldots\quad \frac{I^2(x_{,}\delta y_{,}-y_{,}\delta x_{,})}{(x_{,}-y_{,})^2}+\frac{I\delta I(x_{,}+y_{,})}{x_{,}-y_{,}}=\delta\,.\,\frac{I^2(x_{,}+y_{,})}{2(x_{,}-y_{,})},$$

we have, as the integral (1043), not requiring any constant to be added, the following

$$(1053)\ldots\quad f_0^3(x_{,}-y_{,})^3 f_1=\frac{I^2(x_{,}+y_{,})}{x_{,}-y_{,}}-2\int_{y_{,}}^{x_{,}}\{F(\sqrt{x_{,}})\}^2 x_{,}dx_{,};$$

that is, because

$$(1054)\ldots\quad f_0^2=\frac{2I}{x_{,}-y_{,}},$$

$$(1055)\ldots\quad I(x_{,}-y_{,})^3\,.\,2f_0f_1=I^2(x_{,}+y_{,})-2(x_{,}-y_{,})\int_{y_{,}}^{x_{,}}\left(\frac{\delta I}{\delta x_{,}}\right)^2 x_{,}dx_{,};$$

on putting

$$(1056)\ldots\quad I=i_{x_{,}}-i_{y_{,}},$$

we have

$$(1057)\ldots\quad I(x_{,}-y_{,})^2 f_0f_1=\frac{1}{2}\,\frac{x_{,}+y_{,}}{x_{,}-y_{,}}\,I^2-\int_{y_{,}}^{x_{,}}i_t'^2\,t\,dt,$$

$$\because\ (1058)\ldots\quad \delta\,.\,I(x_{,}-y_{,})^2 f_0f_1=\frac{x_{,}+y_{,}}{x_{,}-y_{,}}\,I\delta I+\frac{I^2(x_{,}\delta y_{,}-y_{,}\delta x_{,})}{(x_{,}-y_{,})^2}-i_{x_{,}}'^2x_{,}\delta x_{,}+i_{y_{,}}'^2y_{,}\delta y_{,}.$$

Hence also

(1059)... $$2f_0f_1=\frac{(x_,+y_,)\int_{y_,}^{x_,}i_t'dt}{(x_,-y_,)^3}-\frac{2\int_{y_,}^{x_,}i_t'^2t\,dt}{(x_,-y_,)^2\int_{y_,}^{x_,}i_t'dt},$$

because

(1060)... $$I=\int_{y_,}^{x_,}i_t'dt.$$

Also

(1061)... $$i_t'=F(\sqrt{t}).$$

Hence in the law of the inverse square, by (997),

(1062)... $$i_t'=\frac{1}{\sqrt{t}}-\frac{1}{2a};$$

∵ (1063)... $$\int_{y_,}^{x_,}i_t'dt=2\sqrt{x_,}-2\sqrt{y_,}-\frac{x_,-y_,}{2a}=(r-r_0)\left(2-\frac{r+r_0}{2a}\right);$$

(1064)... $$i_t'^2t=1-\frac{\sqrt{t}}{a}+\frac{t}{4a^2},$$

∵ (1065)... $$\int_{y_,}^{x_,}i_t'^2t\,dt=(x_,-y_,)-\frac{2}{3a}(x_,^{\frac{3}{2}}-y_,^{\frac{3}{2}})+\frac{x_,^2-y_,^2}{8a^2}$$
$$=(r-r_0)\left\{(r+r_0)\left(1+\frac{r^2+r_0^2}{8a^2}\right)-\frac{2}{3a}(r^2+rr_0+r_0^2)\right\};$$

∵ (1066)... $$2(r+r_0)^3(r-r_0)^2f_0f_1\left(2-\frac{r+r_0}{2a}\right)$$
$$=(r^2+r_0^2)\left(2-\frac{r+r_0}{2a}\right)^2-2(r+r_0)^2\left(1+\frac{r^2+r_0^2}{8a^2}\right)+\frac{4}{3a}(r+r_0)(r^2+rr_0+r_0^2)$$
$$=2(r^2+r_0^2)-4rr_0-\frac{2(r+r_0)}{a}\left\{\frac{r^2+r_0^2}{3}-\frac{2rr_0}{3}\right\}=2(r-r_0)^2\left(1-\frac{r+r_0}{3a}\right);$$

∵ (1067)... $$2f_0f_1=\frac{1-\frac{\rho}{3a}}{1-\frac{\rho}{4a}}\rho^{-3},$$

while

(1068)... $$f_0=\sqrt{\frac{4}{\rho}-\frac{1}{a}};$$

∵ (1069)... $$f_1=\frac{1}{4\rho^3}\frac{\frac{1}{\rho}-\frac{1}{3a}}{\left(\frac{1}{\rho}-\frac{1}{4a}\right)^{\frac{3}{2}}}=\text{coefficient of }\chi^3\text{ in }V.$$

As a verification, when $\frac{1}{a}=0$, we have thus for parabolic motion

(1070)... $$V=\frac{2\chi}{\sqrt{\rho}}+\frac{\chi^3}{4\rho^2\sqrt{\rho}}\text{ nearly;}$$

and in fact this would result also from the development of my old expression for parabolic action,

$$(1071)\ldots \qquad V = 2\sqrt{\rho+\chi} - 2\sqrt{\rho-\chi}.$$

In general the present result agrees with one obtained in some former investigations respecting elliptic action.

When the force is *directly* as the distance

$$(1072)\ldots \qquad i_t' = F(\sqrt{t}) = h - \frac{t}{2};$$

$$\because\ (1073)\ldots \qquad \frac{\int_{y_{,}}^{\prime} i_t'^2\, t\,dt}{x_{,} - y_{,}} = \tfrac{1}{2}h^2(x_{,}+y_{,}) - \frac{h}{3}(x_{,}^2 + x_{,}y_{,} + y_{,}^2) + \frac{1}{16}(x_{,}^3 + x_{,}^2 y_{,} + x_{,}y_{,}^2 + y_{,}^3);$$

$$(1074)\ldots \qquad \frac{I}{x_{,}-y_{,}} = h - \frac{x_{,}+y_{,}}{4};$$

$$\because\ (1075)\ldots \qquad \left(2h - \frac{x_{,}+y_{,}}{2}\right)(x_{,}-y_{,})^2 f_0 f_1 = (x_{,}+y_{,})\left(h - \frac{x_{,}+y_{,}}{4}\right)^2 - (x_{,}+y_{,})h^2$$

$$+\frac{2h}{3}(x_{,}^2 + x_{,}y_{,} + y_{,}^2) - \frac{1}{8}(x_{,}^3 + x_{,}^2 y_{,} + x_{,}y_{,}^2 + y_{,}^3)$$

$$= \frac{h}{6}(x_{,}-y_{,})^2 - \frac{1}{16}\{2(x_{,}^3 + x_{,}^2 y_{,} + x_{,}y_{,}^2 + y_{,}^3) - (x_{,}+y_{,})^3\} = \frac{(x_{,}-y_{,})^2}{2}\left\{\frac{h}{3} - \frac{x_{,}+y_{,}}{8}\right\},$$

$$\because\ (1076)\ldots \qquad 2f_0 f_1 = \frac{\dfrac{h}{3} - \dfrac{x_{,}+y_{,}}{8}}{2h - \dfrac{x_{,}+y_{,}}{2}};$$

& since

$$(1077)\ldots \qquad f_0 = \sqrt{2h - \frac{x_{,}+y_{,}}{2}},$$

we have

$$(1078)\ldots \qquad f_1 = \frac{\dfrac{h}{6} - \dfrac{x_{,}+y_{,}}{16}}{\left(2h - \dfrac{x_{,}+y_{,}}{2}\right)^2}.$$

In these 2 laws f_1 is a function of f_0; is it in other laws of force? If so, making both constant, we ought to have simultaneously

$$(1079)\ldots \qquad f_0^2\,\delta(x_{,}-y_{,}) = 2F(\sqrt{x_{,}})\,.\,\delta x_{,} - 2F(\sqrt{y_{,}})\,.\,\delta y_{,},$$

$$\&\ (1080)\ldots \qquad 6f_0^3 f_1(x_{,}-y_{,})^2(\delta x_{,} - \delta y_{,}) = f_0^4(x_{,}\delta x_{,} - y_{,}\delta y_{,}) - 4\{F(\sqrt{x_{,}})\}^2 x_{,}\delta x_{,} + 4\{F(\sqrt{y_{,}})\} y_{,}\delta y_{,};$$

which give

$$(1081)\ldots \qquad \delta x_{,} = \lambda\{f_0^2 - 2F(\sqrt{y_{,}})\}, \qquad (1082)\ldots \qquad \delta y_{,} = \lambda\{f_0^2 - 2F(\sqrt{x_{,}})\},$$

$$(1083)\ldots \qquad \delta x_{,} - \delta y_{,} = 2\lambda\{F(\sqrt{x_{,}}) - F(\sqrt{y_{,}})\},$$

$$(1084)\ldots \qquad \frac{12 f_0^3 f_1(x_{,}-y_{,})^2\{F(\sqrt{x_{,}}) - F(\sqrt{y_{,}})\}}{\{f_0^2 - 2F(\sqrt{x_{,}})\}\{f_0^2 - 2F(\sqrt{y_{,}})\}} = \{f_0^2 + 2F(\sqrt{x_{,}})\}x_{,} - \{f_0^2 + 2F(\sqrt{y_{,}})\}y_{,};$$

∵ (1085)... $\{\tfrac{3}{2}I^2(x,+y,)-3I''(x,-y,)\}\{F(\sqrt{x,})-F(\sqrt{y,})\}$

$$=\{I+x,F(\sqrt{x,})-y,F(\sqrt{y,})\}\{I-\overline{x,-y,}\,F(\sqrt{x,})\}\{I-\overline{x,-y,}\,F(\sqrt{y,})\};$$

in which

(1086)... $$'' = \int_{y,}^{x,}\{F(\sqrt{x,})\}^2 x,\,dx,:$$

while by (1041) $$I=\int_{y,}^{x,}F(\sqrt{x,})\,dx,.$$

Introducing r, r_0,

(1087)... $\{\tfrac{3}{2}I^2(r^2+r_0^2)-3I''(r^2-r_0^2)\}\{F(r)-F(r_0)\}$

$$=\{I+r^2F(r)-r_0^2F(r_0)\}\{I-\overline{r^2-r_0^2}\,F(r)\}\{I-\overline{r^2-r_0^2}\,F(r_0)\};$$

in which

(1088)... $I=2\int_{r_0}^{r}F(r)\,r\,dr;$ (1089)... $I''=2\int_{r_0}^{r}\{F(r)\}^2r^3\,dr;$

and $F(r)$ is such that its derived function $F'(r)$ represents the law of repulsive force,

(1090)... $$F'(r)=-r^{-n},$$

& ∵

(1091)... $$F(r)=h+\frac{r^{1-n}}{n-1}.$$

Hence

(1092)... $$I=h(r^2-r_0^2)+\frac{2(r^{3-n}-r_0^{3-n})}{(n-1)(3-n)};$$

(1093)... $$I+r^2F(r)-r_0^2F(r_0)=2h(r^2-r_0^2)+\frac{(5-n)(r^{3-n}-r_0^{3-n})}{(n-1)(3-n)};$$

(1094)... $$I-(r^2-r_0^2)F(r)=-\frac{r^{1-n}(r^2-r_0^2)}{n-1}+\frac{2(r^{3-n}-r_0^{3-n})}{(n-1)(3-n)};$$

(1095)... $$I-(r^2-r_0^2)F(r_0)=-\frac{r_0^{1-n}(r^2-r_0^2)}{n-1}+\frac{2(r^{3-n}-r_0^{3-n})}{(n-1)(3-n)};$$

(1096)... $$I''=\frac{h^2}{2}(r^4-r_0^4)+\frac{4h(r^{5-n}-r_0^{5-n})}{(n-1)(5-n)}+\frac{r^{6-2n}-r_0^{6-2n}}{(n-1)^2(3-n)};$$

& ∵ the condition is

(1097)... $$\frac{3(r^{1-n}-r_0^{1-n})}{2(n-1)}\Bigg\{h^2(r^2-r_0^2)^2(r^2+r_0^2)-h^2(r^4-r_0^4)(r^2-r_0^2)$$
$$+\frac{4h(r^4-r_0^4)(r^{3-n}-r_0^{3-n})}{(n-1)(3-n)}-\frac{8h(r^2-r_0^2)(r^{5-n}-r_0^{5-n})}{(n-1)(5-n)}$$
$$+\frac{4(r^2+r_0^2)(r^{3-n}-r_0^{3-n})^2}{(n-1)^2(3-n)^2}-\frac{2(r^2-r_0^2)(r^{6-2n}-r_0^{6-2n})}{(n-1)^2(3-n)}$$
$$=\left\{\frac{2h(r^2-r_0^2)}{(n-1)^2}+\frac{(5-n)(r^{3-n}-r_0^{3-n})}{(n-1)^3(3-n)}\right\}\left\{-r^{1-n}(r^2-r_0^2)+\frac{2(r^{3-n}-r_0^{3-n})}{3-n}\right\}$$
$$\times\left\{-r_0^{1-n}(r^2-r_0^2)+\frac{2(r^{3-n}-r_0^{3-n})}{3-n}\right\};$$

which since it is to be satisfied independently of h resolves itself into the 2 following conditions:

$$(1098)\ldots\quad 3\,(r^{1-n}-r_0^{1-n})\,\{(r^2+r_0^2)\,(r^{3-n}-r_0^{3-n})\,(5-n)-2\,(r^{5-n}-r_0^{5-n})\,(3-n)\}\,\frac{3-n}{5-n}$$
$$=\{2\,(r^{3-n}-r_0^{3-n})-(3-n)\,r^{1-n}\,(r^2-r_0^2)\}\,\{2\,(r^{3-n}-r_0^{3-n})-(3-n)\,r_0^{1-n}\,(r^2-r_0^2)\};$$

$$(1099)\ldots\quad =3\,\frac{3-n}{5-n}\,(r^{1-n}-r_0^{1-n})\,\{2\,(r^2+r_0^2)\,(r^{3-n}-r_0^{3-n})-(3-n)\,(r^2-r_0^2)\,(r^{3-n}+r_0^{3-n})\}.$$

These two conditions merge into one, which may be put under the form

$$(1100)\ldots\quad \frac{3\,(3-n)}{5-n}\,(r^{1-n}-r_0^{1-n})\,\{(n-1)\,(r^{5-n}-r_0^{5-n})+(5-n)\,r^2r_0^2\,(r^{1-n}-r_0^{1-n})\}$$
$$=-\{(n-1)\,r^{3-n}+r_0^2\,.\,\overline{3-n}\,.\,r^{1-n}-2r_0^{3-n}\}\,\{(n-1)\,r_0^{3-n}+r^2\,.\,\overline{3-n}\,.\,r_0^{1-n}-2r^{3-n}\}$$
$$=2\,(n-1)\,(r^{6-2n}+r_0^{6-2n})-r^{3-n}r_0^{3-n}\,\{(n-1)^2+(3-n)^2+4\}+2\,(3-n)\,r^2r_0^2\,(r^{2-2n}+r_0^{2-2n})$$
$$-(n-1)\,(3-n)\,r^{1-n}\,r_0^{1-n}\,(r^4+r_0^4).$$

When $n=2$, this becomes

$$(1101)\ldots\quad (r^{-1}-r_0^{-1})\,\{r^3-r_0^3+3rr_0\,(r_0-r)\}=2\,(r^2+r_0^2)-6rr_0+2\,(r^2+r_0^2)-(r^4+r_0^4)\,r^{-1}r_0^{-1},$$

that is,

$$(1102)\ldots\quad (r-r_0)^4=r^4+r_0^4-4rr_0\,(r^2+r_0^2)+6r^2r_0^2,$$

which is in fact identically true. And when $n=-1$ the condition becomes

$$(1103)\ldots\quad 2\,(r^2-r_0^2)\,\{-2\,(r^6-r_0^6)+6r^2r_0^2\,(r^2-r_0^2)\}=4\,(r^8+r_0^8)-24r^4r_0^4+16r^2r_0^2\,(r^4+r_0^4)$$
$$=-4\,(r^2-r_0^2)^4,$$

which also is identical.

In general

$$(1104)\ldots\quad (r^{1-n}-r_0^{1-n})\,(r^{5-n}-r_0^{5-n})=r^{6-2n}+r_0^{6-2n}-(r^4+r_0^4)\,r^{1-n}r_0^{1-n};$$

& $$(1105)\ldots\quad r^2r_0^2\,(r^{1-n}-r_0^{1-n})^2=r^2r_0^2\,(r^{2-2n}+r_0^{2-2n})-2r^{3-n}r_0^{3-n};$$

∵ the condition is

$$(1106)\ldots\quad 0=A_n\,(r^{6-2n}+r_0^{6-2n})+B_nr^{3-n}r_0^{3-n}+C_nr^2r_0^2\,(r^{2-2n}+r_0^{2-2n})+D_n\,(r^4+r_0^4)\,r^{1-n}r_0^{1-n},$$

in which

$$(1107)\ldots\quad A_n=(n-1)\,\{2\,(5-n)-3\,(3-n)\}=n^2-1;$$

$$(1108)\ldots\quad B_n=2\,(5-n)\,\{3\,(3-n)-n^2+4n-7\}=2\,(n+1)\,(n-2)\,(n-5);$$

$$(1109)\ldots\quad C_n=-(n-3)\,(n-5);\qquad (1110)\ldots\quad D_n=-(n-1)\,(n-2)\,(n-3);$$

the condition may ∵ be thus written

$$(1111)\ldots\quad 0=(r^{6-2n}+r_0^{6-2n})\,(n^2-1)+2\,(n+1)\,(n-2)\,(n-5)\,(rr_0)^{3-n}$$
$$-(n-3)\,(n-5)\,(rr_0)^2\,(r^{2-2n}+r_0^{2-2n})-(n-1)\,(n-2)\,(n-3)\,(rr_0)^{1-n}\,(r^4+r_0^4):$$

or more concisely

$$(1112)\ldots\quad 0=A_nA_n'+B_nB_n'+C_nC_n'+D_nD_n'.$$

When we make $n=2$, we find

$(1113)\ldots\ A_2=3;$ $(1114)\ldots\ B_2=0;$ $(1115)\ldots\ C_2=-3;$ $(1116)\ldots\ D_2=0;$

also

(1117)... $$A_2' = r^2 + r_0^2 = C_0';$$

∵ the condition is satisfied. Again make $n = -1$, we have

(1118)... $A_{-1} = 0$; (1119)... $B_{-1} = 0$; (1120)... $C_{-1} = -24$; (1121)... $D_{-1} = 24$;

also

(1122)... $$C_{-1}' = r^2 r_0^2 (r^4 + r_0^4) = D_{-1}';$$

∵ in this case also the condition is satisfied. Is it in any other case? If $n = 5$, then

(1123)... $A_n = A_5 = 24$; (1124)... $D_5 = -24$;

B_5 & C_5 vanish & we have

(1125)... $$A_5' = r^{-4} + r_0^{-4} = D_5';$$

∵ *the condition appeared to me satisfied in this case also, namely when the force varies inversely as the* 5*th* *power of the distance.* (f_1 however is not a function of f_0.) *However, we must remember that we have multiplied by* $n-5$. In this case

(1126)... $$\frac{2I}{r^2 - r_0^2} = 2h + \frac{1}{2r^2 r_0^2} = f_0^2,$$

(1127)... $$I'' = \frac{r^4 - r_0^4}{4}\left(h^2 + \frac{1}{16 r^4 r_0^4}\right) + h \log \frac{r}{r_0};$$

& since in general, by (1053),

(1128)... $$(r^2 - r_0^2)^3 f_0^3 f_1 = \frac{f_0^4}{4}(r^4 - r_0^4) - 2I'',$$

we have here

(1129)... $$\frac{(r^2 - r_0^2)^2}{r^2 + r_0^2} f_0^3 f_1 = \left(h + \frac{1}{4r^2 r_0^2}\right)^2 - \left(h^2 + \frac{1}{16 r^4 r_0^4}\right) - \frac{2h \log \frac{r}{r_0}}{r^4 - r_0^4} = \frac{h}{2r^2 r_0^2} - \frac{2h \log \frac{r}{r_0}}{r^4 - r_0^4};$$

∵ (1130)... $$f_0^3 f_1 = \frac{h(r^2 + r_0^2)}{2(r^2 - r_0^2)^2}\left\{\frac{1}{r^2 r_0^2} - \frac{4(\log r - \log r_0)}{r^4 - r_0^4}\right\};$$

which does not seem to be a function of rr_0 nor ∵ of f_0. In fact if we perform upon it the operation $r\frac{\delta}{\delta r} - r_0 \frac{\delta}{\delta r_0}$ we do not get 0.

(If we put

(1131)... $$\frac{r}{r_0} = 1 + \epsilon,$$

ϵ being very small, we have

(1132)... $\log r - \log r_0 = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}$; (1133)... $\frac{r^4 - r_0^4}{4} = r_0^4 (\epsilon + \frac{3}{2}\epsilon^2 + \epsilon^3)$;

(1134)... $$\frac{4(\log r - \log r_0) r_0^4}{r^4 - r_0^4} = \frac{1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{3}}{1 + \frac{3\epsilon}{2} + \epsilon^2} = 1 - 2\epsilon + \frac{7\epsilon^2}{3};$$

(1135)... $$\frac{r_0^2}{r^2} = (1+\epsilon)^{-2} = 1 - 2\epsilon + 3\epsilon^2;$$

∵ (1136)... $$f_0^3 f_1 = \frac{h}{3}\frac{(r^2+r_0^2)\epsilon^2}{(r^2-r_0^2)^2 r_0^4} = \frac{h}{6r_0^6} \text{ nearly;}$$

which, by being finite, is so far a verification.)

The only other divisors, besides $n-5$, were $n-1$ & $n-3$ in the foregoing proofs; the cases $n=1$ & $n=3$ are therefore the only cases which would require a separate consideration. For every other case the coefficients A_n, B_n, C_n, D_n are all different from 0; & the condition (1106) is valid and cannot be identically satisfied because the exponent $6-2n$ cannot be equal to $3-n$, nor $4-2n$, nor 2, nor $5-n$, nor $1-n$: ∵ f_1 is not a function of f_0 in any other case.

When $n=3$ then

(1137)... $$f_0^2 = \frac{2I}{r^2-r_0^2} = 2h + \frac{2\log\frac{r}{r_0}}{r^2-r_0^2};$$

(1138)... $$I'' = \frac{h^2}{2}(r^4-r_0^4) + h(r^2-r_0^2) + \tfrac{1}{2}\log\frac{r}{r_0};$$

∵ by (1128)

(1139)... $$\frac{(r^2-r_0^2)^2}{r^2+r_0^2} f_0^3 f_1 = \left(\frac{f_0^2}{2}\right)^2 - \frac{2I''}{r^4-r_0^4} = 2h\left\{\frac{\log\frac{r}{r_0}}{r^2-r_0^2} - \frac{1}{r^2+r_0^2}\right\} + \frac{\log\frac{r}{r_0}}{r^2-r_0^2}\left\{\frac{\log\frac{r}{r_0}}{r^2-r_0^2} - \frac{1}{r^2+r_0^2}\right\}$$
$$= \left(\frac{\log\frac{r}{r_0}}{r^2-r_0^2} + 2h\right)\left(\frac{\log\frac{r}{r_0}}{r^2-r_0^2} - \frac{1}{r^2+r_0^2}\right) = \left(\frac{f_0^2}{2} + h\right)\left(\frac{f_0^2}{2} - h + \frac{1}{r^2+r_0^2}\right);$$

when f_0 is constant we have

(1140)... $$\frac{\delta r}{r} - \frac{\delta r_0}{r_0} = \frac{2\log\frac{r}{r_0}}{r^2-r_0^2}(r\,\delta r - r_0\,\delta r_0) = (f_0^2-2h)(r\,\delta r - r_0\,\delta r_0),$$

which ought to agree, by (1026), with the following

(1141)... $$f_0^2(r\,\delta r - r_0\,\delta r_0) = 2F(r)\,r\,\delta r - 2F(r_0)\,r_0\,\delta r_0 = 2h(r\,\delta r - r_0\,\delta r_0) + \frac{\delta r}{r} - \frac{\delta r_0}{r_0},$$

& it does so.

If now we have simultaneously f_1 constant, we shall have

(1142)... $$f_0^3 f_1\,\delta(r^2-r_0^2)^2 = \left\{\left(\frac{f_0^2}{2}\right)^2 - h^2\right\}\delta(r^2+r_0^2),$$

that is,

(1143)... $$\frac{8f_0^3 f_1 (r^2-r_0^2)^2 (r\,\delta r - r_0\,\delta r_0)}{f_0^4-4h^2} = (r^2-r_0^2)(r\,\delta r + r_0\,\delta r_0) = 2(r\,\delta r - r_0\,\delta r_0)\left\{r^2+r_0^2-\frac{1}{f_0^2-2h}\right\},$$

∵ (1144)... $$r\,\delta r\left(r^2+3r_0^2-\frac{2}{f_0^2-2h}\right) = r_0\,\delta r_0\left\{r_0^2+3r^2-\frac{2}{f_0^2-2h}\right\};$$

& by (1140)

(1145)... $$r\,\delta r\left\{f_0^2-2h-\frac{1}{r}\right\} = r_0\,\delta r_0\left\{f_0^2-2h-\frac{1}{r_0}\right\};$$

we ought ∵ to have (if the property belonged to this law of force)

(1146)... $$(f_0^2-2h)(r^2-r_0^2) + \frac{r^{-1}-r_0^{-1}}{f_0^2-2h} + \frac{(r-r_0)^3}{2rr_0} = 0,$$

that is,

(1147)... $$(f_0^2-2h)^2 rr_0(r+r_0)+\tfrac{1}{2}(r-r_0)^2(f_0^2-2h)=1;$$

but this by (1137) cannot be identically true. We conclude ∵ that in this law of force (the inverse cube) f_1 is not a function of f_0.

For the law of the inverse 1st power, or $n=1$, it seems plain that the calculation would be still more complicated & that it is still less likely we should find f_1 as a function of f_0 alone, involving h but not r nor r_0.

But for the Solar System, or inverse square, $n=2$, not only f_1 but f_m would be a function of ρ and ∵ of f_0; for we have found $V=$ a function of ρ, χ, a. And for the law $n=-1$, or force directly as the distance, f_m would be a function of $r^2+r_0^2$ & ∵ still of f_0. In fact for this law V is a function of χ, $r^2+r_0^2$, h. For in the central ellipse, we may employ the equations

(1148)... $r^2=a^2\cos u^2+b^2\sin u^2$, (1149)... $r_0^2=a^2\cos u_0^2+b^2\sin u_0^2$,

(1150)... $$xx_0+yy_0=\frac{r^2+r_0^2-\chi^2}{2}=a^2\cos u\cos u_0+b^2\sin u\sin u_0,$$

(1151)... $\dfrac{dV}{du}=v^2=a^2\sin u^2+b^2\cos u^2$, (1152)... $2h=v^2+r^2=a^2+b^2$,

(1153)... $a^2=h+\epsilon$, (1154)... $b^2=h-\epsilon$, (1155)... $\dfrac{dV}{du}=h-\epsilon\cos 2u$,

(1156)... $V=h\Delta u-\frac{1}{2}\epsilon\Delta\sin 2u=h\Delta u-\epsilon\cos\Sigma u\cos\Delta u$, (1157)... $r^2=h+\epsilon\cos 2u$,

(1158)... $r_0^2=h+\epsilon\cos 2u_0$, (1159)... $r^2+r_0^2=2h+2\epsilon\cos\Sigma u\cos\Delta u$,

(1160)... $\frac{1}{2}(r^2+r_0^2-\chi^2)=h\cos\Delta u+\epsilon\cos\Sigma u$;

∵ eliminating $\epsilon\Sigma u$ we obtain for the case of the central ellipse a relation between V, $r^2+r_0^2$, χ^2, h, involving no other variables. The partial differential equations also confirm this result; for they permit us to suppose

(1161)... $$\delta V=A\,\delta\xi+B\,\delta\eta,$$

if we put for abbreviation

(1162)... $\xi=r^2+r_0^2-\frac{1}{2}\chi^2$; (1163)... $\eta=\frac{1}{2}\chi^2$.

In fact these suppositions give

(1164)... $\dfrac{\delta V}{\delta r}=2Ar$; (1165)... $\dfrac{\delta V}{\delta r_0}=2Ar_0$; (1166)... $\dfrac{\delta V}{\delta\chi}=(B-A)\chi$;

& change the partial differential equations (925), (926) to the following:

(1167)... $$4A^2r^2+(B-A)^2\chi^2+2A\,(B-A)(r^2-r_0^2+\chi^2)=2h-r^2,$$

(1168)... $$4A^2r_0^2+(B-A)^2\chi^2+2A\,(B-A)(r_0^2-r^2+\chi^2)=2h-r_0^2;$$

which give by subtraction

(1169)... $$4AB=-1,$$

& by addition

(1170)... $$4A^2\xi+4B^2\eta=4h-\xi-\eta;$$

so that it only remains to prove that, A, B being thus determined, the expression $\delta V = A\,\delta\xi + B\,\delta\eta$ is integrable. For this purpose we draw from (1169) expressions of the form

(1171)... $\delta A = A\lambda$, (1172)... $\delta B = -B\lambda$;

& substituting in the differential of (1170) we get for λ the equation

(1173)... $$-8\lambda\,(A^2\xi - B^2\eta) = (4A^2+1)\,\delta\xi + (4B^2+1)\,\delta\eta;$$

∵ (1174)... $\dfrac{\delta A}{\delta\eta} = \dfrac{A\,(4B^2+1)}{8\,(B^2\eta - A^2\xi)}$, (1175)... $\dfrac{\delta B}{\delta\xi} = -\dfrac{B\,(4A^2+1)}{8\,(B^2\eta - A^2\xi)}$,

which are equal by (1169).

We may remark also that the equations (1169), (1170) give

(1176)... $$(4A^2\xi - 4B^2\eta)^2 = 16h^2 - 8h\,(\xi+\eta) + (\xi-\eta)^2;$$

& ∵

(1177)... $$8A^2\xi = 4h - (\xi+\eta) \pm \sqrt{16h^2 - 8h\,(\xi+\eta) + (\xi-\eta)^2};$$

(1178)... $$8B^2\eta = 4h - (\xi+\eta) \mp \sqrt{\qquad\qquad\qquad\qquad}.$$

For the law of the inverse square, if we put

(1179)... $$\delta V = A\,\delta\,(\rho+\chi) + B\,\delta\,(\rho-\chi),$$

we have

(1180)... $\dfrac{\delta V}{\delta r} = \dfrac{\delta V}{\delta r_,} = A + B$, (1181)... $\dfrac{\delta V}{\delta\chi} = A - B$,

and the partial differential equations become

(1182)... $$(A^2-B^2)\,(r^2 - r_0^2 + \chi^2) + 2\,(A^2+B^2)\,r\chi = 2\chi - \frac{r\chi}{a};$$

(1183)... $$(A^2-B^2)\,(r_0^2 - r^2 + \chi^2) + 2\,(A^2+B^2)\,r_0\chi = 2\chi - \frac{r_0\chi}{a}.$$

that is,

(1184)... $$A^2\{(r+\chi)^2 - r_0^2\} - B^2\{(r-\chi)^2 - r_0^2\} = \chi\left(2 - \frac{r}{a}\right),$$

(1185)... $$A^2\{(r_0+\chi)^2 - r^2\} - B^2\{(r_0-\chi)^2 - r^2\} = \chi\left(2 - \frac{r_0}{a}\right),$$

which give

(1186)... $A^2 = \dfrac{1}{\rho+\chi} - \dfrac{1}{4a}$, (1187)... $B^2 = \dfrac{\quad}{\rho-\chi} - \dfrac{\quad}{4a}$,

results which agree with (555) & (556). If then we have in any manner proved that for the law of the inverse square V is a function of ρ, χ, we can easily discover the form of this function by the partial differential equations.

[*Various transformations of the partial differential equations.*]

[41.] To integrate generally those equations by approximation, it seems convenient to introduce the angle ϑ between r and r_0, instead of the chord χ, & thus to put

(1188)... $$\left(\frac{\delta V}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta V}{\delta\vartheta}\right)^2 = 2F\,(r),$$

(1189)... $$\left(\frac{\delta V}{\delta r_0}\right)^2 + \frac{1}{r_0^2}\left(\frac{\delta V}{\delta\vartheta}\right)^2 = 2F\,(r_0).$$

For moderate arcs, V may in general be developed according to odd powers of ϑ in a series of the form

(1190)... $$V = \vartheta\,(u_0 + \vartheta_{,}\,u_1 + \vartheta_{,}^2\,u_2 + \&\text{c.}) = \vartheta u,$$

$\vartheta_{,}$ denoting ϑ^2. Hence

(1191)... $$\frac{\delta V}{\delta \vartheta} = u + 2\vartheta_{,}\,\frac{\delta u}{\delta \vartheta_{,}},$$ (1192)... $$\left(\frac{\delta V}{\delta r}\right)^2 = \vartheta_{,}\left(\frac{\delta u}{\delta r}\right)^2, \quad \&\text{c.},$$

& the equations become

(1193)... $$\left(u + 2\vartheta_{,}\,\frac{\delta u}{\delta \vartheta_{,}}\right)^2 + r^2\vartheta_{,}\left(\frac{\delta u}{\delta r}\right)^2 = 2r^2 F(r);$$

(1194)... $$\left(u + 2\vartheta_{,}\,\frac{\delta u}{\delta \vartheta_{,}}\right)^2 + r_0^2\vartheta_{,}\left(\frac{\delta u}{\delta r_0}\right)^2 = 2r_0^2 F(r_0).\Bigg|$$

However it seems better to integrate on a different plan, by means of other transformations and by the principle of stationary action which I employed last autumn in the problem of atmospheric refraction.*

[42.] Let us ∵ employ, as 3 new auxiliary variables, the perpendicular ϖ let fall from the origin or centre of force on the chord χ of the trajectory and the two projections ξ, ξ_0 of the two radii r, r_0 on this chord; so that ξ, ϖ are the 2 rectangular coordinates of the final & ξ_0, ϖ of the initial point. We shall then have

(1195)... $$\xi = \frac{r^2 - r_0^2 + \chi^2}{2\chi};$$ (1196)... $$\xi_0 = \frac{r^2 - r_0^2 - \chi^2}{2\chi};$$ (1197)... $$\varpi^2 = r^2 - \xi^2 = r_0^2 - \xi_0^2,$$

that is,

(1198)... $$\varpi^2 = (r+\xi)(r-\xi) = \frac{\{(r+\chi)^2 - r_0^2\}\{r_0^2 - (r-\chi)^2\}}{4\chi^2} = \frac{\{(r_0+\chi)^2 - r^2\}\{r^2 - (r_0-\chi)^2\}}{4\chi^2};$$

or, putting for abbreviation

(1199)... $$\varpi^2 = \varpi_{,},$$ (1200)... $$\varpi_{,} = \frac{(r + r_0 + \chi)\,(r + r_0 - \chi)\,(r - r_0 + \chi)\,(-r + r_0 + \chi)}{4\chi^2}.$$

The simplest algebraical way of getting the expressions for ξ, ξ_0, ϖ is to combine the three following equations:

(1201)... $$r^2 = \xi^2 + \varpi^2;$$ (1202)... $$r_0^2 = \xi_0^2 + \varpi^2;$$ (1203)... $$\chi = \xi - \xi_0.$$

To express now $\frac{\delta V}{\delta r}$, $\frac{\delta V}{\delta r_0}$, $\frac{\delta V}{\delta \chi}$ by means of $\frac{\delta V}{\delta \xi}$, $\frac{\delta V}{\delta \xi_0}$, $\frac{\delta V}{\delta \varpi}$ we have

(1204)... $$\frac{\delta V}{\delta \xi} = \frac{\xi}{r}\,\frac{\delta V}{\delta r} + \frac{\delta V}{\delta \chi};$$ (1205)... $$\frac{\delta V}{\delta \xi_0} = \frac{\xi_0}{r_0}\,\frac{\delta V}{\delta r_0} - \frac{\delta V}{\delta \chi};$$

(1206)... $$\frac{\delta V}{\delta \varpi} = \frac{\varpi}{r}\,\frac{\delta V}{\delta r} + \frac{\varpi}{r_0}\,\frac{\delta V}{\delta r_0};$$ ∵ (1207)... $$\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} = \xi\,\frac{\delta V}{r\,\delta r} + \xi_0\,\frac{\delta V}{r_0\,\delta r_0},$$

& (1208)... $$\frac{\delta V}{\varpi\,\delta \varpi} = \frac{\delta V}{r\,\delta r} + \frac{\delta V}{r_0\,\delta r_0},$$ ∵ (1209)... $$\frac{\delta V}{r\,\delta r} = \frac{1}{\xi - \xi_0}\left\{\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi_0}{\varpi}\,\frac{\delta V}{\delta \varpi}\right\},$$

(1210)... $$\frac{\delta V}{r_0\,\delta r_0} = \frac{-1}{\xi - \xi_0}\left\{\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi}{\varpi}\,\frac{\delta V}{\delta \varpi}\right\};$$

[* No trace of this has been found.]

we might hence express $\frac{\delta V}{\delta \chi}$; but the partial differential equations (925), (926) may be put under the form

(1211)... $$\left(\frac{\partial V}{\partial \chi} + \frac{\xi}{r}\frac{\delta V}{\delta r}\right)^2 + \left(\frac{\varpi}{r}\frac{\delta V}{\delta r}\right)^2 = 2F(r) = \{\Psi'(r^2)\}^2,$$

(1212)... $$\left(\frac{\partial V}{\partial \chi} - \frac{\xi_0}{r_0}\frac{\delta V}{\delta r_0}\right)^2 + \left(\frac{\varpi}{r_0}\frac{\delta V}{\delta r_0}\right)^2 = 2F(r_0) = \{\Psi'(r_0^2)\}^2,$$

if we put

(1213)... $$\Psi'(r^2) = \sqrt{2F(r)} = \text{velocity};$$

they become therefore by what precedes, without any separate calculation of $\frac{\delta V}{\delta \chi}$,

(1214)... $$\left(\frac{\delta V}{\delta \xi}\right)^2 + \left(\frac{\varpi}{\xi - \xi_0}\right)^2 \left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi_0}{\varpi}\frac{\delta V}{\delta \varpi}\right)^2 = 2F(r) = \{\Psi'(\xi^2 + \varpi^2)\}^2;$$

(1215)... $$\left(\frac{\delta V}{\delta \xi_0}\right)^2 + \left(\frac{\varpi}{\xi - \xi_0}\right)^2 \left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi}{\varpi}\frac{\delta V}{\delta \varpi}\right)^2 = 2F(r_0) = \{\Psi'(\xi_0^2 + \varpi^2)\}^2.$$

Another way of obtaining these transformed partial differential equations is to employ the *law of variation*

(1216)... $$\delta V = \frac{dx}{dt}\delta x - \frac{dx_0}{dt}\delta x_0 + \frac{dy}{dt}\delta y - \frac{dy_0}{dt}\delta y_0 + \frac{dz}{dt}\delta z - \frac{dz_0}{dt}\delta z_0,$$

combined with the condition that V is a function of r, r_0, χ or of ξ, ξ_0, ϖ. For this last condition gives the law of uniform description of areas, under the form

(1217)... $$x\frac{\delta V}{\delta y} - y\frac{\delta V}{\delta x} + x_0\frac{\delta V}{\delta y_0} - y_0\frac{\delta V}{\delta x_0} = 0,$$

that is,

(1218)... $$x\frac{dy}{dt} - y\frac{dx}{dt} = x_0\frac{dy_0}{dt} - y_0\frac{dx_0}{dt};$$

& if in this we suppose

(1219)... $x = \xi$; (1220)... $y = \varpi$; (1221)... $x_0 = \xi_0$; (1222)... $y_0 = \varpi$,

& observe that we can evidently consider the z's as $= 0$, the orbit being so evidently plane, we find

(1223)... $\xi\frac{dy}{dt} - \xi_0\frac{dy_0}{dt} = \varpi\left(\frac{dx}{dt} - \frac{dx_0}{dt}\right)$; & (1224)... $\xi\frac{\delta V}{\delta y} + \xi_0\frac{\delta V}{\delta y_0} = \varpi\left(\frac{\delta V}{\delta x} + \frac{\delta V}{\delta x_0}\right)$;

also we may suppose

(1225)... $\delta x = \delta \xi$, (1226)... $\delta x_0 = \delta \xi_0$, (1227)... $\delta y = \delta y_0 = \delta \varpi$,

& then the law of varying action becomes

(1228)... $$\delta V = \frac{dx}{dt}\delta\xi - \frac{dx_0}{dt}\delta\xi_0 + \left(\frac{dy}{dt} - \frac{dy_0}{dt}\right)\delta\varpi,$$

& gives

(1229)... $\frac{\delta V}{\delta \xi} = \frac{dx}{dt}$; (1230)... $-\frac{\delta V}{\delta \xi_0} = \frac{dx_0}{dt}$; (1231)... $\frac{\delta V}{\delta \varpi} = \frac{dy}{dt} - \frac{dy_0}{dt}$;

so that by (1223) we shall have

(1232)... $\frac{dy}{dt} = \frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi_0 \delta V}{\varpi \delta \varpi}\right)$; (1233)... $\frac{dy_0}{dt} = \frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi}{\varpi}\frac{\delta V}{\delta \varpi}\right)$:

thus $\frac{\delta V}{\delta \xi}$ and $\frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi_0}{\varpi}\frac{\delta V}{\delta \varpi}\right)$ are the two components of final velocities in the 2 rectangular directions of ξ, ϖ: and $-\frac{\delta V}{\delta \xi_0}$ & $\frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi}{\varpi}\frac{\delta V}{\delta \varpi}\right)$ are the components of initial velocity in the same 2 rectangular directions: the partial differential equations (1214) (1215) are ∵ in this manner explained and confirmed anew.

To shew, more algebraically, that we may suppose $\delta x = \delta \xi$, $\delta x_0 = \delta \xi_0$, $\delta y = \delta y_0 = \delta \varpi$, or that we may put the law of varying action under the form (1228), we may observe that the general equations

(1234)... $\xi^2 + \varpi^2 = x^2 + y^2 + z^2$, (1235)... $\xi_0^2 + \varpi^2 = x_0^2 + y_0^2 + z_0^2$,

(1236)... $$(\xi - \xi_0)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2,$$

give here

(1237)... $\xi(\delta\xi - \delta x) + \varpi(\delta\varpi - \delta y) = 0$; (1238)... $\xi_0(\delta\xi_0 - \delta x_0) + \varpi(\delta\varpi - \delta y_0) = 0$;

(1239)... $$\delta\xi - \delta\xi_0 = \delta x - \delta x_0;$$

∵ eliminating δy, δy_0 from (1216) and suppressing the z's, the law of varying action becomes

(1240)... $$\varpi \delta V = \varpi\left(\frac{dx}{dt}\delta x - \frac{dx_0}{dt}\delta x_0 + \frac{dy}{dt}\delta\varpi - \frac{dy_0}{dt}\delta\varpi\right) + \xi\frac{dy}{dt}(\delta\xi - \delta x) - \xi_0\frac{dy_0}{dt}(\delta\xi_0 - \delta x_0)$$
$$= \varpi\left(\frac{dx}{dt}\delta\xi - \frac{dx_0}{dt}\delta\xi_0 + \frac{dy}{dt}\delta\varpi - \frac{dy_0}{dt}\delta\varpi\right) + \left(\xi\frac{dy}{dt} - \varpi\frac{dx}{dt}\right)(\delta\xi - \delta x) - \left(\xi_0\frac{dy_0}{dt} - \varpi\frac{dx_0}{dt}\right)(\delta\xi_0 - \delta x_0),$$

which reduces itself to (1228) by (1239) and (1223).

If then we put s or $s - s_0$ to denote the arc of the trajectory of which χ or $\xi - \xi_0$ is the chord, we have (rigorously and generally)

(1241)... $\frac{\delta V}{\delta \xi} = \Psi'(\xi^2 + \varpi^2)\frac{dx}{ds}$; (1242)... $\frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi_0}{\varpi}\frac{\delta V}{\delta \varpi}\right) = \Psi'(\xi^2 + \varpi^2)\frac{dy}{ds}$;

(1243)... $-\frac{\delta V}{\delta \xi_0} = \Psi'(\xi_0^2 + \varpi^2)\frac{dx_0}{ds}$; (1244)... $\frac{\varpi}{\xi - \xi_0}\left(\frac{\delta V}{\delta \xi} + \frac{\delta V}{\delta \xi_0} - \frac{\xi}{\varpi}\frac{\delta V}{\delta \varpi}\right) = \Psi'(\xi_0^2 + \varpi^2)\frac{dy_0}{ds}$.

These last expressions are general and rigorous for central forces: they apply to astronomical refraction.

When the arc is small we may, as a 1st approximation, neglect the final and initial components of velocity perpendicular to the chord and so reduce the partial differential equations to the following:

(1245)... $\frac{\delta V}{\delta \xi} = \Psi'(\xi^2 + \varpi^2)$; (1246)... $\frac{\delta V}{\delta \xi_0} = -\Psi'(\xi_0^2 + \varpi^2)$:

which gives by integration

(1247)... $$V = V_1 = \int_{\xi_0}^{\xi}\Psi'(x^2 + \varpi^2)\,dx:$$

no constant being added because V vanishes with $\xi - \xi_0$. This last expression is the value which V would have if the planet moved from the initial to the final point along the chord instead of the arc with the same law of velocity; & ∵, by the law of stationary action, the error of the expression must bear always an evanescent ratio to the square of the chord $\xi-\xi_0$; so that if we call this error V_2, $\frac{\delta V_2}{\delta\xi}$ & $\frac{\delta V_2}{\delta\xi_0}$ must be small of an order higher than the first. We may therefore substitute V_1 for V in the expressions (1242) & (1243) for the components of final and initial velocities perpendicular to the chord; and the results will be true to the 1st order at least. Making this substitution we find

(1248)... $$\frac{dy}{ds}=\frac{\varpi}{(\xi-\xi_0)\Psi}\left\{\Delta\Psi-2\xi_0\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx\right\};$$

in which

(1249)... $$\Delta\Psi=\Psi(\xi^2+\varpi^2)-\Psi(\xi_0^2+\varpi^2);$$

and similarly

(1250)... $$\frac{dy_{\prime}}{ds}=\frac{\varpi}{(\xi-\xi_0)\Psi_0}\left\{\Delta\Psi-2\xi\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx\right\}.$$

To simplify these expressions we may observe that the function $\Delta\Psi-2\xi_0\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx$, considered as depending on ξ, vanishes when $\xi=\xi_0$ and has its differential coefficient

$$=2(\xi-\xi_0)\Psi'(\xi^2+\varpi^2);$$

$\because$ (1251)... $$\Delta\Psi-2\xi_0\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx=2\int_{\xi_0}^{\xi}(x-\xi_0)\Psi'(x^2+\varpi^2)\,dx.$$

Or, because the differential coefficient of the same function taken with respect to ξ_0 is

$$-2\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx,$$

we may put the function under the form

(1252)... $$\Delta\Psi-2\xi_0\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx=2\left(\int_{\xi_0}^{\xi}dx_0\right)\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx.$$

(Verification

$$\left(\int_0^{\xi}dx_0\right)\int_{x_0}^{\xi}x^m\,dx=\int_0^{\xi}\frac{\xi^{m+1}-x_0^{m+1}}{m+1}\,dx_0=\frac{\xi^{m+2}}{m+1}-\frac{\xi^{m+2}}{(m+1)(m+2)}=\frac{\xi^{m+2}}{m+2}=\int^{\xi}xx^m\,dx.)$$

Also integrating by parts we find

(1253)... $$2(x-\xi_0)\Psi'(x^2+\varpi^2)=\frac{d}{dx}\{(x-\xi_0)^2\Psi'(x^2+\varpi^2)\}-2(x-\xi_0)^2x\Psi''(x^2+\varpi^2).$$

$\because$ (1254)... $$2\int_{\xi_0}^{\xi}(x-\xi_0)\Psi'(x^2+\varpi^2)\,dx=(\xi-\xi_0)^2\Psi'(\xi^2+\varpi^2)-2\int_{\xi_0}^{\xi}(x-\xi_0)^2x\Psi''(x^2+\varpi^2)\,dx.$$

Hence confining ourselves to the first order of accuracy, relatively to $\xi-\xi_0$, (1248) gives

(1255)... $$\frac{dy}{ds}=\varpi(\xi-\xi_0)\frac{\Psi'(\xi^2+\varpi^2)}{\Psi(\xi^2+\varpi^2)};$$

the correction, from (1248), being

$$-\frac{2\varpi\int_{\xi_0}^{\xi}(x-\xi_0)^2x\Psi''(x^2+\varpi^2)\,dx}{(\xi-\xi_0)\Psi},$$

but being of the order of $(\xi-\xi_0)^2$ & ∵ probably comparable with that which arises from V_2. And since $\frac{dy}{ds}$ is the final inclination of arc to chord & ∵ = ultimately $\left(\frac{\text{chord of arc}}{\text{diameter of curvature}}\right)$, we must have for all central forces the formula

$$(1256)\ldots\quad \textit{convex curvature} = \frac{2\varpi\Psi'(\xi^2+\varpi^2)}{\Psi(\xi^2+\varpi^2)} = \frac{\delta}{\delta\varpi}\log\Psi(\xi^2+\varpi^2)$$
$$= \frac{\varpi}{r}\frac{\delta}{\delta r}\log v = \frac{\varpi}{v^2}\cdot\frac{\delta\,.\,v^2}{\delta\,.\,r^2} = \frac{\varpi}{r}\frac{F'(r)}{2F(r)},$$

ϖ being perpendicular from centre on tangent & v being velocity considered as a function of the radius vector r. That is,

$$(1257)\ldots\quad \tfrac{1}{2}\,(\text{chord of } \textit{concave curvature}) = -\frac{2F(r)}{F'(r)} = \frac{\text{velocity}^2}{\text{attractive force}}:$$

and in fact this result is evidently true by the 1st principles of central forces, (see *Principia*): because by those dynamical principles the deflection $= \frac{1}{2}$ force × time$^2 = \frac{1}{2}\frac{\text{force}}{\text{velocity}^2}\times\text{arc}^2$; & also by geometrical principles deflection = arc^2/chord of curvature.

In like manner

$$(1258)\ldots\quad \Delta\Psi - 2\xi\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx = -2\int_{\xi_0}^{\xi}(\xi-x)\,\Psi'(x^2+\varpi^2)\,dx$$
$$= -(\xi-\xi_0)^2\,\Psi'(\xi_0^2+\varpi^2) - 2\int_{\xi_0}^{\xi}(\xi-x)^2\,x\Psi''(x^2+\varpi^2)\,dx;$$

∵ (1250) gives, as a rigorous transformation of itself, though subject to its own inaccuracy,

$$(1259)\ldots\quad \frac{dy_0}{ds} = -\frac{\varpi(\xi-\xi_0)\,\Psi'(\xi_0^2+\varpi^2)}{\Psi(\xi_0^2+\varpi^2)} - \frac{2\varpi\int_{\xi_0}^{\xi}(\xi-x)^2\,x\Psi''\,dx}{(\xi-\xi_0)},$$

& to the accuracy of the 1st order

$$(1260)\ldots\quad \frac{dy_0}{ds} = -\frac{\varpi(\xi-\xi_0)\,\Psi'(\xi_0^2+\varpi^2)}{\Psi(\xi_0^2+\varpi^2)}.$$

In this order of approximation ∵ we find

$$(1261)\ldots\quad \frac{dy_0}{ds} = -\frac{dy}{ds},$$

as we ought.

We have therefore, to the accuracy of the third order at least, by (1241) &c.,

$$(1262)\ldots\quad \frac{\delta V}{\delta\xi} = \Psi(\xi^2+\varpi^2) - \frac{\varpi^2}{2(\xi-\xi_0)^2\Psi}\left\{\Delta\Psi - 2\xi_0\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx\right\}^2;$$

$$(1263)\ldots\quad \frac{\delta V}{\delta\xi_0} = -\Psi(\xi_0^2+\varpi^2) + \frac{\varpi^2}{2(\xi-\xi_0)^2\Psi_0}\left\{\Delta\Psi - 2\xi\int_{\xi_0}^{\xi}\Psi'(x^2+\varpi^2)\,dx\right\}^2:$$

that is, because

$$(1264)\ldots\quad V = \int_{\xi_0}^{\xi}\Psi(x^2+\varpi^2)\,dx + V_2,$$

we have

$$(1265)\ldots \qquad \frac{\delta V_2}{\delta \xi} \quad \frac{\varpi^2 \left\{\Delta\Psi' - 2\xi_0 \int_{\xi_0}^{\xi} \Psi''(x^2+\varpi^2)\,dx\right\}^2}{2(\xi-\xi_0)^2\,\Psi'(\xi^2+\varpi^2)}$$

$$\&\ (1266)\ldots \qquad \frac{\delta V_2}{\delta \xi_0} : \quad \frac{\varpi^2 \left\{\Delta\Psi' - 2\xi \int_{\xi_0}^{\xi} \Psi''(x^2+\varpi^2)\,dx\right\}^2}{2(\xi-\xi_0)^2\,\Psi'(\xi_0^2+\varpi^2)},$$

$\Delta\Psi'$ having the meaning (1249).

If there be any function V_2 of the three independent variables ξ, ξ_0, ϖ which accurately satisfy these approximate equations and if we put for abbreviation

$$(1267)\ldots \qquad \int_{\xi_0}^{\xi} \Psi''(x^2+\varpi^2)\,dx = I,$$

so that the equations become

$$(1268)\ldots \quad \frac{\delta V_2}{\delta\xi} = -\frac{\varpi^2(\Delta\Psi' - 2\xi_0 I)^2}{2\Delta\xi^2 \,.\, \Psi'}, \qquad (1269)\ldots \quad \frac{\delta V_2}{\delta\xi_0} - \frac{\varpi^2(\Delta\Psi' - 2\xi I)^2}{2\Delta\xi^2 \,.\, \Psi'_0}$$

and if we observe that when ϖ is constant,

$$(1270)\ldots \qquad \delta I = \Psi''\delta\xi - \Psi''_0\delta\xi_0,$$

& $\because$

$$(1271)\ldots \qquad \delta(\Delta\Psi' - 2\xi_0 I) = 2(\xi-\xi_0)\Psi''\delta\xi - 2I\,\delta\xi_0,$$

$$(1272)\ldots \qquad \delta(\Delta\Psi' - 2\xi I) = -2I\,\delta\xi + 2\Delta\xi\,\Psi''_0\delta\xi_0,$$

we find, by comparing the expressions for $\frac{\delta^2 V_2}{\delta\xi\,\delta\xi_0}$, the following condition which must be satisfied:

$$(1273)\ldots \quad (\Delta\Psi' - 2\xi_0 I)(2I\Delta\xi - \Delta\Psi' + 2\xi_0 I)\,\Psi'_0 = \frac{\Delta\xi^3 \,.\, \Psi' \,.\, \Psi'_0}{\varpi^2}\,\frac{\delta^2 V_2}{\delta\xi\,\delta\xi_0}$$
$$= (\Delta\Psi' - 2\xi I)(-2I\Delta\xi - \Delta\Psi' + 2\xi I)\,\Psi';$$

& in fact these expressions are equal to $-(\Delta\Psi' - 2\xi I)(\Delta\Psi' - 2\xi_0 I)$, multiplied by Ψ'_0 and by Ψ'; they are therefore nearly but not exactly equal. Perhaps, however, on assigning to $\frac{dy}{ds}$, $\frac{dy_0}{ds}$ the values (1248) (1250), which we shall call for abbreviation y', y'_0, & changing $\frac{dx}{ds}$, $\frac{dx_0}{ds}$ to $\sqrt{1-y'^2}$, $\sqrt{1-y_0'^2}$ in (1241) (1243), we may get two rigorously constructed though only approximate expressions for $\frac{\delta V}{\delta\xi}$, $\frac{\delta V}{\delta\xi_0}$. If so we must have

$$(1274)\ldots \qquad -\frac{\Psi' \,.\, y'}{\sqrt{1-y'^2}}\,\frac{\delta y'}{\delta\xi_0} = \frac{\Psi'_0 \,.\, y'_0}{\sqrt{1-y_0'^2}}\,\frac{\delta y'_0}{\delta\xi};$$

but even this seems unlikely. Still we can easily approximate by the present method to V_2; but it seems better to resume, as follows, the partial differential equations of [41.].*

---

* [Since $\frac{\partial}{\partial\xi}\frac{\Delta\Psi' - 2\xi I}{\xi-\xi_0} = -\frac{\Delta\Psi' - 2\xi_0 I}{(\xi-\xi_0)^2}$, and since $\Delta\Psi' - 2\xi I = 0$ when $\xi = \xi_0$, it easily follows that the first and last terms of (1273) are of the order $(\Delta\xi)^4$. This shows that the method adopted is true up to and including $(\xi-\xi_0)^2$ for V. (1274) is of course rigorous.]

[43.] Let (1188) be put under the form

(1275)... $$\frac{\delta V}{\delta r} = \sqrt{2F(r) - \frac{1}{r^2}\left(\frac{\delta V}{\delta \vartheta}\right)^2}$$

or (1276)... $$q = \sqrt{2F(y) - \frac{p^2}{y^2}}.$$

Then, considering p as a function of x, y, r_0, q will be a function of the same &

(1277)... $$\frac{\delta q}{\delta x} = \frac{\frac{p}{y^2}\frac{\delta p}{\delta x}}{\sqrt{2F - \frac{p^2}{y^2}}} = \frac{\delta p}{\delta y}.$$

Now

(1278)... $\delta p = \frac{\delta p}{\delta y}\delta y + \frac{\delta p}{\delta x}\delta x + \frac{\delta p}{\delta r_0}\delta r_0$, $\because$ (1279)... $\delta p = \frac{\delta p}{\delta r_0}\delta r_0 + \frac{\delta p}{\delta x}\cdot\left(\delta x - \frac{p}{qy^2}\delta y\right)$;

$\because$ we are to integrate the 2 simultaneous equations

(1280)... $dx = \frac{p\,dy}{y^2 q}$, (1281)... $dp = 0$,

and after getting integrals of the form

(1282)... $\Psi_1(x, y, p) = \text{const.}$, (1283)... $\Psi_2(x, y, p) = \text{const.}$,

to establish any arbitrary relation between Ψ_1, Ψ_2 & r_0. In other words the integral of (1275) is

(1284)... $$\Phi(\Psi_1, \Psi_2, r_0) = 0.$$

Now (1281) gives at once

(1285)... $$p = \text{constant} = \Psi_1;$$

and then (1282) gives

(1286)... $$x - \int \frac{p\,dy}{qy^2} = \text{const.} = \Psi_2.$$

Hence (1284) becomes

(1287)... $$\vartheta = x = \int \frac{p\,dy}{qy^2} + \text{funct.}\,(p, r_0);$$

that is,

(1288)... $\vartheta =$ a known function of $\left(r, \frac{\delta V}{\delta \vartheta}\right)$ + an arbitrary function of $\left(r_0, \frac{\delta V}{\delta \vartheta}\right)$;

similarly

(1289)... $\vartheta =$ a known function of $\left(r_0, \frac{\delta V}{\delta \vartheta}\right)$ + an arbitrary function of $\left(r, \frac{\delta V}{\delta \vartheta}\right)$,

& it seems evident that the known function of each equation must determine the arbitrary function of the other, subject at least to the possibility of each containing an arbitrary function of $\frac{\delta V}{\delta \vartheta}$.

[End of manuscript.]

II.

ON A GENERAL METHOD IN DYNAMICS

BY WHICH THE STUDY OF THE MOTIONS OF ALL FREE SYSTEMS OF ATTRACTING OR REPELLING POINTS IS REDUCED TO THE SEARCH AND DIFFERENTIATION OF ONE CENTRAL RELATION, OR CHARACTERISTIC FUNCTION*

Received April 1,—Read April 10, 1834.

[*Philosophical Transactions of the Royal Society*, Part II for 1834, pp. 247–308.]

CONTENTS

PAGE
Introductory Remarks. 104
Integration of the Equations of Motion of a System, Characteristic Function of such Motion, and Law of varying Action. 105
Verifications of the foregoing Integrals. 109
Introduction of relative or polar Coordinates, or other marks of position of a System. . 113
Separation of the relative motion of a system from the motion of its centre of gravity; characteristic function for such relative motion, and law of its variation. . . . 121
On Systems of two Points, in general; Characteristic Function of the motion of any Binary System. 128
On the undisturbed Motion of a Planet or Comet about the Sun: Dependence of the Characteristic Function of such Motion, on the chord and the sum of the Radii. . . 135
On Systems of three Points, in general; and on their Characteristic Functions. . . . 140
General method of improving an approximate expression for the Characteristic Function of motion of a System in any Problem of Dynamics. 147
Application of the foregoing method to the case of a Ternary or Multiple System, with any laws of attraction or repulsion, and with one predominant mass. 150
Rigorous transition from the theory of Binary to that of Multiple Systems, by means of the disturbing part of the whole Characteristic Function; and approximate expressions for the perturbations. 155
General introduction of the Time into the expression of the Characteristic Function in any dynamical problem. 160

---

* [Communicated by Captain Beaufort, R.N., F.R.S. Afterwards Admiral Sir Francis Beaufort (1774–1857), he was the son of Daniel Augustus Beaufort who took a prominent part in the formation of the Royal Irish Academy. Beaufort was hydrographer to the Navy and his name is still remembered for his introduction of the scale of wind force and the tabulated system of weather registration in common use.]

Introductory Remarks.

The theoretical development* of the laws of motion of bodies is a problem of such interest and importance, that it has engaged the attention of all the most eminent mathematicians, since the invention of dynamics as a mathematical science by Galileo, and especially since the wonderful extension which was given to that science by Newton. Among the successors of those illustrious men, Lagrange has perhaps done more than any other analyst to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem. But the science of force, or of power acting by law in space and time, has undergone already another revolution,† and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginable conditions, which Lagrange so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods. And the method proposed in the present essay, for the deductive study of the motions of attracting or repelling systems, will perhaps be received with indulgence, as an attempt to assist in carrying forward so high an inquiry.

In the methods commonly employed, the determination of the motion of a free point in space, under the influence of accelerating forces, depends on the integration of three equations in ordinary differentials of the second order; and the determination of the motions of a system of free points, attracting or repelling one another, depends on the integration of a system of such equations, in number threefold the number of the attracting or repelling points, unless we previously diminish by unity this latter number, by considering only relative motions. Thus, in the solar system, when we consider only the mutual attractions of the sun and of the ten known planets,‡ the determination of the motions of the latter about the former is reduced, by the usual methods, to the integration of a system of thirty ordinary differential equations of the second order, between the coordinates and the time; or, by a transformation of Lagrange, § to the integration of a system of sixty ordinary differential equations of the first order, between the time and the elliptic elements: by which integrations, the thirty varying coordinates, or the sixty varying elements, are to be found as functions of the time. In the method of the present essay, this problem is reduced to the search and differentiation of a single function, which satisfies two partial differential equations of the first order and of the second degree: and every other dynamical problem, respecting the motions of any system, however numerous, of attracting or repelling points, (even if we suppose those points restricted by any conditions of connexion consistent with the law of living force,) is reduced, in like manner, to the study of one central function, of which the form marks out and characterizes the properties of the moving system, and is to be determined by a pair of partial differential equations of the first order, combined with some simple considerations. The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another: and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function,* the unfolding of one central relation.

---

* [On the development of Theoretical Dynamics from Lagrange to Hamilton, see Cayley's *Report to the British Association*, Dublin (1858), p. 1.]

† [Hamilton is here referring to Boscovich's hypothesis on the constitution of matter (1743). Cf. Graves, *Life of Hamilton*, Vol. I, p. 593, Vol. II, pp. 85–86, 88.]

‡ [The planets known at the date of this paper were Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and four asteroids Ceres, Pallas, Vesta and Juno, the last being discovered in 1807. Neptune was not discovered until 1846.]

§ [*Méc. Anal.* 3rd ed. Tome II, pp. 112–143.]

The present essay does not pretend to treat fully of this extensive subject,—a task which may require the labours of many years and many minds; but only to suggest the thought and propose the path to others. Although, therefore, the method may be used in the most varied dynamical researches, it is at present only applied to the orbits and perturbations of a system with any laws of attraction or repulsion, and with one predominant mass or centre of predominant energy; and only so far, even in this one research, as appears sufficient to make the principle itself understood. It may be mentioned here, that this dynamical principle is only another form of that idea which has already been applied to optics in the *Theory of systems of rays*, and that an intention of applying it to the motions of systems of bodies was announced† at the publication of that theory. And besides the idea itself, the manner of calculation also, which has been thus exemplified in the sciences of optics and dynamics, seems not confined to those two sciences, but capable of other applications; and the peculiar combination which it involves, of the principles of variations with those of partial differentials, for the determination and use of an important class of integrals, may constitute, when it shall be matured by the future labours of mathematicians, a separate branch of analysis.‡

WILLIAM R. HAMILTON.

Observatory, Dublin,
March 1834.

Integration of the Equations of Motion of a System, Characteristic Function of such Motion, and Law of varying Action.

1. The known differential equations of motion of a system of free points, repelling or attracting one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula:

$$\Sigma \, . \, m\,(x''\delta x + y''\delta y + z''\delta z) = \delta U. \qquad (1.)$$

In this formula the sign of summation Σ extends to all the points of the system; m is, for any one such point, the constant called its mass; x'', y'', z'' are its component accelerations, or the second differential coefficients of its rectangular coordinates x, y, z, taken with respect to the time; δx, δy, δz are any arbitrary infinitesimal displacements which the point can be imagined to receive in the same three rectangular directions; and δU is the infinitesimal variation corresponding, of a function U of the masses and mutual distances of the several points of the system, of which the form depends on the laws of their mutual actions, by the equation

---

* Lagrange and, after him, Laplace and others, have employed a single function to express the different forces of a system, and so to form in an elegant manner the differential equations of its motion. By this conception, great simplicity has been given to the statement of the problem of dynamics; but the solution of that problem, or the expression of the motions themselves, and of their integrals, depends on a very different and hitherto unimagined function, as it is the purpose of this essay to show.

† *Transactions of the Royal Irish Academy*, Vol. xv, p. 80. [*Mathematical Papers*, Vol. I, p. 9.] A notice of this dynamical principle was also lately given in an article "On a general Method of expressing the Paths of Light and of the Planets," published in the *Dublin University Review* for October 1833. [*Mathematical Papers*, Vol. I, p. 311.]

‡ [Hamilton afterwards developed this suggestion in his Calculus of Principal Relations. See pp. 297–410.]

$$U = \Sigma \,.\, mm_{,} f(r), \tag{2.}$$

r being the distance between any two points m, $m_{,}$, and the function $f(r)$ being such that its derivative or differential coefficient $f'(r)$ expresses the law of their repulsion, being negative in the case of attraction. The function which has been here called U may be named the *force-function* of a system: it is of great utility in theoretical mechanics, into which it was introduced by Lagrange,* and it furnishes the following elegant forms for the differential equations of motion, included in the formula (1.):

$$\begin{aligned}
m_1 x_1'' &= \frac{\delta U}{\delta x_1}; & m_2 x_2'' &= \frac{\delta U}{\delta x_2}; & \ldots \quad m_n x_n'' &= \frac{\delta U}{\delta x_n};\\
m_1 y_1'' &= \frac{\delta U}{\delta y_1}; & m_2 y_2'' &= \frac{\delta U}{\delta y_2}; & \ldots \quad m_n y_n'' &= \frac{\delta U}{\delta y_n};\\
m_1 z_1'' &= \frac{\delta U}{\delta z_1}; & m_2 z_2'' &= \frac{\delta U}{\delta z_2}; & \ldots \quad m_n z_n'' &= \frac{\delta U}{\delta z_n};
\end{aligned} \tag{3.}$$

the second members of these equations being the partial differential coefficients of the first order of the function U. But notwithstanding the elegance and simplicity of this known manner of stating the principal problem of dynamics, the difficulty of solving that problem, or even of expressing its solution, has hitherto appeared insuperable; so that only seven intermediate integrals, or integrals of the first order, with as many arbitrary constants, have hitherto been found for these general equations of motion of a system of n points, instead of $3n$ intermediate and $3n$ final integrals, involving ultimately $6n$ constants: nor has any integral [of relative motion] been found which does not need to be integrated again. No general solution has been obtained assigning (as a complete solution ought to do) $3n$ relations between the n masses $m_1, m_2, \ldots m_n$, the $3n$ varying coordinates $x_1, y_1, z_1, \ldots x_n, y_n, z_n$, the varying time t, and the $6n$ initial data of the problem, namely, the initial coordinates $a_1, b_1, c_1, \ldots a_n, b_n, c_n$, and their initial rates of increase, $a_1', b_1', c_1', \ldots a_n', b_n', c_n'$; the quantities called here initial being those which correspond to the arbitrary origin of time. It is, however, possible (as we shall see) to express these long-sought relations by the partial differential coefficients of a new central or radical function, to the search and employment of which the difficulty of mathematical dynamics becomes henceforth reduced.

2. If we put for abridgement

$$T = \tfrac{1}{2} \Sigma \,.\, m(x'^2 + y'^2 + z'^2), \tag{4.}$$

so that $2T$ denotes, as in the *Mécanique Analytique*, the whole living force† of the system;

---

* [Lagrange used the negative of the function U, under the name of V. (Cf. *Méc. Anal.* 3rd ed. Tome I, p. 290.)]

† [The term *vis viva* was first used by Leibniz. The term *energy* was introduced by Thomas Young (*Natural Philosophy*, Lecture VIII) and the term *work* by Coriolis. Hamilton writing to P. G. Tait in 1862 said: "*Energy* and *Work*, in the *old* English meaning, are things not unfamiliar to me. But I have only the dimmest views of the modern meanings attached to those terms." Cf. Graves, *Life of Hamilton*, Vol. III, p. 150.]

(x', y', z', being here, according to the analogy of our foregoing notation, the rectangular components of velocity of the point m, or the first differential coefficients of its coordinates taken with respect to the time;) an easy and well-known combination of the differential equations of motion, obtained by changing in the formula (1.) the variations to the differentials of the coordinates, may be expressed in the following manner,

$$dT = dU, \tag{5.}$$

and gives, by integration, the celebrated law of living force, under the form

$$T = U + H. \tag{6.}$$

In this expression, which is one of the seven known integrals already mentioned, the quantity H is independent of the time, and does not alter in the passage of the points of the system from one set of positions to another. We have, for example, an initial equation of the same form, corresponding to the origin of time, which may be written thus,

$$T_0 = U_0 + H. \tag{7.}$$

The quantity H may, however, receive any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data; but the increment of H, thus obtained, is evidently connected with the analogous increments of the functions T and U, by the relation

$$\Delta T = \Delta U + \Delta H, \tag{8.}$$

which, for the case of infinitesimal variations, may conveniently be written thus,

$$\delta T = \delta U + \delta H; \tag{9.}$$

and this last relation, when multiplied by dt, and integrated, conducts to an important result. For it thus becomes, by (4.) and (1.),

$$\int \Sigma \,.\, m\,(dx \,.\, \delta x' + dy \,.\, \delta y' + dz \,.\, \delta z') = \int \Sigma \,.\, m\,(dx' \,.\, \delta x + dy' \,.\, \delta y + dz' \,.\, \delta z) + \int \delta H \,.\, dt, \tag{10.}$$

that is, by the principles of the calculus of variations,

$$\delta V = \Sigma \,.\, m\,(x'\delta x + y'\delta y + z'\delta z) - \Sigma \,.\, m\,(a'\delta a + b'\delta b + c'\delta c) + t\,\delta H, \tag{A.}$$

if we denote by V the integral

$$V = \int \Sigma \,.\, m\,(x'dx + y'dy + z'dz) = \int_0^t 2T\,dt, \tag{B.}$$

namely, the accumulated living force, called often the action of the system, from its initial to its final position.

If, then, we consider (as it is easy to see that we may) the action V as a function of the initial and final coordinates, and of the quantity H, we shall have, by (A.), the following groups of equations; first, the group,

$$\left.\begin{array}{lll}
\dfrac{\delta V}{\delta x_1} = m_1 x_1'; & \dfrac{\delta V}{\delta x_2} = m_2 x_2'; & \dfrac{\delta V}{\delta x_n} = m_n x_n'; \\
\dfrac{\delta V}{\delta y_1} = m_1 y_1'; & \dfrac{\delta V}{\delta y_2} = m_2 y_2'; & \dfrac{\delta V}{\delta y_n} = m_n y_n'; \\
\dfrac{\delta V}{\delta z_1} = m_1 z_1'; & \dfrac{\delta V}{\delta z_2} = m_2 z_2'; & \dfrac{\delta V}{\delta z_n} = m_n z_n'.
\end{array}\right\} \tag{C.}$$

Secondly, the group,

$$\begin{array}{llll} \frac{\delta V}{\delta a_1} = -m_1 a'_1; & \frac{\delta V}{\delta a_2} = -m_2 a'_2; & \dots & \frac{\delta V}{\delta a_n} = -m_n a'_n; \\ \frac{\delta V}{\delta b_1} = -m_1 b'_1; & \frac{\delta V}{\delta b_2} = -m_2 b'_2; & \dots & \frac{\delta V}{\delta b_n} = -m_n b'_n; \\ \frac{\delta V}{\delta c_1} = -m_1 c'_1; & \frac{\delta V}{\delta c_2} = -m_2 c'_2; & \dots & \frac{\delta V}{\delta c_n} = -m_n c'_n; \end{array} \tag{D.}$$

d finally, the equation,

$$\frac{\delta V}{\delta H} = t. \tag{E.}$$

So that if this function V were known, it would only remain to eliminate H between the $3n+1$ equations (C.) and (E.), in order to obtain all the $3n$ intermediate integrals, or between (D.) and (E.) to obtain all the $3n$ final integrals of the differential equations of motion; that is, ultimately, to obtain the $3n$ sought relations between the $3n$ varying coordinates and the time, involving also the masses and the $6n$ initial data above mentioned; the discovery of which relations would be (as we have said) the general solution of the general problem of dynamics. We have, therefore, at least reduced that general problem to the search and differentiation of a single function V, which we shall call on this account the CHARACTERISTIC FUNCTION of motion of a system; and the equation (A.), expressing the fundamental law of its variation, we shall call the *equation of the characteristic function*, or the LAW OF VARYING ACTION.

3. To show more clearly that the action or accumulated living force of a system, or in other words, the integral of the product of the living force by the element of the time, may be regarded as a function of the $6n+1$ quantities already mentioned, namely, of the initial and final coordinates, and of the quantity H, we may observe, that whatever depends on the manner and time of motion of the system may be considered as such a function; because the initial form of the law of living force, when combined with the $3n$ known or unknown relations between the time, the initial data, and the varying coordinates, will always furnish $3n+1$ relations, known or unknown, to connect the time and the initial components of velocities with the initial and final coordinates, and with H. Yet from not having formed the conception of the action as a *function* of this kind, the consequences that have been here deduced from the formula (A.) for the variation of that definite integral appear to have escaped the notice of Lagrange, and of the other illustrious analysts who have written on theoretical mechanics; although they were in possession of a formula for the variation of this integral not greatly differing from ours. For although Lagrange and others, in treating of the motion of a system, have shown that the variation of this definite integral vanishes when the extreme coordinates and the constant H are given, they appear to have deduced from this result only the well-known law of *least action*; namely, that if the points or bodies of a system be imagined to move from a given set of initial to a given set of final positions, not as they do nor even as they could move consistently with the general dynamical laws or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configurations which constitutes the law of living force; and if, besides, this geometrically imaginable, but dynamically impossible motion, be made to differ infinitely *little* from the actual manner of

motion of the system, between the given extreme positions; then the varied value of the definite integral called action, or the accumulated living force of the system in the motion thus imagined, will differ infinitely *less* from the actual value of that integral. But when this well-known law of least, or as it might be better called, of *stationary action*, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the calculus of variations, the differential equations of motion of the second order, which can always be otherwise found. It seems, therefore, to be with reason that Lagrange*, Laplace, and Poisson have spoken lightly of the utility of this principle in the present state of dynamics. A different estimate, perhaps, will be formed of that other principle which has been introduced in the present paper, under the name of the *law of varying action*, in which we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system, and (in general) the quantity H, and which serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

Verifications of the foregoing Integrals.

4. A verification, which ought not to be neglected, and at the same time an illustration of this new principle, may be obtained by deducing the known differential equations of motion from our system of intermediate integrals, and by showing the consistence of these again with our final integral system. As preliminary to such verification, it is useful to observe that the final equation (6.) of living force, when combined with the system (C.), takes this new form,†

$$\tfrac{1}{2}\Sigma\,.\,\frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}=U+H; \qquad \text{(F).}$$

and that the initial equation (7.) of living force becomes by (D.)

$$\tfrac{1}{2}\Sigma\,.\,\frac{1}{m}\left\{\left(\frac{\delta V}{\delta a}\right)^2+\left(\frac{\delta V}{\delta b}\right)^2+\left(\frac{\delta V}{\delta c}\right)^2\right\}=U_0+H. \qquad \text{(G.)}$$

These two partial differential equations, initial and final, of the first order and the second degree, must both be identically satisfied by the characteristic function V: they furnish (as we shall find) the principal means of discovering the form of that function, and are of essential importance in its theory.‡ If the form of this function were known, we might eliminate $3n-1$ of the $3n$ initial coordinates between the $3n$ equations (C.); and although we cannot yet perform the actual process of this elimination, we are entitled to assert that it would remove along with the others the remaining initial coordinate, and would conduct to the equation (6.) of final living force, which might then be transformed into the equation (F.). In like manner we may conclude that all the $3n$ final coordinates could be eliminated together from the $3n$ equations (D.), and that the result would be the initial equation (7.) of living force, or the transformed equation (G.). We may therefore consider the law of living force, which assisted us in discovering the properties of our characteristic function V, as included reciprocally in those properties, and as resulting by elimination, in every particular case, from the systems (C.) and (D.); and in treating of either of these systems, or in conducting any other dynamical investigation by the method of this characteristic function, we are at liberty to employ the partial differential equations (F.) and (G.), which that function must necessarily satisfy.

---

* [For Lagrange's remarks on the principle of least action, see *Méc. Anal.* 3rd ed. Tome I, pp. 229, 230.]

† [These equations are exactly analogous to the equations $\Omega=0$, $\Omega'=0$ of the Third Supplement. *Mathematical Papers*, Vol. I, pp. 170, 485.]

‡ [For Jacobi's criticism, see Appendix, p. 613.]

It will now be easy to deduce, as we proposed, the known equations of motion (3.) of the second order, by differentiation and elimination of constants, from our intermediate integral system (C.), (E.), or even from a part of that system, namely, from the group (C.), when combined with the equation (F.). For we thus obtain

$$\left.\begin{aligned}
m_1 x_1'' = \frac{d}{dt}\frac{\delta V}{\delta x_1} &= x_1' \frac{\delta^2 V}{\delta x_1^2} + x_2' \frac{\delta^2 V}{\delta x_1 \delta x_2} + \ldots + x_n' \frac{\delta^2 V}{\delta x_1 \delta x_n} \\
&\quad + y_1' \frac{\delta^2 V}{\delta x_1 \delta y_1} + y_2' \frac{\delta^2 V}{\delta x_1 \delta y_2} + \ldots + y_n' \frac{\delta^2 V}{\delta x_1 \delta y_n} \\
&\quad + z_1' \frac{\delta^2 V}{\delta x_1 \delta z_1} + z_2' \frac{\delta^2 V}{\delta x_1 \delta z_2} + \ldots + z_n' \frac{\delta^2 V}{\delta x_1 \delta z_n} \\
&= \frac{1}{m_1}\frac{\delta V}{\delta x_1}\frac{\delta^2 V}{\delta x_1^2} + \frac{1}{m_2}\frac{\delta V}{\delta x_2}\frac{\delta^2 V}{\delta x_1 \delta x_2} + \ldots + \frac{1}{m_n}\frac{\delta V}{\delta x_n}\frac{\delta^2 V}{\delta x_1 \delta x_n} \\
&\quad + \frac{1}{m_1}\frac{\delta V}{\delta y_1}\frac{\delta^2 V}{\delta x_1 \delta y_1} + \frac{1}{m_2}\frac{\delta V}{\delta y_2}\frac{\delta^2 V}{\delta x_1 \delta y_2} + \ldots + \frac{1}{m_n}\frac{\delta V}{\delta y_n}\frac{\delta^2 V}{\delta x_1 \delta y_n} \\
&\quad + \frac{1}{m_1}\frac{\delta V}{\delta z_1}\frac{\delta^2 V}{\delta x_1 \delta z_1} + \frac{1}{m_2}\frac{\delta V}{\delta z_2}\frac{\delta^2 V}{\delta x_1 \delta z_2} + \ldots + \frac{1}{m_n}\frac{\delta V}{\delta z_n}\frac{\delta^2 V}{\delta x_1 \delta z_n} \\
&= \frac{\delta}{\delta x_1}\Sigma \,.\, \frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right\} = \frac{\delta}{\delta x_1}(U+H);
\end{aligned}\right\} \qquad (11.)$$

that is, we obtain

$$m_1 x_1'' = \frac{\delta U}{\delta x_1}. \qquad (12.)$$

And in like manner we might deduce, by differentiation, from the integrals (C.) and from (F.) all the other known differential equations of motion, of the second order, contained in the set marked (3.); or, more concisely, we may deduce at once the formula (1.), which contains all those known equations, by observing that the intermediate integrals (C.), when combined with the relation (F.), give*

$$\begin{aligned}
\Sigma \,.\, m\,(x''\delta x + y''\delta y + z''\delta z) &= \Sigma\left(\frac{d}{dt}\frac{\delta V}{\delta x}\,.\,\delta x + \frac{d}{dt}\frac{\delta V}{\delta y}\,.\,\delta y + \frac{d}{dt}\frac{\delta V}{\delta z}\,.\,\delta z\right) \\
&= \Sigma \,.\, \frac{1}{m}\left(\frac{\delta V}{\delta x}\frac{\delta}{\delta x} + \frac{\delta V}{\delta y}\frac{\delta}{\delta y} + \frac{\delta V}{\delta z}\frac{\delta}{\delta z}\right)\Sigma\left(\frac{\delta V}{\delta x}\delta x + \frac{\delta V}{\delta y}\delta y + \frac{\delta V}{\delta z}\delta z\right) \\
&= \Sigma\left(\delta x\frac{\delta}{\delta x} + \delta y\frac{\delta}{\delta y} + \delta z\frac{\delta}{\delta z}\right)\Sigma \,.\, \frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2\right. \\
&= \Sigma\left(\delta x\frac{\delta}{\delta x} + \delta y\frac{\delta}{\delta y} + \delta z\frac{\delta}{\delta z}\right)(U+H) \\
&= \delta U.
\end{aligned} \qquad (13.)$$

5. Again, we were to show that our intermediate integral system, composed of the equations (C.) and (E.), with the $3n$ arbitrary constants $a_1, b_1, c_1, \ldots a_n, b_n, c_n$, (and involving also the

---

* $\left[m_r x_r'' \delta x_r = \frac{d}{dt}\frac{\delta V}{\delta x_r}\delta x_r = \Sigma \frac{\delta^2 V}{\delta x_r \delta x_s} x_s' \delta x_r = \Sigma \frac{1}{m_s}\frac{\delta V}{\delta x_s}\frac{\delta^2 V}{\delta x_s \delta x_r}\delta x_r = \Sigma \frac{\delta}{\delta x_r}\left\{\frac{1}{2m_s}\left(\frac{\delta V}{\delta x_s}\right)^2\right\}\delta x_r = \Sigma \frac{\delta U}{\delta x_r}\delta x_r = \delta U.\right]$

auxiliary constant H,) is consistent with our final integral system of equations (D.) and (E.), which contain $3n$ other arbitrary constants, namely, $a'_1, b'_1, c'_1, \ldots a'_n, b'_n, c'_n$. The immediate differentials of the equations (C.), (D.), (E.), taken with respect to the time, are, for the first group,

$$\begin{array}{lll}
\dfrac{d}{dt}\dfrac{\delta V}{\delta x_1}=m_1x''_1; & \dfrac{d}{dt}\dfrac{\delta V}{\delta x_2}=m_2x''_2; & \dfrac{d}{dt}\dfrac{\delta V}{\delta x_n}=m_nx''_n; \\
\dfrac{d}{dt}\dfrac{\delta V}{\delta y_1}=m_1y''_1; & \dfrac{d}{dt}\dfrac{\delta V}{\delta y_2}=m_2y''_2; & \dfrac{d}{dt}\dfrac{\delta V}{\delta y_n}=m_ny''_n; \\
\dfrac{d}{dt}\dfrac{\delta V}{\delta z_1}=m_1z''_1; & \dfrac{d}{dt}\dfrac{\delta V}{\delta z_2}=m_2z''_2; & \dfrac{d}{dt}\dfrac{\delta V}{\delta z_n}=m_nz''_n;
\end{array} \qquad \text{(H.)}$$

for the second group,

$$\begin{array}{lll}
\dfrac{d}{dt}\dfrac{\delta V}{\delta a_1}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta a_2}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta a_n}=0; \\
\dfrac{d}{dt}\dfrac{\delta V}{\delta b_1}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta b_2}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta b_n}=0; \\
\dfrac{d}{dt}\dfrac{\delta V}{\delta c_1}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta c_2}=0; & \dfrac{d}{dt}\dfrac{\delta V}{\delta c_n}=0;
\end{array} \qquad \text{(I.)}$$

and finally, for the last equation,

$$\frac{d}{dt}\frac{\delta V}{\delta H}=1. \qquad \text{(K.)}$$

By combining the equations (C.) with their differentials (H.), and with the relation (F.), we deduced, in the foregoing number, the known equations of motion (3.); and we are now to show the consistence of the same intermediate integrals (C.) with the group of differentials (I.), which have been deduced from the final integrals.

The first equation of the group (I.) may be developed thus:

$$\begin{aligned}
0 &= x'_1\frac{\delta^2 V}{\delta a_1\,\delta x_1}+x'_2\frac{\delta^2 V}{\delta a_1\,\delta x_2}+\ldots+x'_n\frac{\delta^2 V}{\delta a_1\,\delta x_n} \\
&\quad +y'_1\frac{\delta^2 V}{\delta a_1\,\delta y_1}+y'_2\frac{\delta^2 V}{\delta a_1\,\delta y_2}+\ldots+y'_n\frac{\delta^2 V}{\delta a_1\,\delta y_n} \\
&\quad +z'_1\frac{\delta^2 V}{\delta a_1\,\delta z_1}+z'_2\frac{\delta^2 V}{\delta a_1\,\delta z_2}+\ldots+z'_n\frac{\delta^2 V}{\delta a_1\,\delta z_n},
\end{aligned} \qquad \text{(14.)}$$

and the others may be similarly developed. In order, therefore, to show that they are satisfied by the group (C.), it is sufficient to prove that the following equations are true,

$$\left.\begin{aligned}
0&=\frac{\delta}{\delta a_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}, \\
0&=\frac{\delta}{\delta b_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}, \\
0&=\frac{\delta}{\delta c_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\},
\end{aligned}\right\} \qquad \text{(L.)}$$

the integer i receiving any value from 1 to n inclusive; which may be shown at once, and the required verification thereby be obtained, if we merely take the variation of the relation (F.) with respect to the initial coordinates, as in the former verification we took its variation with respect to the final coordinates, and so obtained results which agreed with the known equations of motion, and which may be thus collected,

$$\left.\begin{aligned}\frac{\delta}{\delta x_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}&=\frac{\delta U}{\delta x_i};\\ \frac{\delta}{\delta y_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}&=\frac{\delta U}{\delta y_i};\\ \frac{\delta}{\delta z_i}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}&=\frac{\delta U}{\delta z_i}.\end{aligned}\right\}\qquad\text{(M.)}$$

The same relation (F.), by being varied with respect to the quantity H, conducts to the expression

$$\frac{\delta}{\delta H}\Sigma\cdot\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2+\left(\frac{\delta V}{\delta z}\right)^2\right\}=1;\qquad\text{(N.)}$$

and this, when developed, agrees with the equation (K.), which is a new verification of the consistence of our foregoing results. Nor would it have been much more difficult, by the help of the foregoing principles, to have integrated directly our integrals of the first order, and so to have deduced in a different way our final integral system.

6. It may be considered as still another verification of our own general integral equations, to show that they include not only the known law of living force, or the integral expressing that law, but also the six other known integrals of the first order, which contain the law of motion of the centre of gravity, and the law of description of areas. For this purpose, it is only necessary to observe that it evidently follows from the conception of our characteristic function V, that this function depends on the initial and final positions of the attracting or repelling points of a system, not as referred to any foreign standard, but only as compared with one another; and therefore that this function will not vary, if without making any real change in either initial or final configuration, or in the relation of these to each other, we alter at once all the initial and all the final positions of the points of the system, by any common motion, whether of translation or of rotation.* Now by considering three coordinate translations, we obtain the three following partial differential equations of the first order, which the function V must satisfy,

$$\left.\begin{aligned}\Sigma\frac{\delta V}{\delta x}+\Sigma\frac{\delta V}{\delta a}&=0;\\ \Sigma\frac{\delta V}{\delta y}+\Sigma\frac{\delta V}{\delta b}&=0;\\ \Sigma\frac{\delta V}{\delta z}+\Sigma\frac{\delta V}{\delta c}&=0;\end{aligned}\right.\qquad\text{(O.)}$$

and by considering three coordinate rotations, we obtain these three other relations between the partial differential coefficients of the same order of the same characteristic function,

---

* [The function V is obviously independent of the choice of coordinate axes and the motions considered are equivalent to changes in the coordinate axes. Hence the value of V does not alter under such motions.]

$$\left.\begin{aligned}\Sigma\left(x\frac{\delta V}{\delta y}-y\frac{\delta V}{\delta x}\right)+\Sigma\left(a\frac{\delta V}{\delta b}-b\frac{\delta V}{\delta a}\right)=0;\\ \Sigma\left(y\frac{\delta V}{\delta z}-z\frac{\delta V}{\delta y}\right)+\Sigma\left(b\frac{\delta V}{\delta c}-c\frac{\delta V}{\delta b}\right)=0;\\ \Sigma\left(z\frac{\delta V}{\delta x}-x\frac{\delta V}{\delta z}\right)+\Sigma\left(c\frac{\delta V}{\delta a}-a\frac{\delta V}{\delta c}\right)=0;\end{aligned}\right\}\qquad\text{(P.)}$$

and if we change the final [differential] coefficients of V to the final components of momentum, and the initial coefficients to the initial components taken negatively, according to the dynamical properties of this function expressed by the integrals (C.) and (D.), we shall change these partial differential equations (O.), (P.), to the following,

$$\Sigma\,.\,mx'=\Sigma\,.\,ma';\quad \Sigma\,.\,my'=\Sigma\,.\,mb';\quad \Sigma\,.\,mz'=\Sigma\,.\,mc';\qquad(15.)$$

and

$$\left.\begin{aligned}\Sigma\,.\,m\,(xy'-yx')&=\Sigma\,.\,m\,(ab'-ba');\\ \Sigma\,.\,m\,(yz'-zy')&=\Sigma\,.\,m\,(bc'-cb');\\ \Sigma\,.\,m\,(zx'-xz')&=\Sigma\,.\,m\,(ca'-ac').\end{aligned}\right\}\qquad(16.)$$

In this manner, therefore, we can deduce from the properties of our characteristic function the six other known integrals above mentioned, in addition to that seventh which contains the law of living force, and which assisted in the discovery of our method.

Introduction of relative or polar Coordinates, or other marks of position of a System.

7. The property of our characteristic function, by which it depends only on the internal or mutual relations between the positions initial and final of the points of an attracting or repelling system, suggests an advantage in employing internal or relative coordinates; and from the analogy of other applications of algebraical methods to researches of a geometrical kind, it may be expected that polar and other marks of position will also often be found useful. Supposing, therefore, that the $3n$ final coordinates $x_1, y_1, z_1, \dots x_n, y_n, z_n$ have been expressed as functions of $3n$ other variables, $\eta_1, \eta_2, \dots \eta_{3n}$, and that the $3n$ initial coordinates have in like manner been expressed as functions of $3n$ similar quantities, which we shall call $e_1, e_2, \dots e_{3n}$, we shall proceed to assign a general method for introducing these new marks of position into the expressions of our fundamental relations.

For this purpose we have only to transform the law of varying action, or the fundamental formula (A.), by transforming the two sums,

$$\Sigma\,.\,m\,(x'\delta x+y'\delta y+z'\delta z),\quad\text{and}\quad \Sigma\,.\,m\,(a'\delta a+b'\delta b+c'\delta c),$$

which it involves, and which are respectively equivalent to the following more developed expressions,

$$\left.\begin{aligned}\Sigma\,.\,m\,(x'\delta x+y'\delta y+z'\delta z)=m_1\,(x_1'\delta x_1+y_1'\delta y_1+z_1'\delta z_1)\\ +m_2\,(x_2'\delta x_2+y_2'\delta y_2+z_2'\delta z_2)\\ +\&\text{c.}+m_n\,(x_n'\delta x_n+y_n'\delta y_n+z_n'\delta z_n);\end{aligned}\right\}\qquad(17.)$$

$$\left.\begin{aligned}\Sigma\,.\,m\,(a'\delta a+b'\delta b+c'\delta c)=m_1\,(a_1'\delta a_1+b_1'\delta b_1+c_1'\delta c_1)\\ +m_2\,(a_2'\delta a_2+b_2'\delta b_2+c_2'\delta c_2)\\ +\&\text{c.}+m_n\,(a_n'\delta a_n+b_n'\delta b_n+c_n'\delta c_n).\end{aligned}\right\}\qquad(18.)$$

Now x_i being by supposition a function of the $3n$ new marks of position $\eta_1, \ldots \eta_{3n}$, its variation δx_i, and its differential coefficient x'_i, may be thus expressed:

$$\delta x_i = \frac{\delta x_i}{\delta \eta_1}\,\delta\eta_1 + \frac{\delta x_i}{\delta \eta_2}\,\delta\eta_2 + \ldots + \frac{\delta x_i}{\delta \eta_{3n}}\,\delta\eta_{3n}; \qquad (19.)$$

$$x'_i = \frac{\delta x_i}{\delta \eta_1}\,\eta'_1 + \frac{\delta x_i}{\delta \eta_2}\,\eta'_2 + \ldots + \frac{\delta x_i}{\delta \eta_{3n}}\,\eta'_{3n}; \qquad (20.)$$

and similarly for y_i and z_i. If, then, we consider x'_i as a function, by (20.), of $\eta'_1, \ldots \eta'_{3n}$, involving also in general $\eta_1, \ldots \eta_{3n}$, and if we take its partial differential coefficients of the first order with respect to $\eta'_1, \ldots \eta'_{3n}$, we find the relations,

$$\frac{\delta x'_i}{\delta \eta'_1} = \frac{\delta x_i}{\delta \eta_1}; \quad \frac{\delta x'_i}{\delta \eta'_2} = \frac{\delta x_i}{\delta \eta_2}; \qquad \frac{\delta x'_i}{\delta \eta'_{3n}} = \frac{\delta x_i}{\delta \eta_{3n}}, \qquad (21.)$$

and therefore we obtain these new expressions for the variations δx_i, δy_i, δz_i,

$$\begin{aligned}
\delta x_i &= \frac{\delta x'_i}{\delta \eta'_1}\,\delta\eta_1 + \frac{\delta x'_i}{\delta \eta'_2}\,\delta\eta_2 + \ldots + \frac{\delta x'_i}{\delta \eta'_{3n}}\,\delta\eta_{3n},\\
\delta y_i &= \frac{\delta y'_i}{\delta \eta'_1}\,\delta\eta_1 + \frac{\delta y'_i}{\delta \eta'}\,\delta\eta_2 + \ldots + \frac{\delta y'_i}{\delta \eta'_{3n}}\,\delta\eta_{3n}, \qquad (22.)\\
\delta z_i &= \frac{\delta z'_i}{\delta \eta'_1}\,\delta\eta_1 + \frac{\delta z'_i}{\delta \eta'_2}\,\delta\eta_2 + \ldots + \frac{\delta z'_i}{\delta \eta'_{3n}}\,\delta\eta_{3n}.
\end{aligned}$$

Substituting these expressions (22.) for the variations in the sum (17.), we easily transform it into the following,

$$\begin{aligned}
\Sigma\,.\,m\,(x'\delta x + y'\delta y + z'\delta z) &= \Sigma\,.\,m\left(x'\frac{\delta x'}{\delta\eta'_1} + y'\frac{\delta y'}{\delta\eta'_1} + z'\frac{\delta z'}{\delta\eta'_1}\right).\,\delta\eta_1\\
&\quad + \Sigma\,.\,m\left(x'\frac{\delta x'}{\delta\eta'_2} + y'\frac{\delta y'}{\delta\eta'_2} + z'\frac{\delta z'}{\delta\eta'_2}\right).\,\delta\eta_2\\
&\quad + \&\text{c.} + \Sigma\,.\,m\left(x'\frac{\delta x'}{\delta\eta'_{3n}} + y'\frac{\delta y'}{\delta\eta'_{3n}} + z'\frac{\delta z'}{\delta\eta'_{3n}}\right).\,\delta\eta_{3} \qquad (23.)\\
&= \frac{\delta T}{\delta\eta'_1}\,\delta\eta_1 + \frac{\delta T}{\delta\eta'_2}\,\delta\eta_2 + \ldots + \frac{\delta T}{\delta\eta'_{3n}}\,\delta\eta_{3n};
\end{aligned}$$

T being the same quantity as before, namely, the half of the final living force of the system, but being now considered as a function of $\eta'_1, \ldots \eta'_{3n}$, involving also the masses, and in general $\eta_1, \ldots \eta_{3n}$, and obtained by substituting for the quantities x', y', z' their values of the form (20.) in the equation of definition

$$T = \tfrac{1}{2}\Sigma\,.\,m\,(x'^2 + y'^2 + z'^2). \qquad (4.)$$

In like manner we find this transformation for the sum (18.),

$$\Sigma\,.\,m\,(a'\delta a + b'\delta b + c'\delta c) = \frac{\delta T_0}{\delta e'_1}\,\delta e_1 + \frac{\delta T_0}{\delta e'_2}\,\delta e_2 + \ldots + \frac{\delta T_0}{\delta e'_{3n}}\,\delta e_{3n}. \qquad (24.)$$

The law of varying action, or the formula (A.), becomes therefore, when expressed by the present more general coordinates or marks of position,[*]

$$\delta V = \Sigma \,.\, \frac{\delta T}{\delta \eta'} \delta \eta - \Sigma \,.\, \frac{\delta T_0}{\delta e'} \delta e + t \delta H; \tag{Q.}$$

and instead of the groups (C.) and (D.), into which, along with the equation (E.), this law resolved itself before, it gives now these other groups,

$$\frac{\delta V}{\delta \eta_1} = \frac{\delta T}{\delta \eta'_1}; \quad \frac{\delta V}{\delta \eta_2} = \frac{\delta T}{\delta \eta'_2}; \quad \ldots \quad \frac{\delta V}{\delta \eta_{3n}} = \frac{\delta T}{\delta \eta'_{3n}} \tag{R.}$$

and

$$\frac{\delta V}{\delta e_1} = -\frac{\delta T_0}{\delta e'_1}; \quad \frac{\delta V}{\delta e_2} = -\frac{\delta T_0}{\delta e'_2}; \quad \ldots \quad \frac{\delta V}{\delta e_{3n}} = -\frac{\delta T_0}{\delta e'_{3n}}. \tag{S.}$$

The quantities $e_1, e_2, \ldots e_{3n}$, and $e'_1, e'_2, \ldots e'_{3n}$, are now the initial data respecting the manner of motion of the system; and the $3n$ final integrals, connecting these $6n$ initial data, and the n masses, with the time t, and with the $3n$ final or varying quantities $\eta_1, \eta_2, \ldots \eta_{3n}$, which mark the varying positions of the n moving points of the system, are now to be obtained by eliminating the auxiliary constant H between the $3n + 1$ equations (S.) and (E.); while the $3n$ intermediate integrals, or integrals of the first order, which connect the same varying marks of position and their first differential coefficients with the time, the masses, and the initial marks of position, are the result of elimination of the same auxiliary constant H between the equations (R.) and (E.). Our fundamental formula, and intermediate and final integrals, can therefore be very simply expressed with any new sets of coordinates; and the partial differential equations (F.), (G.), which our characteristic function V must satisfy, and which are, as we have said, essential in the theory of that function, can also easily be expressed with any such transformed coordinates, by merely combining the final and initial expressions of the law of living force,

$$T = U + H, \tag{6.}$$

$$T_0 = U_0 + H, \tag{7.}$$

with the new groups (R.) and (S.). For this purpose we must now consider the function U, of the masses and mutual distances of the several points of the system, as depending on the new

---

[*] [Equation (Q.) can be obtained most simply as follows without introducing the cartesian coordinates.

Let T be a function of the η's and η''s, being homogeneous of the second degree in the latter, and satisfying $T = U + H$, where U is a function of the η's, and H a constant along each of the curves considered. Writing

$$V = 2\int_0^t T\,dt = \int_0^t \Sigma \frac{\delta T}{\delta \eta'} d\eta,$$

we have

$$\delta V = \int_0^t \Sigma \delta \left(\frac{\delta T}{\delta \eta'}\right) d\eta + \int_0^t \Sigma \frac{\delta T}{\delta \eta'} \delta d\eta$$

$$= \Sigma \frac{\delta T}{\delta \eta'} \delta \eta - \Sigma \frac{\delta T_0}{\delta e'} \delta e + \int_0^t \Sigma \delta \left(\frac{\delta T}{\delta \eta'}\right) d\eta - \int_0^t \Sigma d \left(\frac{\delta T}{\delta \eta'}\right) \delta \eta.$$

But

$$\Sigma \frac{\delta T}{\delta \eta'} \eta' = 2T = T + U + H,$$

and therefore

$$\Sigma \delta \left(\frac{\delta T}{\delta \eta'}\right) \eta' = \Sigma \frac{\delta T}{\delta \eta} \delta \eta + \Sigma \frac{\delta U}{\delta \eta} \delta \eta + \delta H,$$

so that we find

$$\delta V = \Sigma \frac{\delta T}{\delta \eta'} \delta \eta - \Sigma \frac{\delta T_0}{\delta e'} \delta e + t \delta H - \int_0^t \Sigma \left(\frac{d}{dt} \frac{\delta T}{\delta \eta'} - \frac{\delta T}{\delta \eta} - \frac{\delta U}{\delta \eta}\right) \delta \eta \, dt.$$

This, by the principle of least action, leads to the equations (Z.) and equation (Q.) follows immediately.]

marks of position $\eta_1, \eta_2, \dots \eta_{3n}$; and the analogous function U_0, as depending similarly on the initial quantities $e_1, e_2, \dots e_{3n}$; we must also suppose that T is expressed (as it may) as a function of its own coefficients $\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \dots \frac{\delta T}{\delta \eta'_{3n}}$, which will always be, with respect to these, homogeneous of the second dimension, and may also involve explicitly the quantities $\eta_1, \eta_2, \dots \eta_{3n}$; and that T_0 is expressed as a similar function of its coefficients $\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \dots \frac{\delta T_0}{\delta e'_{3n}}$; so that

$$\left.\begin{aligned} T &= F\left(\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \dots \frac{\delta T}{\delta \eta'_{3n}}\right), \\ T_0 &= F\left(\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \dots \frac{\delta T_0}{\delta e'_{3n}}\right) \end{aligned}\right\} \tag{25.}$$

and that then these coefficients of T and T_0 are changed to their values (R.) and (S.), so as to give, instead of (F.) and (G.), two other transformed equations, namely,

$$F\left(\frac{\delta V}{\delta \eta_1}, \frac{\delta V}{\delta \eta_2}, \dots \frac{\delta V}{\delta \eta_{3n}}\right) = U + H, \tag{T.}$$

and, on account of the homogeneity and dimension of T_0,

$$F\left(\frac{\delta V}{\delta e_1}, \frac{\delta V}{\delta e_2}, \dots \frac{\delta V}{\delta e_{3n}}\right) = U_0 + H. \tag{U.}$$

8. Nor is there any difficulty in deducing analogous transformations for the known differential equations of motion of the second order, of any system of free points, by taking the variation of the new form (T.) of the law of living force, and by attending to the dynamical meanings of the coefficients of our characteristic function. For if we observe that the final living force $2T$, when considered as a function of $\eta_1, \eta_2, \dots \eta_{3n}$, and of $\eta'_1, \eta'_2, \dots \eta'_{3n}$, is necessarily homogeneous of the second dimension with respect to the latter set of variables, and must therefore satisfy the condition

$$2T = \eta'_1 \frac{\delta T}{\delta \eta'_1} + \eta'_2 \frac{\delta T}{\delta \eta'_2} + \dots + \eta'_{3n} \frac{\delta T}{\delta \eta'_{3n}}, \tag{26.}$$

we shall perceive that its total variation,

$$\left.\begin{aligned} \delta T = {} & \frac{\delta T}{\delta \eta_1} \delta \eta_1 + \frac{\delta T}{\delta \eta_2} \delta \eta_2 + \dots + \frac{\delta T}{\delta \eta_{3n}} \delta \eta_{3n} \\ & + \frac{\delta T}{\delta \eta'_1} \delta \eta'_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta'_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta'_{3n}, \end{aligned}\right\} \tag{27.}$$

may be put under the form

$$\left.\begin{aligned} \delta T &= \eta'_1 \delta \frac{\delta T}{\delta \eta'_1} + \eta'_2 \delta \frac{\delta T}{\delta \eta'_2} + \dots + \eta'_{3n} \delta \frac{\delta T}{\delta \eta'_{3n}} \\ & \quad - \frac{\delta T}{\delta \eta_1} \delta \eta_1 - \frac{\delta T}{\delta \eta_2} \delta \eta_2 - \dots - \frac{\delta T}{\delta \eta_{3n}} \delta \eta_{3n} \\ &= \Sigma \,.\, \eta' \delta \frac{\delta T}{\delta \eta'} - \Sigma \,.\, \frac{\delta T}{\delta \eta} \delta \eta \\ &= \Sigma \left(\eta' \delta \frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta} \delta \eta \right), \end{aligned}\right\} \tag{28.}$$

and therefore that the total variation of the new partial differential equation (T.) may be thus written,

$$\Sigma\left(\eta'\delta\frac{\delta V}{\delta\eta}-\frac{\delta T}{\delta\eta}\delta\eta\right)=\Sigma\,.\frac{\delta U}{\delta\eta}\delta\eta+\delta H: \tag{V.}$$

in which, if we observe that $\eta'=\frac{d\eta}{dt}$, and that the quantities of the form η are the only ones which vary with the time, we shall see that

$$\Sigma\,.\,\eta'\delta\frac{\delta V}{\delta\eta}=\Sigma\left(\frac{d}{dt}\frac{\delta V}{\delta\eta}\,.\,\delta\eta+\frac{d}{dt}\frac{\delta V}{\delta e}\,.\,\delta e\right)+\frac{d}{dt}\frac{\delta V}{\delta H}\,.\,\delta H, \tag{29.}$$

because the identical equation $\delta dV=d\delta V$ gives, when developed,

$$\begin{aligned}&\Sigma\left(\delta\frac{\delta V}{\delta\eta}\,.\,d\eta+\delta\frac{\delta V}{\delta e}\,.\,de\right)+\delta\frac{\delta V}{\delta H}\,.\,dH\\ &=\Sigma\left(d\frac{\delta V}{\delta\eta}\,.\,\delta\eta+d\frac{\delta V}{\delta e}\,.\,\delta e\right)+d\frac{\delta V}{\delta H}\,.\,\delta H.\end{aligned} \tag{30.}$$

Decomposing, therefore, the expression (V.), for the variation of half the living force, into as many separate equations as it contains independent variations, we obtain, not only the equation

$$\frac{d}{dt}\frac{\delta V}{\delta H}=1, \tag{K.}$$

which had already presented itself, and the group

$$\frac{d}{dt}\frac{\delta V}{\delta e_1}=0,\quad \frac{d}{dt}\frac{\delta V}{\delta e_2}=0,\qquad \frac{d}{dt}\frac{\delta V}{\delta e_{3n}}=0, \tag{W.}$$

which might have been at once obtained by differentiation from the final integrals (S.), but also a group of $3n$ other equations of the form

$$\frac{d}{dt}\frac{\delta V}{\delta\eta}-\frac{\delta T}{\delta\eta}=\frac{\delta U}{\delta\eta}, \tag{X.}$$

which give, by the intermediate integrals (R.),

$$\frac{d}{dt}\frac{\delta T}{\delta\eta'}-\frac{\delta T}{\delta\eta}=\frac{\delta U}{\delta\eta}: \tag{Y.}$$

that is, more fully,

$$\begin{aligned}&\frac{d}{dt}\frac{\delta T}{\delta\eta_1'}-\frac{\delta T}{\delta\eta_1}=\frac{\delta U}{\delta\eta_1};\\ &\frac{d}{dt}\frac{\delta T}{\delta\eta_2'}-\frac{\delta T}{\delta\eta_2}=\frac{\delta U}{\delta\eta_2};\\ &\\ &\frac{d}{dt}\frac{\delta T}{\delta\eta_{3n}'}-\frac{\delta T}{\delta\eta_{3n}}=\frac{\delta U}{\delta\eta_{3n}}.\end{aligned} \tag{Z.}$$

These last transformations of the differential equations of motion of the second order, of an attracting or repelling system, coincide in all respects (a slight difference of notation excepted,) with the elegant canonical forms in the *Mécanique Analytique* of Lagrange*; but it seemed worth while to deduce them here anew, from the properties of our characteristic function. And if we were to suppose (as it has often been thought convenient and even necessary to do,) that the n points of a system are not entirely free, nor subject only to their own mutual attractions or repulsions, but connected by any geometrical conditions, and influenced by any foreign agencies, consistent with the law of conservation of living force; so that the number of independent marks of position should be now less numerous, and the force-function U less simple than before; it might still be proved, by a reasoning very similar to the foregoing, that on these suppositions also (which, however, the dynamical spirit is tending more and more to exclude,*) the accumulated living force or action V of the system is a *characteristic motion-function* of the kind already explained; having the same law and formula of variation, which are susceptible of the same transformations; obliged to satisfy in the same way a final and an initial relation between its partial differential coefficients of the first order; conducting, by the variation of one of these two relations, to the same canonical forms assigned by Lagrange for the differential equations of motion; and furnishing, on the same principles as before, their intermediate and their final integrals. To those imaginable cases, indeed, in which the law of living force no longer holds, our method also would not apply; but it appears to be the growing conviction of the persons who have meditated the most profoundly on the mathematical dynamics of the universe, that these are cases suggested by insufficient views of the mutual actions of body.

---

* [*Méc. Anal.* 3rd ed. Tome I, pp. 290–292.]

9. It results from the foregoing remarks, that in order to apply our method of the characteristic function to any problem of dynamics respecting any moving system, the known law of living force is to be combined with our law of varying action; and that the general expression of this latter law is to be obtained in the following manner. We are first to express the quantity T, namely, the half of the living force of the system, as a function (which will always be homogeneous of the second dimension,) of the differential coefficients or rates of increase η_1', η_2', &c., of any rectangular coordinates, or other marks of position of the system: we are next to take the variation of this homogeneous function with respect to those rates of increase, and to change the variations of those rates $\delta\eta_1'$, $\delta\eta_2'$, &c., to the variations $\delta\eta_1$, $\delta\eta_2$, &c., of the marks of position themselves; and then to subtract the initial from the final value of the result, and to equate the remainder to $\delta V - t\delta H$.† A slight consideration will show that this general rule or process for obtaining the variation of the characteristic function V is applicable even when the marks of position η_1, η_2, &c. are not all independent of each other; which will happen when they have been made, from any motive of convenience, more numerous than the rectangular coordinates of the several points of the system. For if we suppose that the $3n$ rectangular coordinates $x_1, y_1, z_1, \ldots x_n, y_n, z_n$ have been expressed by any transformation as functions of $3n+k$ other marks of position, $\eta_1, \eta_2, \ldots \eta_{3n+k}$, which must therefore be connected by k equations of condition,

$$\left.\begin{array}{l} 0=\phi_1(\eta_1, \eta_2, \ldots \eta_{3n+k}), \\ 0=\phi_2(\eta_1, \eta_2, \ldots \eta_{3n+k}), \\ \\ 0=\phi_k(\eta_1, \eta_2, \ldots \eta_{3n+k}), \end{array}\right\} \tag{31.}$$

---

* [See Reference to Boscovich, p. 104.]

† $\left[\Sigma \frac{\partial T}{\partial \eta'} \delta\eta - \Sigma \frac{\partial T}{\partial e'} \delta e = \delta V - t\delta H.\right]$

giving k of the new marks of position as functions of the remaining $3n$,

$$\left.\begin{aligned}
\eta_{3n+1} &= \psi_1(\eta_1, \eta_2, \dots \eta_{3n}),\\
\eta_{3n+2} &= \psi_2(\eta_1, \eta_2, \dots \eta_{3n}),\\
&\\
\eta_{3n+k} &= \psi_k(\eta_1, \eta_2, \dots \eta_{3n}),
\end{aligned}\right\} \qquad (32.)$$

the expression

$$T = \tfrac{1}{2}\Sigma \,.\, m\,(x'^2 + y'^2 + z'^2) \qquad (4.)$$

will become, by the introduction of these new variables, a homogeneous function of the second dimension of the $3n+k$ rates of increase $\eta'_1, \eta'_2, \dots \eta'_{3n+k}$, involving also in general $\eta_1, \eta_2, \dots \eta_{3n+k}$, and having a variation which may be thus expressed:

$$\left.\begin{aligned}
\delta T = {}& \left(\frac{\delta T}{\delta \eta'_1}\right)\delta\eta'_1 + \left(\frac{\delta T}{\delta \eta'_2}\right)\delta\eta'_2 + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}}\right)\delta\eta'_{3n+k}\\
&+ \left(\frac{\delta T}{\delta \eta_1}\right)\delta\eta_1 + \left(\frac{\delta T}{\delta \eta_2}\right)\delta\eta_2 + \dots + \left(\frac{\delta T}{\delta \eta_{3n+k}}\right)\delta\eta_{3n+k};
\end{aligned}\right\} \qquad (33.)$$

or in this other way,

$$\left.\begin{aligned}
\delta T = {}& \frac{\delta T}{\delta \eta'_1}\delta\eta'_1 + \frac{\delta T}{\delta \eta'_2}\delta\eta'_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}}\delta\eta'_{3n}\\
&+ \frac{\delta T}{\delta \eta_1}\delta\eta_1 + \frac{\delta T}{\delta \eta_2}\delta\eta_2 + \dots + \frac{\delta T}{\delta \eta_{3n}}\delta\eta_{3n},
\end{aligned}\right\} \qquad (34.)$$

on account of the relations (32.), which give, when differentiated with respect to the time,

$$\begin{aligned}
\eta'_{3n+1} &= \eta'_1\frac{\delta\psi_1}{\delta\eta_1} + \eta'_2\frac{\delta\psi_1}{\delta\eta_2} + \dots + \eta'_{3n}\frac{\delta\psi_1}{\delta\eta_{3n}},\\
\eta'_{3n+2} &= \eta'_1\frac{\delta\psi_2}{\delta\eta_1} + \eta'_2\frac{\delta\psi_2}{\delta\eta_2} + \dots + \eta'_{3n}\frac{\delta\psi_2}{\delta\eta_{3n}},\\
&\\
\eta'_{3n+k} &= \eta'_1\frac{\delta\psi_k}{\delta\eta_1} + \eta'_2\frac{\delta\psi_k}{\delta\eta_2} + \dots + \eta'_{3n}\frac{\delta\psi_k}{\delta\eta_{3n}};
\end{aligned} \qquad (35.)$$

and therefore, attending only to the variations of quantities of the form η',

$$\begin{aligned}
\delta\eta'_{3n+1} &= \frac{\delta\psi_1}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_1}{\delta\eta_2}\delta\eta'_2 + \dots + \frac{\delta\psi_1}{\delta\eta_{3n}}\delta\eta'_{3n},\\
\delta\eta'_{3n+2} &= \frac{\delta\psi_2}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_2}{\delta\eta_2}\delta\eta'_2 + \dots + \frac{\delta\psi_2}{\delta\eta_{3n}}\delta\eta'_{3n},\\
&\\
\delta\eta'_{3n+k} &= \frac{\delta\psi_k}{\delta\eta_1}\delta\eta'_1 + \frac{\delta\psi_k}{\delta\eta_2}\delta\eta'_2 + \dots + \frac{\delta\psi_k}{\delta\eta_{3n}}\delta\eta'_{3n}.
\end{aligned} \qquad (36.)$$

Comparing the two expressions (33.) and (34.), we find by (36.) the relations

$$\left.\begin{aligned}
\frac{\delta T}{\delta\eta'_1} &= \left(\frac{\delta T}{\delta\eta'_1}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_1} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_1} + \dots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_1},\\
\frac{\delta T}{\delta\eta'_2} &= \left(\frac{\delta T}{\delta\eta'_2}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_2} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_2} + \dots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_2},\\
&\dots\dots\\
\frac{\delta T}{\delta\eta'_{3n}} &= \left(\frac{\delta T}{\delta\eta'_{3n}}\right) + \left(\frac{\delta T}{\delta\eta'_{3n+1}}\right)\frac{\delta\psi_1}{\delta\eta_{3n}} + \left(\frac{\delta T}{\delta\eta'_{3n+2}}\right)\frac{\delta\psi_2}{\delta\eta_{3n}} + \dots + \left(\frac{\delta T}{\delta\eta'_{3n+k}}\right)\frac{\delta\psi_k}{\delta\eta_{3n}};
\end{aligned}\right\} \qquad (37.)$$

which give, by (32.),

$$\frac{\delta T}{\delta \eta_1'}\,\delta\eta_1+\frac{\delta T}{\delta \eta_2'}\,\delta\eta_2+\ldots+\frac{\delta T}{\delta \eta_{3n}'}\,\delta\eta_{3n}=\left(\frac{\delta T}{\delta \eta_1'}\right)\delta\eta_1+\left(\frac{\delta T}{\delta \eta_2'}\right)\delta\eta_2+\ldots+\left(\frac{\delta T}{\delta \eta_{3n+k}'}\right)\delta\eta_{3n+k}; \qquad (38.)$$

we may therefore put the expression (Q.) under the following more general form,

$$\delta V=\Sigma\,.\left(\frac{\delta T}{\delta\eta'}\right)\delta\eta-\Sigma\,.\left(\frac{\delta T_0}{\delta e'}\right)\delta e+t\,\delta H, \qquad (\mathrm{A}^1.)$$

the coefficients $\left(\frac{\delta T}{\delta\eta'}\right)$ being formed by treating all the $3n+k$ quantities $\eta_1', \eta_2', \ldots \eta_{3n+k}'$, as independent; which was the extension above announced, of the rule for forming the variation of the characteristic function V.

We cannot, however, immediately decompose this new expression (A¹.) for δV, as we did the expression (Q.), by treating all the variations $\delta\eta$, δe as independent; but we may decompose it so, if we previously combine it with the final equations of condition (31.), and with the analogous initial equations of condition, namely,

$$\left.\begin{aligned}0&=\Phi_1(e_1, e_2, \ldots e_{3n+k}),\\0&=\Phi_2(e_1, e_2, \ldots e_{3n+k}),\\&\\0&=\Phi_k(e_1, e_2, \ldots e_{3n+k}),\end{aligned}\right\} \qquad (39.)$$

which we may do by adding the variations of the connecting functions $\phi_1, \ldots \phi_k, \Phi_1, \ldots \Phi_k$, multiplied respectively by factors to be determined, $\lambda_1, \ldots \lambda_k, \Lambda_1, \ldots \Lambda_k$. In this manner the law of varying action takes this new form,

$$\delta V=\Sigma\,.\left(\frac{\delta T}{\delta\eta'}\right)\delta\eta-\Sigma\,.\left(\frac{\delta T_0}{\delta e'}\right)\delta e+t\,\delta H+\Sigma\,.\,\lambda\,\delta\phi+\Sigma\,.\,\Lambda\,\delta\Phi; \qquad (\mathrm{B}^1.)$$

and decomposes itself into $6n+2k+1$ separate expressions, for the partial differential coefficients of the first order of the characteristic function V, namely, into the following,

$$\left.\begin{aligned}\frac{\delta V}{\delta\eta_1}&=\left(\frac{\delta T}{\delta\eta_1'}\right)+\lambda_1\frac{\delta\phi_1}{\delta\eta_1}+\lambda_2\frac{\delta\phi_2}{\delta\eta_1}+\ldots+\lambda_k\frac{\delta\phi_k}{\delta\eta_1},\\\frac{\delta V}{\delta\eta_2}&=\left(\frac{\delta T}{\delta\eta_2'}\right)+\lambda_1\frac{\delta\phi_1}{\delta\eta_2}+\lambda_2\frac{\delta\phi_2}{\delta\eta_2}+\ldots+\lambda_k\frac{\delta\phi_k}{\delta\eta_2},\\&\cdots\cdots\\\frac{\delta V}{\delta\eta_{3n+k}}&\quad\left.\frac{\delta T}{\delta\eta_{3n+k}'}\right)+\lambda_1\frac{\delta\phi_1}{\delta\eta_{3n+k}}+\ldots+\lambda_k\frac{\delta\phi_k}{\delta\eta_{3n+k}};\end{aligned}\right. \qquad (\mathrm{C}^1.)$$

and

$$\left.\begin{aligned}\frac{\delta V}{\delta e_1}&=-\left(\frac{\delta T_0}{\delta e_1'}\right)+\Lambda_1\frac{\delta\Phi_1}{\delta e_1}+\Lambda_2\frac{\delta\Phi_2}{\delta e_1}+\ldots+\Lambda_k\frac{\delta\Phi_k}{\delta e_1},\\\frac{\delta V}{\delta e_2}&=-\left(\frac{\delta T_0}{\delta e_2'}\right)+\Lambda_1\frac{\delta\Phi_1}{\delta e_2}+\Lambda_2\frac{\delta\Phi_2}{\delta e_2}+\ldots+\Lambda_k\frac{\delta\Phi_k}{\delta e_2},\\&\cdots\cdots\\\frac{\delta V}{3n+k}&=-\left(\frac{\delta T_0}{\delta e_{3n+k}'}\right)\quad{}_1\frac{\delta\Phi_1}{\delta e_{3n+k}}+\ldots+\Lambda_k\frac{\delta\Phi_k}{\delta e_{3n+k}};\end{aligned}\right. \qquad (\mathrm{D}^1.)$$

besides the old equation (E.). The analogous introduction of multipliers in the canonical forms of Lagrange, for the differential equations of motion of the second order, by which a sum such as $\Sigma . \lambda \frac{\delta \phi}{\delta \eta}$ is added to $\frac{\delta U}{\delta \eta}$ in the second member of the formula (Y.), is also easily justified on the principles of the present essay.

Separation of the relative motion of a system from the motion of its centre of gravity; characteristic function for such relative motion, and law of its variation.

10. As an example of the foregoing transformations, and at the same time as an important application, we shall now introduce relative coordinates, $x_{,}, y_{,}, z_{,}$, referred to an internal origin $x_{,,}, y_{,,}, z_{,,}$; that is, we shall put

$$x_i = x_{,i} + x_{,,}, \quad y_i = y_{,i} + y_{,,}, \quad z_i = z_{,i} + z_{,,}, \tag{40.}$$

and in like manner

$$a_i = a_{,i} + a_{,,}, \quad b_i = b_{,i} + b_{,,}, \quad c_i = c_{,i} + c_{,,}; \tag{41.}$$

together with the differentiated expressions

$$x'_i = x'_{,i} + x'_{,,}, \quad y'_i = y'_{,i} + y'_{,,}, \quad z'_i = z'_{,i} + z'_{,,}, \tag{42.}$$

and

$$a'_i = a'_{,i} + a'_{,,}, \quad b'_i = b'_{,i} + b'_{,,}, \quad c'_i = c'_{,i} + c'_{,,}. \tag{43.}$$

Introducing the expressions (42.) for the rectangular components of velocity, we find that the value given by (4.) for the living force $2T$ decomposes itself into the three following parts,

$$\begin{aligned} 2T = \Sigma . m (x'^2 + y'^2 + z'^2) &= \Sigma . m (x'^2_{,} + y'^2_{,} + z'^2_{,}) \\ &+ 2 (x'_{,,} \Sigma . m x'_{,} + y'_{,,} \Sigma . m y'_{,} + z'_{,,} \Sigma . m z'_{,}) + (x'^2_{,,} + y'^2_{,,} + z'^2_{,,}) \Sigma m; \end{aligned} \tag{44.}$$

if then we establish, as we may, the three equations of condition,

$$\Sigma . m x_{,} = 0, \quad \Sigma . m y_{,} = 0, \quad \Sigma . m z_{,} = 0, \tag{45.}$$

which give by (40.),

$$x_{,,} = \frac{\Sigma . mx}{\Sigma m}, \quad y_{,,} = \frac{\Sigma . my}{\Sigma m}, \quad z_{,,} = \frac{\Sigma . mz}{\Sigma m}, \tag{46.}$$

so that $x_{,,}, y_{,,}, z_{,,}$ are now the coordinates of the point which is called the centre of gravity of the system, we may reduce the function T to the form

$$T = T_{,} + T_{,,}, \tag{47.}$$

in which

$$T_{,} = \tfrac{1}{2} \Sigma . m (x'^2_{,} + y'^2_{,} + z'^2_{,}), \tag{48.}$$

and

$$T_{,,} = \tfrac{1}{2} (x'^2_{,,} + y'^2_{,,} + z'^2_{,,}) \Sigma m. \tag{49.}$$

By this known decomposition, the whole living force $2T$ of the system is resolved into the two parts $2T_{,}$ and $2T_{,,}$, of which the former, $2T_{,}$, may be called the *relative living force*, being that which results solely from the relative velocities of the points of the system, in their motions about their common centre of gravity $x_{,,}, y_{,,}, z_{,,}$; while the latter part, $2T_{,,}$, results only from the absolute motion of that centre of gravity in space, and is the same as if all the masses of the system were united in that common centre. At the same time, the law of living force,

$T = U + H$, (6.), resolves itself by the law of motion of the centre of gravity into the two following separate equations,

$$T_{,} = U + H_{,}, \tag{50.}$$

and

$$T_{,,} = H_{,,}; \tag{51.}$$

$H_{,}$ and $H_{,,}$ being two new constants independent of the time t, and such that their sum

$$H_{,} + H_{,,} = H. \tag{52.}$$

And we may in like manner decompose the action, or accumulated living force V, which is equal to the definite integral $\int_0^t 2T\,dt$, into the two following analogous parts,

$$V = V_{,} + V_{,,}, \tag{E^1.}$$

determined by the two equations,

$$V_{,} = \int_0^t 2T_{,}\,dt, \tag{F^1.}$$

and

$$V_{,,} = \int_0^t 2T_{,,}\,dt. \tag{G^1.}$$

The last equation gives by (51.),

$$V_{,,} = 2H_{,,}t; \tag{53.}$$

a result which, by the law of motion of the centre of gravity, may be thus expressed,[*]

$$V_{,,} = \sqrt{(x_{,,} - a_{,,})^2 + (y_{,,} - b_{,,})^2 + (z_{,,} - c_{,,})^2} \,.\, \sqrt{2H_{,,}\Sigma m}: \tag{H^1.}$$

$a_{,,}$, $b_{,,}$, $c_{,,}$ being the initial coordinates of the centre of gravity, so that

$$a_{,,} = \frac{\Sigma \,.\, ma}{\Sigma m}, \quad b_{,,} = \frac{\Sigma \,.\, mb}{\Sigma m}, \quad c_{,,} = \frac{\Sigma \,.\, mc}{\Sigma m}. \tag{54.}$$

And for the variation δV of the whole function V, the rule of the last number gives

$$\left.\begin{aligned} \delta V = \Sigma \,.\, m\,(x'_{,}\delta x_{,} - a'_{,}\delta a_{,} + y'_{,}\delta y_{,} - b'_{,}\delta b_{,} + z'_{,}\delta z_{,} - c'_{,}\delta c_{,}) \\ + (x'_{,,}\delta x_{,,} - a'_{,,}\delta a_{,,} + y'_{,,}\delta y_{,,} - b'_{,,}\delta b_{,,} + z'_{,,}\delta z_{,,} - c'_{,,}\delta c_{,,})\,\Sigma m \\ + t\delta H + \lambda_1 \Sigma \,.\, m\,\delta x_{,} + \lambda_2 \Sigma \,.\, m\,\delta y_{,} + \lambda_3 \Sigma \,.\, m\,\delta z_{,} \\ + \Lambda_1 \Sigma \,.\, m\,\delta a_{,} + \Lambda_2 \Sigma \,.\, m\,\delta b_{,} + \Lambda_3 \Sigma \,.\, m\,\delta c_{,}; \end{aligned}\right| \tag{I^1.}$$

while the variation of the part $V_{,,}$, determined by the equation (H^1.), is easily shown to be equivalent to the part

$$\delta V_{,,} = (x'_{,,}\delta x_{,,} - a'_{,,}\delta a_{,,} + y'_{,,}\delta y_{,,} - b'_{,,}\delta b_{,,} + z'_{,,}\delta z_{,,} - c'_{,,}\delta c_{,,})\,\Sigma m + t\delta H_{,,}; \tag{K^1.}$$

the variation of the other part $V_{,}$ may therefore be thus expressed,

$$\left.\begin{aligned} \delta V_{,} = \Sigma \,.\, m\,(x'_{,}\delta x_{,} - a'_{,}\delta a_{,} + y'_{,}\delta y_{,} - b'_{,}\delta b_{,} + z'_{,}\delta z_{,} - c'_{,}\delta c_{,}) \\ + t\delta H_{,} + \lambda_1 \Sigma \,.\, m\,\delta x_{,} + \lambda_2 \Sigma \,.\, m\,\delta y_{,} + \lambda_3 \Sigma \,.\, m\,\delta z_{,} \\ + \Lambda_1 \Sigma \,.\, m\,\delta a_{,} + \Lambda_2 \Sigma \,.\, m\,\delta b_{,} + \Lambda_3 \Sigma \,.\, m\,\delta c_{,}: \end{aligned}\right| \tag{L^1.}$$

and it resolves itself into the following separate expressions, in which the part $V_{,}$ is considered

---

[*] [If v is the velocity of the centre of gravity,

$$(x_{,,} - a_{,,})^2 + (y_{,,} - b_{,,})^2 + (z_{,,} - c_{,,})^2 = v^2t^2 = 2H_{,,}t^2/\Sigma m.]$$

as a function of the $6n+1$ quantities $x_{,i}, y_{,i}, z_{,i}, a_{,i}, b_{,i}, c_{,i}, H_{,}$, of which, however, only $6n-5$ are really independent:

first group,

$$\begin{aligned} &\frac{\delta V_{,}}{\delta x_{,1}} = m_1 x'_{,1} + \lambda_1 m_1; &&\frac{\delta V_{,}}{\delta x_{,n}} = m_n x'_{,n} + \lambda_1 m_n; \\ &\frac{\delta V_{,}}{\delta y_{,1}} = m_1 y'_{,1} + \lambda_2 m_1; &&\frac{\delta V_{,}}{\delta y_{,n}} = m_n y'_{,n} + \lambda_2 m_n; \\ &\frac{\delta V_{,}}{\delta z_{,1}} = m_1 z'_{,1} + \lambda_3 m_1; &&\frac{\delta V_{,}}{\delta z} = m_n z'_{,n} + \lambda_3 m_n; \end{aligned} \qquad (\mathrm{M}^1.)$$

second group,

$$\begin{aligned} &\frac{\delta V_{,}}{\delta a_{,1}} = -m_1 a'_{,1} + \Lambda_1 m_1; &&\frac{\delta V_{,}}{\delta a_{,n}} = -m_n a'_{,n} + \Lambda_1 m_n; \\ &\frac{\delta V_{,}}{\delta b_{,1}} = -m_1 b'_{,1} + \Lambda_2 m_1; &&\frac{\delta V_{,}}{\delta b_{,n}} = -m_n b'_{,n} + \Lambda_2 m_n; \\ &\frac{\delta V_{,}}{\delta c_{,1}} = -m_1 c'_{,1} + \Lambda_3 m_1; &&\frac{\delta V_{,}}{\delta c} = -m_n c'_{,n} + \Lambda_3 m_n; \end{aligned} \qquad (\mathrm{N}^1.)$$

and finally,

$$\frac{\delta V_{,}}{\delta H} = t. \qquad (\mathrm{O}^1.)$$

With respect to the six multipliers $\lambda_1, \lambda_2, \lambda_3, \Lambda_1, \Lambda_2, \Lambda_3$, which were introduced by the 3 final equations of condition (45.), and by the 3 analogous initial equations of condition,

$$\Sigma . ma_{,} = 0, \quad \Sigma . mb_{,} = 0, \quad \Sigma . mc_{,} = 0; \qquad (55.)$$

we have, by differentiating these conditions,

$$\Sigma . mx'_{,} = 0, \quad \Sigma . my'_{,} = 0, \quad \Sigma . mz'_{,} = 0, \qquad (56.)$$

and

$$\Sigma . ma'_{,} = 0, \quad \Sigma . mb'_{,} = 0, \quad \Sigma . mc'_{,} = 0; \qquad (57.)$$

and therefore

$$\lambda_1 = \frac{\Sigma \frac{\delta V_{,}}{\delta x_{,}}}{\Sigma m}, \quad \lambda_2 = \frac{\Sigma \frac{\delta V_{,}}{\delta y_{,}}}{\Sigma m}, \quad \lambda_3 = \frac{\Sigma \frac{\delta V_{,}}{\delta z_{,}}}{\Sigma m}, \qquad (58.)$$

and

$$\Lambda_1 = \frac{\Sigma \frac{\delta V_{,}}{\delta a_{,}}}{\Sigma m}, \quad \Lambda_2 = \frac{\Sigma \frac{\delta V_{,}}{\delta b_{,}}}{\Sigma m}, \quad \Lambda_3 = \frac{\Sigma \frac{\delta V_{,}}{\delta c_{,}}}{\Sigma m}. \qquad (59.)$$

11. As an example of the determination of these multipliers, we may suppose that the part $V_{,}$, of the whole action V, has been expressed, before differentiation, as a function of $H_{,}$, and of these other $6n-6$ independent quantities

$$\left.\begin{aligned} &x_{,1} - x_{,n} = \xi_1, \quad x_{,2} - x_{,n} = \xi_2, \quad &&x_{,n-1} - x_{,n} = \xi_{n-1}, \\ &y_{,1} - y_{,n} = \eta_1, \quad y_{,2} - y_{,n} = \eta_2, \quad &&y_{,n-1} - y_{,n} = \eta_{n-1}, \\ &z_{,1} - z_{,n} = \zeta_1, \quad z_{,2} - z_{,n} = \zeta_2, \quad &&z_{,n-1} - z_{,n} = \zeta_{n-1}, \end{aligned}\right\} \qquad (60.)$$

and

$$\left.\begin{aligned} &a_{,1} - a_{,n} = \alpha_1, \quad a_{,2} - a_{,n} = \alpha_2, \quad &&a_{,n-1} - a_{,n} = \alpha_{n-1}, \\ &b_{,1} - b_{,n} = \beta_1, \quad b_{,2} - b_{,n} = \beta_2, \quad \ldots &&b_{,n-1} - b_{,n} = \beta_{n-1}, \\ &c_{,1} - c_{,n} = \gamma_1, \quad c_{,2} - c_{,n} = \gamma_2, \quad \ldots &&c_{,n-1} - c_{,n} = \gamma_{n-1}; \end{aligned}\right\} \qquad (61.)$$

that is, of the *differences* only of the *centrobaric* coordinates*; or, in other words, as a function of the coordinates (initial and final) of $n-1$ points of the system, referred to the n^{th} point, as an internal or moveable origin: because the centrobaric coordinates $x_{,i}, y_{,i}, z_{,i}, a_{,i}, b_{,i}, c_{,i}$ may themselves, by the equations of condition, be expressed as functions of these, namely,

$$x_{,i}=\xi_i-\frac{\Sigma\,.\,m\xi}{\Sigma m} \quad y_{,i}=\eta_i-\frac{\Sigma\,.\,m\eta}{\Sigma m}; \quad z_{,i}=\zeta_i-\frac{\Sigma\,.\,m\zeta}{\Sigma m}, \tag{62.}$$

and in like manner,

$$a_{,i}=\alpha_i-\frac{\Sigma\,.\,m\alpha}{\Sigma m}, \quad b_{,i}=\beta_i-\frac{\Sigma\,.\,m\beta}{\Sigma m} \quad c_{,i}=\gamma_i-\frac{\Sigma\,.\,m\gamma}{\Sigma m}; \tag{63.}$$

in which we are to observe, that the six quantities $\xi_n, \eta_n, \zeta_n, \alpha_n, \beta_n, \gamma_n$ must be considered as separately vanishing. When $V_,$ has been thus expressed as a function of the centrobaric coordinates, involving their differences only, it will evidently satisfy the six partial differential equations,†

$$\left.\begin{aligned} &\Sigma\frac{\delta V_,}{\delta x_,}=0, \quad \Sigma\frac{\delta V_,}{\delta y_,}=0, \quad \Sigma\frac{\delta V_,}{\delta z_,}=0, \\ &\Sigma\frac{\delta V_,}{\delta a_,}=0, \quad \Sigma\frac{\delta V_,}{\delta b_,}=0, \quad \Sigma\frac{\delta V_,}{\delta c_,}= \end{aligned}\right\} \tag{P^1.}$$

after this preparation, therefore, of the function $V_,$, the six multipliers determined by (58.) and (59.) will vanish, so that we shall have

$$\lambda_1=0, \quad \lambda_2=0, \quad \lambda_3=0, \quad \Lambda_1=0, \quad \Lambda_2=0, \quad \Lambda_3=0, \tag{64.}$$

and the groups (M¹.) and (N¹.) will reduce themselves to the two following:

$$\begin{aligned} &\frac{\delta V_,}{\delta x_{,1}}=m_1x'_{,1}; \quad \frac{\delta V_,}{\delta x_{,2}}=m_2x'_{,2}; \quad \frac{\delta V_,}{\delta x_{,n}} \\ &\frac{\delta V_,}{\delta y_{,1}}=m_1y'_{,1}; \quad \frac{\delta V_,}{\delta y_{,2}}= \quad {}'_{,2}; \quad \frac{\delta V_,}{\delta y_{,n}}=m_ny'_{,n}; \\ &\frac{\delta V_,}{\delta z_{,1}}=m_1z'_{,1}; \quad \frac{\delta V_,}{\delta z_{,2}}=m_2z'_{,2}; \quad \frac{\delta V_,}{\delta z_{,n}}=m_nz'_{,n}; \end{aligned} \tag{Q^1.}$$

and

$$\left.\begin{aligned} &\frac{\delta V_,}{\delta a_{,1}}=-m_1a'_{,1}; \quad \frac{\delta V_,}{\delta a_{,2}}=-m_2a'_{,2}; \quad \frac{\delta V.}{\delta a.}=-m_na'_{,n}; \\ &\frac{\delta V_,}{\delta b_{,1}}=-m_1b'_{,1}; \quad \frac{\delta V_,}{\delta b_{,2}}=-m_2b'_{,2}; \quad \frac{\delta V_,}{\delta b_{,n}}=-m_nb'_{,n}; \\ &\frac{\delta V_,}{\delta c_{,1}}=-m_1c'_{,1}; \quad \frac{\delta V.}{\delta c.}=-m_2c_{,2}; \quad \frac{\delta V.}{\delta c_{,n}}=-m_nc'_{,n}; \end{aligned}\right\rbrace \tag{R^1.}$$

analogous in all respects to the groups (C.) and (D.). We find, therefore, for the relative motion of a system about its own centre of gravity, equations of the same form as those which we had obtained before for the absolute motion of the same system of points in space. And we see that in investigating such relative motion only, it is useful to confine ourselves to the part $V_,$ of our

---

* [The term *centrobaric* was first introduced by Möbius.]

† [If $V_,$ is expressed as a function of the $\xi, \eta, \zeta, \alpha, \beta, \gamma$,

$$\frac{\delta V_,}{\delta x_{,i}}=\frac{\delta V_,}{\delta \xi_i}\ (i=1, 2, \ldots, n-1), \quad \frac{\delta V_,}{\delta x_{,n}}=-\Sigma\frac{\delta V_,}{\delta \xi}.]$$

whole characteristic function, that is, to the *relative action* of the system, or accumulated living force of the motion about the centre of gravity; and to consider this part as the *characteristic function* of such relative motion, in a sense analogous to that which has been already explained.

This relative action, or part $V_{,}$, may, however, be otherwise expressed, and even in an infinite variety of ways, on account of the six equations of condition which connect the $6n$ centrobaric coordinates; and every different preparation of its form will give a different set of values for the six multipliers $\lambda_1, \lambda_2, \lambda_3, \Lambda_1, \Lambda_2, \Lambda_3$. For example, we might eliminate, by a previous preparation, the six centrobaric coordinates of the point m_n from the expression of $V_{,}$, so as to make this expression involve only the centrobaric coordinates of the other $n-1$ points of the system, and then we should have

$$\frac{\delta V_{,}}{\delta x_{,n}}=0,\quad \frac{\delta V_{,}}{\delta y_{,n}}=0,\quad \frac{\delta V_{,}}{\delta z_{,n}}=0,\quad \frac{\delta V_{,}}{\delta a_{,n}}=0,\quad \frac{\delta V_{,}}{\delta b_{,n}}=0,\quad \frac{\delta V_{,}}{\delta c_{,n}}=0, \tag{S^1.}$$

and therefore, by the six last equations of the groups (M^1.) and (N^1.), the multipliers would take the values

$$\lambda_1=-x'_{,n},\quad \lambda_2=-y'_{,n},\quad \lambda_3=-z'_{,n},\quad \Lambda_1=a'_{,n},\quad \Lambda_2=b'_{,n},\quad \Lambda_3=c'_{,n}, \tag{65.}$$

and would reduce, by (60.) and (61.), the preceding $6n-6$ equations of the same groups (M^1.) and (N^1.), to the forms

$$\left.\begin{array}{llll}
\dfrac{\delta V_{,}}{\delta x_{,1}}=m_1\xi'_1, & \dfrac{\delta V_{,}}{\delta x_{,2}}=m_2\xi'_2, & \cdots & \dfrac{\delta V_{,}}{\delta x_{,n-1}}=m_{n-1}\xi'_{n-1},\\
\dfrac{\delta V_{,}}{\delta y_{,1}}=m_1\eta'_1, & \dfrac{\delta V_{,}}{\delta y_{,2}}=m_2\eta'_2, & \cdots & \dfrac{\delta V_{,}}{\delta y_{,n-1}}=m_{n-1}\eta'_{n-1},\\
\dfrac{\delta V_{,}}{\delta z_{,1}}=m_1\zeta'_1, & \dfrac{\delta V_{,}}{\delta z_{,2}}=m_2\zeta'_2, & \cdots & \dfrac{\delta V_{,}}{\delta z_{,n-1}}=m_{n-1}\zeta'_{n-1},
\end{array}\right\} \tag{T^1.}$$

and

$$\left.\begin{array}{llll}
\dfrac{\delta V_{,}}{\delta a_{,1}}=-m_1\alpha'_1, & \dfrac{\delta V_{,}}{\delta a_{,2}}=-m_2\alpha'_2, & \cdots & \dfrac{\delta V_{,}}{\delta a_{,n-1}}=-m_{n-1}\alpha'_{n-1},\\
\dfrac{\delta V_{,}}{\delta b_{,1}}=-m_1\beta'_1, & \dfrac{\delta V_{,}}{\delta b_{,2}}=-m_2\beta'_2, & \cdots & \dfrac{\delta V_{,}}{\delta b_{,n-1}}=-m_{n-1}\beta'_{n-1},\\
\dfrac{\delta V_{,}}{\delta c_{,1}}=-m_1\gamma'_1, & \dfrac{\delta V_{,}}{\delta c_{,2}}=-m_2\gamma'_2, & \cdots & \dfrac{\delta V_{,}}{\delta c_{,n-1}}=-m_{n-1}\gamma'_{n-1}.
\end{array}\right\} \tag{U^1.}$$

12. We might also express the relative action $V_{,}$, not as a function of the centrobaric, but of some other internal coordinates, or marks of relative position. We might, for instance, express it and its variation as functions of the $6n-6$ independent internal coordinates $\xi, \eta, \zeta, \alpha, \beta, \gamma$ already mentioned, and of their variations, defining these without any reference to the centre of gravity, by the equations

$$\left.\begin{array}{lll}
\xi_i=x_i-x_n, & \eta_i=y_i-y_n, & \zeta_i=z_i-z_n,\\
\alpha_i=a_i-a_n, & \beta_i=b_i-b_n, & \gamma_i=c_i-c_n.
\end{array}\right\} \tag{66.}$$

For all such transformations of $\delta V_{,}$ it is easy to establish a rule or law, which may be called the *law of varying relative action* (exactly analogous to the rule (B^1.)), namely, the following:

$$\delta V_{,}=\Sigma\,.\left(\frac{\delta T_{,}}{\delta \eta'_{,}}\right)\delta\eta_{,}-\Sigma\,.\left(\frac{\delta T_{,0}}{\delta e'_{,}}\right)\delta e_{,}+t\,\delta H_{,}+\Sigma\,.\,\lambda_{,}\,\delta\phi_{,}+\Sigma\,.\,\Lambda_{,}\,\delta\Phi_{,}; \tag{V^1.}$$

which implies that we are to express the half $T_,$ of the relative living force of the system as a function of the rates of increase $\eta'_,$ of any marks of relative position; and after taking its variation with respect to these rates, to change their variations to the variations of the marks of position themselves; then to subtract the initial from the final value of the result, and to add the variations of the final and initial functions $\phi_,$, $\Phi_,$, which enter into the equations of condition, if any, of the form $\phi_, = 0$, $\Phi_, = 0$, (connecting the final and initial marks of relative position,) multiplied respectively by undetermined factors $\lambda_,$, $\Lambda_,$; and lastly, to equate the whole result to $\delta V_, - t\delta H_,$, $H_,$ being the quantity independent of the time in the equation (50.) of relative living force, and $V_,$ being the relative action, of which we desired to express the variation. It is not necessary to dwell here on the demonstration of this new rule (V^1.), which may easily be deduced from the principles already laid down; or by the calculus of variations from the law of relative living force, combined with the differential equations of the second order of relative motion.

But to give an example of its application, let us resume the problem already mentioned, namely to express $\delta V_,$ by means of the $6n-5$ independent variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, $\delta\alpha_i$, $\delta\beta_i$, $\delta\gamma_i$, $\delta H_,$. For this purpose we shall employ a known transformation of the relative living force $2T_,$, multiplied by the sum of the masses of the system, namely the following:

$$2T_,\Sigma m = \Sigma \,.\, m_i m_k \{(x'_i - x'_k)^2 + (y'_i - y'_k)^2 + (z'_i - z'_k)^2\}: \tag{67.}$$

the sign of summation Σ extending, in the second member, to all the combinations of points two by two, which can be formed without repetition. This transformation gives, by (66.),

$$\left.\begin{aligned} 2T_,\Sigma m = {} & m_n \Sigma_, \,.\, m\,(\xi'^2 + \eta'^2 + \zeta'^2) \\ & + \Sigma_, \,.\, m_i m_k \{(\xi'_i - \xi'_k)^2 + (\eta'_i - \eta'_k)^2 + (\zeta'_i - \zeta'_k)^2\}; \end{aligned}\right\} \tag{68.}$$

the sign of summation $\Sigma_,$ extending only to the first $n-1$ points of the system. Applying, therefore, our general rule or law of varying relative action, and observing that the $6n-6$ internal coordinates ξ, η, ζ, α, β, γ are independent, we find the following new expression:

$$\begin{aligned} \delta V_, = t\delta H_, & + \frac{m_n}{\Sigma m} \,.\, \Sigma_, \,.\, m\,(\xi'\delta\xi - \alpha'\delta\alpha + \eta'\delta\eta - \beta'\delta\beta + \zeta'\delta\zeta - \gamma'\delta\gamma) \\ & + \frac{1}{\Sigma m} \,.\, \Sigma_, \,.\, m_i m_k \{(\xi'_i - \xi'_k)(\delta\xi_i - \delta\xi_k) + (\eta'_i - \eta'_k)(\delta\eta_i - \delta\eta_k) + (\zeta'_i - \zeta'_k)(\delta\zeta_i - \delta\zeta_k)\} \\ & - \frac{1}{\Sigma m} \,.\, \Sigma_, \,.\, m_i m_k \{(\alpha'_i - \alpha'_k)(\delta\alpha_i - \delta\alpha_k) + (\beta'_i - \beta'_k)(\delta\beta_i - \delta\beta_k) + (\gamma'_i - \gamma'_k)(\delta\gamma_i - \delta\gamma_k)\}: \end{aligned} \tag{W^1.}$$

which gives, besides the equation (O^1.), the following groups:

$$\left.\begin{aligned} \frac{\delta V_,}{\delta\xi_i} &= \frac{m_i}{\Sigma m} \,.\, \Sigma \,.\, m\,(\xi'_i - \xi') = m_i\left(\xi'_i - \frac{\Sigma_, m\xi'}{\Sigma m}\right), \\ \frac{\delta V_,}{\delta\eta_i} &= \frac{m_i}{\Sigma m} \,.\, \Sigma \,.\, m\,(\eta'_i - \eta') = m_i\left(\eta'_i - \frac{\Sigma_, m\eta'}{\Sigma m}\right), \\ \frac{\delta V_,}{\delta\zeta_i} &= \frac{m_i}{\Sigma m} \,.\, \Sigma \,.\, m\,(\zeta'_i - \zeta') = m_i\left(\zeta'_i - \frac{\Sigma_, m\zeta'}{\Sigma m}\right), \end{aligned}\right\} \tag{X^1.}$$

and

$$\left.\begin{aligned}\frac{\delta V_{,}}{\delta\alpha_i}&=\frac{-m_i}{\Sigma m}\,.\,\Sigma\,.\,m\,(\alpha'_i-\alpha')=-m_i\left(\alpha'_i-\frac{\Sigma_{,}m\alpha'}{\Sigma m}\right),\\ \frac{\delta V_{,}}{\delta\beta_i}&=\frac{-m_i}{\Sigma m}\,.\,\Sigma\,.\,m\,(\beta'_i-\beta')=-m_i\left(\beta'_i-\frac{\Sigma_{,}m\beta'}{\Sigma m}\right),\\ \frac{\delta V_{,}}{\delta\gamma_i}&=\frac{-m_i}{\Sigma m}\,.\,\Sigma\,.\,m\,(\gamma'_i-\gamma')=-m_i\left(\gamma'_i-\frac{\Sigma_{,}m\gamma'}{\Sigma m}\right);\end{aligned}\right\}\qquad(\mathrm{Y}^1.)$$

results which may be thus summed up:

$$\left.\begin{aligned}\delta V_{,}=t\,\delta H_{,}&+\Sigma_{,}\,.\,m\,(\xi'\delta\xi-\alpha'\delta\alpha+\eta'\delta\eta-\beta'\delta\beta+\zeta'\delta\zeta-\gamma'\delta\gamma)\\ &-\frac{1}{\Sigma m}(\Sigma_{,}m\xi'\,.\,\Sigma_{,}m\,\delta\xi+\Sigma_{,}m\eta'\,.\,\Sigma_{,}m\,\delta\eta+\Sigma_{,}m\zeta'\,.\,\Sigma_{,}m\,\delta\zeta)\\ &+\frac{1}{\Sigma m}(\Sigma_{,}m\alpha'\,.\,\Sigma_{,}m\,\delta\alpha+\Sigma_{,}m\beta'\,.\,\Sigma_{,}m\,\delta\beta+\Sigma_{,}m\gamma'\,.\,\Sigma_{,}m\,\delta\gamma),\end{aligned}\right\}\qquad(\mathrm{Z}^1.)$$

and might have been otherwise deduced by our rule, from this other known transformation of $T_{,}$,

$$T_{,}=\tfrac{1}{2}\Sigma_{,}\,.\,m\,(\xi'^2+\eta'^2+\zeta'^2)-\frac{(\Sigma_{,}m\xi')^2+(\Sigma_{,}m\eta')^2+(\Sigma_{,}m\zeta')^2}{2\Sigma m}.\qquad(69.)$$

And to obtain, with any set of internal or relative marks of position, the two partial differential equations which the characteristic function $V_{,}$ of relative motion must satisfy, and which offer (as we shall find) the chief means of discovering its form, namely, the equations analogous to those marked (F.) and (G.), we have only to eliminate the rates of increase of the marks of position of the system, which determine the final and initial components of the relative velocities of its points, by the law of varying relative action, from the final and initial expressions of the law of relative living force; namely, from the following equations:

$$T_{,}=U+H_{,},\qquad(50.)$$

and

$$T_{,0}=U_0+H_{,}.\qquad(70.)$$

The law of areas, or the property respecting rotation which was expressed by the partial differential equations (P.), will also always admit of being expressed in relative coordinates, and will assist in discovering the form of the characteristic function $V_{,}$; by showing that this function involves only such internal coordinates (in number $6n-9$) as do not alter by any common rotation of all points final and initial, round the centre of gravity, or round any other internal origin; that origin being treated as fixed, and the quantity $H_{,}$ as constant, in determining the effects of this rotation. The general problem of dynamics, respecting the motions of a free system of n points attracting or repelling one another, is therefore reduced, in the last analysis, by the method of the present essay, to the research and differentiation of a function $V_{,}$, depending on $6n-9$ internal or relative coordinates,* and on the quantity $H_{,}$, and satisfying a pair of partial differential equations of the first order and second degree; in integrating which equations, we are to observe, that at the assumed origin of the motion, namely at the moment when $t=0$, the final or variable coordinates are equal to their initial values, and the partial differential coefficient $\frac{\delta V_{,}}{\delta H_{,}}$ vanishes; and, that at a moment infinitely little distant, the differential alterations of the coordinates have ratios connected with the other partial differential coefficients of the characteristic function $V_{,}$, by the law of varying relative action. It may be here observed, that, although the consideration of the point, called usually the centre of gravity, is very simply suggested by the process of the tenth number, yet this internal centre is even more simply indicated by our early corollaries from the law of varying action; which show that the components of relative final velocities, in any system of attracting or repelling points, may be expressed by the differences of quantities of the form $\frac{1}{m}\frac{\delta V}{\delta x}, \frac{1}{m}\frac{\delta V}{\delta y}, \frac{1}{m}\frac{\delta V}{\delta z}$: and therefore that in calculating these relative velocities, it is advantageous to introduce the final sums Σmx, Σmy, Σmz, and, for an analogous reason, the initial sums Σma, Σmb, Σmc, among the marks of the extreme positions of the system, in the expression of the characteristic function V; because, in differentiating that expression for the calculation of relative velocities, those sums may be treated as constant.

---

* [The relative configuration of the $2(n-1)$ initial and final positions of the planets is defined by $6n-12$ mutual distances and in addition we have the three coordinates of any one of them relative to the Sun.]

On Systems of two Points, in general; Characteristic Function of the motion of any Binary System.

13. To illustrate the foregoing principles, which extend to any free system of points, however numerous, attracting or repelling one another, let us now consider, in particular, a system of two such points. For such a system, the known *force-function* U becomes, by (2.),

$$U = m_1 m_2 f(r), \tag{71.}$$

r being the mutual distance

$$r = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} \tag{72.}$$

between the two points m_1, m_2, and $f(r)$ being a function of this distance such that its derivative or differential coefficient $f'(r)$ expresses the law of their repulsion or attraction, according as it is positive or negative. The known differential equations of motion, of the second order, are now, by (1.), comprised in the following formula:

$$m_1(x''_1\delta x_1+y''_1\delta y_1+z''_1\delta z_1)+m_2(x''_2\delta x_2+y''_2\delta y_2+z''_2\delta z_2)=m_1m_2\delta f(r); \tag{73.}$$

they are therefore, separately,

$$\left.\begin{aligned} x''_1 &= m_2\frac{\delta f(r)}{\delta x_1}, \quad y''_1 = m_2\frac{\delta f(r)}{\delta y_1}, \quad z''_1 = m_2\frac{\delta f(r)}{\delta z_1}, \\ x''_2 &= m_1\frac{\delta f(r)}{\delta x_2}, \quad y''_2 = m_1\frac{\delta f(r)}{\delta y_2}, \quad z''_2 = m_1\frac{\delta f(r)}{\delta z_2}. \end{aligned}\right\} \tag{74.}$$

The problem of integrating these equations consists in proposing to assign, by their means, six relations between the time t, the masses m_1, m_2, the six varying coordinates $x_1, y_1, z_1, x_2, y_2, z_2$, and their initial values and initial rates of increase $a_1, b_1, c_1, a_2, b_2, c_2, a'_1, b'_1, c'_1, a'_2, b'_2, c'_2$. If we knew these six final integrals, and combined them with the initial form of the law of living force, or of the known intermediate integral

$$\tfrac{1}{2}m_1(x_1'^2+y_1'^2+z_1'^2)+\tfrac{1}{2}m_2(x_2'^2+y_2'^2+z_2'^2)=m_1m_2f(r)+H; \tag{75.}$$

that is, with the following formula,

$$\tfrac{1}{2}m_1(a_1'^2+b_1'^2+c_1'^2)+\tfrac{1}{2}m_2(a_2'^2+b_2'^2+c_2'^2)=m_1m_2f(r_0)+H, \tag{76.}$$

in which r_0 is the initial distance

$$r_0=\sqrt{(a_1-a_2)^2+(b_1-b_2)^2+(c_1-c_2)^2}, \tag{77.}$$

and H is a constant quantity, introduced by integration; we could, by the combination of these seven relations, determine the time t, and the six initial components of velocity a_1', b_1', c_1', a_2', b_2', c_2', as functions of the twelve final and initial coordinates $x_1, y_1, z_1, x_2, y_2, z_2, a_1, b_1, c_1, a_2, b_2, c_2$, and of the quantity H, (involving also the masses:) we could therefore determine whatever else depends on the manner and time of motion of this system of two points, as a function of the same extreme coordinates and of the same quantity H. In particular, we could determine the action, or accumulated living force of the system, namely,

$$V=m_1\int_0^t(x_1'^2+y_1'^2+z_1'^2)\,dt+m_2\int_0^t(x_2'^2+y_2'^2+z_2'^2)\,dt, \tag{A^2.}$$

as a function of those thirteen quantities $x_1, y_1, z_1, x_2, y_2, z_2, a_1, b_1, c_1, a_2, b_2, c_2, H$: and might then calculate the variation of this function,

$$\begin{aligned}\delta V=&\frac{\delta V}{\delta x_1}\delta x_1+\frac{\delta V}{\delta y_1}\delta y_1+\frac{\delta V}{\delta z_1}\delta z_1+\frac{\delta V}{\delta x_2}\delta x_2+\frac{\delta V}{\delta y_2}\delta y_2+\frac{\delta V}{\delta z_2}\delta z_2\\&+\frac{\delta V}{\delta a_1}\delta a_1+\frac{\delta V}{\delta b_1}\delta b_1+\frac{\delta V}{\delta c_1}\delta c_1+\frac{\delta V}{\delta a_2}\delta a_2+\frac{\delta V}{\delta b_2}\delta b_2+\frac{\delta V}{\delta c_2}\delta c_2\\&+\frac{\delta V}{\delta H}\delta H.\end{aligned} \tag{B^2.}$$

But the essence of our method consists in *forming previously the expression of this variation, by our law of varying action*, namely,

$$\left.\begin{aligned}\delta V=&m_1(x_1'\delta x_1-a_1'\delta a_1+y_1'\delta y_1-b_1'\delta b_1+z_1'\delta z_1-c_1'\delta c_1)\\&+m_2(x_2'\delta x_2-a_2'\delta a_2+y_2'\delta y_2-b_2'\delta b_2+z_2'\delta z_2-c_2'\delta c_2)\\&+t\delta H;\end{aligned}\right\} \tag{C^2.}$$

and in *considering V as a characteristic function of the motion*, from the form of which may be deduced all the intermediate and all the final integrals of the known differential equations, by resolving the expression (C^2.) into the following separate groups, (included in (C.) and (D.),)

$$\left.\begin{aligned}&\frac{\delta V}{\delta x_1}=m_1x_1',\ \frac{\delta V}{\delta y_1}=m_1y_1',\ \frac{\delta V}{\delta z_1}=m_1z_1',\\&\frac{\delta V}{\delta x_2}=m_2x_2',\ \frac{\delta V}{\delta y_2}=m_2y_2',\ \frac{\delta V}{\delta z_2}=m_2z_2';\end{aligned}\right\} \tag{D^2.}$$

and

$$\left.\begin{aligned}&\frac{\delta V}{\delta a_1}=-m_1a_1',\ \frac{\delta V}{\delta b_1}=-m_1b_1',\ \frac{\delta V}{\delta c_1}=-m_1c_1',\\&\frac{\delta V}{\delta a_2}=-m_2a_2',\ \frac{\delta V}{\delta b_2}=-m_2b_2',\ \frac{\delta V}{\delta c_2}=-m_2c_2';\end{aligned}\right\} \tag{E^2.}$$

besides this other equation, which had occurred before,

$$\frac{\delta V}{\delta H}=t. \tag{E.}$$

By this new method, the difficulty of integrating the six known equations of motion of the second order (74.) is reduced to the search and differentiation of a single function V; and to find the form of this function, we are to employ the following pair of partial differential equations of the first order:

$$\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta x_1}\right)^2+\left(\frac{\delta V}{\delta y_1}\right)^2+\left(\frac{\delta V}{\delta z_1}\right)^2\right\}+\frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta x_2}\right)^2+\left(\frac{\delta V}{\delta y_2}\right)^2+\left(\frac{\delta V}{\delta z_2}\right)^2\right\}=m_1m_2f(r)+H, \tag{F^2.}$$

$$\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta a_1}\right)^2+\left(\frac{\delta V}{\delta b_1}\right)^2+\left(\frac{\delta V}{\delta c_1}\right)^2\right\}+\frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta a_2}\right)^2+\left(\frac{\delta V}{\delta b_2}\right)^2+\left(\frac{\delta V}{\delta c_2}\right)^2\right\}=m_1m_2f(r_0)+H, \tag{G^2.}$$

combined with some simple considerations. And it easily results from the principles already laid down, that the integral of this pair of equations, adapted to the present question, is

$$V=\sqrt{(x_{\prime\prime}-a_{\prime\prime})^2+(y_{\prime\prime}-b_{\prime\prime})^2+(z_{\prime\prime}-c_{\prime\prime})^2}\,.\sqrt{2H_{\prime\prime}\,(m_1+m_2)}+\frac{m_1m_2}{m_1+m_2}\left(h\vartheta+\int_{r_0}^{r}\rho\,dr\right); \tag{H^2.}$$

in which $x_{\prime\prime}, y_{\prime\prime}, z_{\prime\prime}, a_{\prime\prime}, b_{\prime\prime}, c_{\prime\prime}$ denote the coordinates, final and initial, of the centre of gravity of the system,

$$\begin{aligned}
x_{\prime\prime}&=\frac{m_1x_1+m_2x_2}{m_1+m_2}, & y_{\prime\prime}&=\frac{m_1y_1+m_2y_2}{m_1+m_2}, & z_{\prime\prime}&=\frac{m_1z_1+m_2z_2}{m_1+m_2};\\
a_{\prime\prime}&=\frac{m_1a_1+m_2a_2}{m_1+m_2}, & b_{\prime\prime}&=\frac{m_1b_1+m_2b_2}{m_1+m_2}, & c_{\prime\prime}&=\frac{m_1c_1+m_2c_2}{m_1+m_2},
\end{aligned} \tag{78.}$$

and ϑ is the angle between the final and initial distances r, r_0: we have also put for abridgement

$$\rho=\pm\sqrt{2\,(m_1+m_2)\left(f(r)+\frac{H_{\prime}}{m_1m_2}\right)-\frac{h^2}{r^2}}, \tag{79.}$$

the upper or the lower sign to be used, according as the distance r is increasing or decreasing; and have introduced three auxiliary quantities $h, H_{\prime}, H_{\prime\prime}$, to be determined by this condition,*

$$0=\vartheta+\int_{r_0}^{r}\frac{\delta\rho}{\delta h}\,dr, \tag{I^2.}$$

combined with the two following,

$$\begin{aligned}
\frac{m_1m_2}{m_1+m_2}\int_{r_0}^{r}\frac{\delta\rho}{\delta H_{\prime}}\,dr&=\sqrt{(x_{\prime\prime}-a_{\prime\prime})^2+(y_{\prime\prime}-b_{\prime\prime})^2+(z_{\prime\prime}-c_{\prime\prime})^2}\,.\sqrt{\frac{m_1+m_2}{2H_{\prime\prime}}},\\
H_{\prime}+H_{\prime\prime}&=H;
\end{aligned} \tag{K^2.}$$

which auxiliary quantities, although in one view they are functions of the twelve extreme coordinates, are yet to be treated as constant in calculating the three definite integrals, or limits of sums of numerous small elements,

$$\int_{r_0}^{r}\rho\,dr,\quad \int_{r_0}^{r}\frac{\delta\rho}{\delta h}\,dr,\quad \int_{r_0}^{r}\frac{\delta\rho}{\delta H_{\prime}}\,dr.$$

---

* [Cf. Appendix, Note 2, pp. 613–621.]

The form (H^2.), for the *characteristic function of a binary system*, may be regarded as a central or radical relation, which includes the whole theory of the motion of such a system; so that all the details of this motion may be deduced from it by the application of our general method. But because the theory of binary systems has been brought to great perfection already, by the labours of former writers, it may suffice to give briefly here a few instances of such deduction.

14. The form (H^2.), for the characteristic function of a binary system, involves explicitly, when ρ is changed to its value (79.), the twelve quantities $x_{,,}, y_{,,}, z_{,,}, a_{,,}, b_{,,}, c_{,,}, r, r_0, \vartheta, h, H_{,}, H_{,,}$, (besides the masses m_1, m_2 which are always considered as given;) its variation may therefore be thus expressed:

$$\begin{aligned}\delta V = {} & \frac{\delta V}{\delta x_{,,}}\delta x_{,,} + \frac{\delta V}{\delta y_{,,}}\delta y_{,,} + \frac{\delta V}{\delta z_{,,}}\delta z_{,,} + \frac{\delta V}{\delta a_{,,}}\delta a_{,,} + \frac{\delta V}{\delta b_{,,}}\delta b_{,,} + \frac{\delta V}{\delta c_{,,}}\delta c_{,,} \\ & + \frac{\delta V}{\delta r}\delta r + \frac{\delta V}{\delta r_0}\delta r_0 + \frac{\delta V}{\delta \vartheta}\delta\vartheta + \frac{\delta V}{\delta h}\delta h + \frac{\delta V}{\delta H_{,}}\delta H_{,} + \frac{\delta V}{\delta H_{,,}}\delta H_{,,}\,.\end{aligned} \qquad (\mathrm{L}^2.)$$

In this expression, if we put for abridgement

$$\lambda = \sqrt{\frac{2H_{,,}(m_1+m_2)}{(x_{,,}-a_{,,})^2+(y_{,,}-b_{,,})^2+(z_{,,}-c_{,,})^2}}, \qquad (80.)$$

we shall have

$$\begin{aligned}&\frac{\delta V}{\delta x_{,,}} = \lambda(x_{,,}-a_{,,}), \quad \frac{\delta V}{\delta y_{,,}} = \lambda(y_{,,}-b_{,,}), \quad \frac{\delta V}{\delta z_{,,}} = \lambda(z_{,,}-c_{,,}), \\ &\frac{\delta V}{\delta a_{,,}} = \lambda(a_{,,}-x_{,,}), \quad \frac{\delta V}{\delta b_{,,}} = \lambda(b_{,,}-y_{,,}), \quad \frac{\delta V}{\delta c_{,,}} = \lambda(c_{,,}-z_{,,});\end{aligned} \qquad (\mathrm{M}^2.)$$

and if we put

$$\rho_0 = \pm\sqrt{2(m_1+m_2)\left(f(r_0)+\frac{H_{,}}{m_1 m_2}\right)-\frac{h^2}{r_0^2}}, \qquad (81.)$$

the sign of the radical being determined by the same rule as that of ρ, we shall have

$$\frac{\delta V}{\delta r} = \frac{m_1 m_2 \rho}{m_1+m_2}, \quad \frac{\delta V}{\delta r_0} = \frac{-m_1 m_2 \rho_0}{m_1+m_2}, \quad \frac{\delta V}{\delta \vartheta} = \frac{m_1 m_2 h}{m_1+m_2}; \qquad (\mathrm{N}^2.)$$

besides, by the equations of condition (I^2.), (K^2.), we have

$$\frac{\delta V}{\delta h} = 0, \qquad (\mathrm{O}^2.)$$

and

$$\frac{\delta V}{\delta H_{,,}} = \frac{\delta V}{\delta H_{,}} = \int_{r_0}^{r} \frac{dr}{\rho}, \quad \delta H_{,} + \delta H_{,,} = \delta H. \qquad (\mathrm{P}^2.)$$

The expression (L^2.) may therefore be thus transformed:

$$\begin{aligned}\delta V = {} & \lambda\{(x_{,,}-a_{,,})(\delta x_{,,}-\delta a_{,,}) + (y_{,,}-b_{,,})(\delta y_{,,}-\delta b_{,,}) + (z_{,,}-c_{,,})(\delta z_{,,}-\delta c_{,,})\} \\ & + \frac{m_1 m_2}{m_1+m_2}(\rho\,\delta r - \rho_0\,\delta r_0 + h\,\delta\vartheta) + \int_{r_0}^{r}\frac{dr}{\rho}\,.\,\delta H;\end{aligned} \qquad (\mathrm{Q}^2.)$$

and may be resolved by our general method into twelve separate expressions for the final and initial components of velocities, namely,

$$\left.\begin{aligned}
x_1' &= \frac{1}{m_1}\frac{\delta V}{\delta x_1} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime}-a_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta x_1} + h\frac{\delta\vartheta}{\delta x_1}\right),\\
y_1' &= \frac{1}{m_1}\frac{\delta V}{\delta y_1} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime}-b_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta y_1} + h\frac{\delta\vartheta}{\delta y_1}\right),\\
z_1' &= \frac{1}{m_1}\frac{\delta V}{\delta z_1} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime}-c_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho\frac{\delta r}{\delta z_1} + h\frac{\delta\vartheta}{\delta z_1}\right),\\
x_2' &= \frac{1}{m_2}\frac{\delta V}{\delta x_2} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime}-a_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta x_2} + h\frac{\delta\vartheta}{\delta x_2}\right),\\
y_2' &= \frac{1}{m_2}\frac{\delta V}{\delta y_2} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime}-b_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta y_2} + h\frac{\delta\vartheta}{\delta y_2}\right),\\
z_2' &= \frac{1}{m_2}\frac{\delta V}{\delta z_2} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime}-c_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho\frac{\delta r}{\delta z_2} + h\frac{\delta\vartheta}{\delta z_2}\right),
\end{aligned}\right\} \qquad (R^2.)$$

and

$$\left.\begin{aligned}
a_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta a_1} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime}-a_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta a_1} - h\frac{\delta\vartheta}{\delta a_1}\right),\\
b_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta b_1} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime}-b_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta b_1} - h\frac{\delta\vartheta}{\delta b_1}\right),\\
c_1' &= \frac{-1}{m_1}\frac{\delta V}{\delta c_1} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime}-c_{\prime\prime}) + \frac{m_2}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta c_1} - h\frac{\delta\vartheta}{\delta c_1}\right),\\
a_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta a_2} = \frac{\lambda}{m_1+m_2}(x_{\prime\prime}-a_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta a_2} - h\frac{\delta\vartheta}{\delta a_2}\right),\\
b_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta b_2} = \frac{\lambda}{m_1+m_2}(y_{\prime\prime}-b_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta b_2} - h\frac{\delta\vartheta}{\delta b_2}\right),\\
c_2' &= \frac{-1}{m_2}\frac{\delta V}{\delta c_2} = \frac{\lambda}{m_1+m_2}(z_{\prime\prime}-c_{\prime\prime}) + \frac{m_1}{m_1+m_2}\left(\rho_0\frac{\delta r_0}{\delta c_2} - h\frac{\delta\vartheta}{\delta c_2}\right);
\end{aligned}\right\} \qquad (S^2.)$$

besides the following expression for the time of motion of the system:

$$t = \frac{\delta V}{\delta H} = \int_{r_0}^{r}\frac{dr}{\rho}, \qquad (T^2.)$$

which gives by (K².), and by (79.), (80.),

$$t = \frac{m_1+m_2}{\lambda}. \qquad (U^2.)$$

The six equations (R².) give the six intermediate integrals, and the six equations (S².) give the six final integrals of the six known differential equations of motion (74.) for any binary system, if we eliminate or determine the three auxiliary quantities h, $H_{\prime}$, $H_{\prime\prime}$, by the three conditions (I².) (T².) (U².). Thus, if we observe that the distances r, r_0, and the included angle ϑ, depend only on relative coordinates, which may be thus denoted,

$$\left.\begin{aligned}
x_1-x_2=\xi, \quad y_1-y_2=\eta, \quad z_1-z_2=\zeta,\\
a_1-a_2=\alpha, \quad b_1-b_2=\beta, \quad c_1-c_2=\gamma
\end{aligned}\right| \qquad (82.)$$

we obtain by easy combinations the three following intermediate integrals for the centre of gravity of the system:

$$x'_{,,}t = x_{,,} - a_{,,}, \quad y'_{,,}t = y_{,,} - b_{,,}, \quad z'_{,,}t = z_{,,} - c_{,,}, \tag{83.}$$

and the three following final integrals,

$$a'_{,,}t = x_{,,} - a_{,,}, \quad b'_{,,}t = y_{,,} - b_{,,}, \quad c'_{,,}t = z_{,,} - c_{,,}, \tag{84.}$$

expressing the well-known law of the rectilinear and uniform motion of that centre. We obtain also the three following intermediate integrals for the relative motion of one point of the system about the other:

$$\left.\begin{aligned} \xi' &= \rho \frac{\delta r}{\delta \xi} + h \frac{\delta \vartheta}{\delta \xi}, \\ \eta' &= \rho \frac{\delta r}{\delta \eta} + h \frac{\delta \vartheta}{\delta \eta}, \\ \zeta' &= \rho \frac{\delta r}{\delta \zeta} + h \frac{\delta \vartheta}{\delta \zeta}, \end{aligned}\right\} \tag{85.}$$

and the three following final integrals,

$$\left.\begin{aligned} \alpha' &= \rho_0 \frac{\delta r_0}{\delta \alpha} - h \frac{\delta \vartheta}{\delta \alpha}, \\ \beta' &= \rho_0 \frac{\delta r_0}{\delta \beta} - h \frac{\delta \vartheta}{\delta \beta}, \\ \gamma' &= \rho_0 \frac{\delta r_0}{\delta \gamma} - h \frac{\delta \vartheta}{\delta \gamma}; \end{aligned}\right\} \tag{86.}$$

in which the auxiliary quantities h, H, are to be determined by (I^2.), (T^2.), and in which the dependence of r, r_0, ϑ, on ξ, η, ζ, α, β, γ, is expressed by the following equations:

$$\begin{gathered} r = \sqrt{\xi^2 + \eta^2 + \zeta^2}, \quad r_0 = \sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\ r r_0 \cos \vartheta = \xi\alpha + \eta\beta + \zeta\gamma. \end{gathered} \tag{87.}$$

If then we put, for abridgement,

$$A = \frac{\rho}{r} + \frac{h}{r^2 \tan \vartheta}, \quad B = \frac{h}{r r_0 \sin \vartheta}, \quad C = \frac{-\rho_0}{r_0} + \frac{h}{r_0^2 \tan \vartheta}, \tag{88.}$$

we shall have these three intermediate integrals,

$$\xi' = A\xi - B\alpha, \quad \eta' = A\eta - B\beta, \quad \zeta' = A\zeta - B\gamma, \tag{89.}$$

and these three final integrals,

$$\alpha' = B\xi - C\alpha, \quad \beta' = B\eta - C\beta, \quad \gamma' = B\zeta - C\gamma, \tag{90.}$$

of the equations of relative motion. These integrals give,

$$\begin{gathered} \xi\eta' - \eta\xi' = \alpha\beta' - \beta\alpha' = B(\alpha\eta - \beta\xi), \\ \eta\zeta' - \zeta\eta' = \beta\gamma' - \gamma\beta' = B(\beta\zeta - \gamma\eta), \\ \zeta\xi' - \xi\zeta' = \gamma\alpha' - \alpha\gamma' = B(\gamma\xi - \alpha\zeta), \end{gathered} \tag{91.}$$

and

$$\zeta(\alpha\beta' - \beta\alpha') + \xi(\beta\gamma' - \gamma\beta') + \eta(\gamma\alpha' - \alpha\gamma') = 0; \tag{92.}$$

they contain therefore the known law of equable description of areas, and the law of a plane relative orbit. If we take for simplicity this plane for the plane $\xi\eta$, the quantities ζ, ζ', γ, γ' will vanish; and we may put,

$$\left.\begin{array}{lll}\xi = r\cos\theta, & \eta = r\sin\theta, & \zeta = 0,\\ \alpha = r_0\cos\theta_0, & \beta = r_0\sin\theta_0, & \gamma = 0,\end{array}\right\} \qquad (93.)$$

and

$$\left.\begin{array}{lll}\xi' = r'\cos\theta - \theta' r\sin\theta, & \eta' = r'\sin\theta + \theta' r\cos\theta, & \zeta' = 0,\\ \alpha' = r_0'\cos\theta_0 - \theta_0' r_0\sin\theta_0, & \beta' = r_0'\sin\theta_0 + \theta_0' r_0\cos\theta_0, & \gamma' = 0,\end{array}\right\} \qquad (94.)$$

the angles θ, θ_0 being counted from some fixed line in the plane, and being such that their difference

$$\theta - \theta_0 = \vartheta. \qquad (95.)$$

These values give

$$\xi\eta' - \eta\xi' = r^2\theta', \quad \alpha\beta' - \beta\alpha' = r_0^2\theta_0', \quad \alpha\eta - \beta\xi = rr_0\sin\vartheta, \qquad (96.)$$

and therefore, by (88.) and (91.),

$$r^2\theta' = r_0^2\theta_0' = h; \qquad (97.)$$

the quantity $\frac{1}{2}h$ is therefore the constant areal velocity in the relative motion of the system; a result which is easily seen to be independent of the directions of the three rectangular coordinates. The same values, (93.), (94.), give

$$\left.\begin{array}{lll}\xi\cos\theta + \eta\sin\theta = r, & \xi'\cos\theta + \eta'\sin\theta = r', & \alpha\cos\theta + \beta\sin\theta = r_0\cos\vartheta,\\ \alpha\cos\theta_0 + \beta\sin\theta_0 = r_0, & \alpha'\cos\theta_0 + \beta'\sin\theta_0 = r_0', & \xi\cos\theta_0 + \eta\sin\theta_0 = r\cos\vartheta,\end{array}\right\} \qquad (98.)$$

and therefore, by the intermediate and final integrals, (89.), (90.),

$$r' = \rho, \quad r_0' = \rho_0; \qquad (99.)$$

results which evidently agree with the condition (T^2.), and which give by (79.) and (81.), for all directions of coordinates,

$$r'^2 + \frac{h^2}{r^2} - 2(m_1 + m_2)f(r) = r_0'^2 + \frac{h^2}{r_0^2} - 2(m_1 + m_2)f(r_0) = 2H_{,}\left(\frac{1}{m_1} + \frac{1}{m_2}\right); \qquad (100.)$$

the other auxiliary quantity $H_{,}$ is therefore also a constant, independent of the time, and enters as such into the constant part in the expression for $\left(r'^2 + \frac{h^2}{r^2}\right)$ the square of the relative velocity. The equation of condition (I^2.), connecting these two constants h, $H_{,}$, with the extreme lengths of the radius vector r, and with the angle ϑ described by this radius in revolving from its initial to its final direction, is the equation of the plane relative orbit; and the other equation of condition (T^2.), connecting the same two constants with the same extreme distances and with the time, gives the law of the velocity of mutual approach or recess.

We may remark that the part $V_{,}$ of the whole characteristic function V, which represents the relative action and determines the relative motion in the system, namely,

$$V_{,} = \frac{m_1 m_2}{m_1 + m_2}\left(h\vartheta + \int_{r_0}^{r} \rho\, dr\right), \qquad (V^2.)$$

may be put, by (I^2.), under the form

$$= \frac{m_1 m_2}{m_1 + m_2}\int_{r_0}^{r}\left(\rho - h\frac{\delta\rho}{\delta h}\right) dr, \qquad (W^2.)$$

or finally, by (79.),

$$V_{,}=2\int_{r_0}^{r}\frac{m_1m_2f(r)+H_{,}}{\rho}\,dr; \tag{X^2.}$$

the condition (I^2.) may also itself be transformed, by (79.), as follows:

$$\vartheta=h\int_{r_0}^{r}\frac{dr}{r^2\rho}: \tag{Y^2.}$$

results which all admit of easy verifications. The partial differential equations connected with the law of relative living force, which the characteristic function $V_{,}$ of relative motion must satisfy, may be put under the following forms:

$$\left.\begin{aligned}\left(\frac{\delta V_{,}}{\delta r}\right)^2+\frac{1}{r^2}\left(\frac{\delta V_{,}}{\delta\vartheta}\right)^2&=\frac{2m_1m_2}{m_1+m_2}(U+H_{,}),\\ \left(\frac{\delta V_{,}}{\delta r_0}\right)^2+\frac{1}{r_0^2}\left(\frac{\delta V_{,}}{\delta\vartheta}\right)^2&=\frac{2m_1m_2}{m_1+m_2}(U_0+H_{,});\end{aligned}\right\} \tag{Z^2.}$$

and if the first of the equations of this pair have its variation taken with respect to r and ϑ, attention being paid to the dynamical meanings of the coefficients of the characteristic function, it will conduct (as in former instances) to the known differential equations of motion of the second order.

On the undisturbed Motion of a Planet or Comet about the Sun: Dependence of the Characteristic Function of such Motion, on the chord and the sum of the Radii.

15. To particularize still further, let

$$f(r)=\frac{1}{r}, \tag{101.}$$

that is, let us consider a binary system, such as a planet or comet and the sun, with the Newtonian law of attraction; and let us put, for abridgement,

$$m_1+m_2=\mu,\quad \frac{h^2}{\mu}=p,\quad \frac{-m_1m_2}{2H_{,}}=\mathrm{a}. \tag{102.}$$

The characteristic function $V_{,}$ of relative motion may now be expressed as follows:

$$V_{,}=\frac{m_1m_2}{\sqrt{\mu}}\left(\vartheta\sqrt{p}+\int_{r_0}^{r}\pm\sqrt{\frac{2}{r}-\frac{1}{\mathrm{a}}-\frac{p}{r^2}}\,.\,dr\right); \tag{A^3.}$$

in which p is to be considered as a function of the extreme radii vectores r, r_0, and of their included angle ϑ, involving also the quantity a, or the connected quantity $H_{,}$, and determined by the condition

$$\vartheta=\int_{r_0}^{r}\frac{\pm\,dr}{r^2\sqrt{\frac{2}{rp}-\frac{1}{\mathrm{a}p}-\frac{1}{r^2}}}, \tag{B^3.}$$

that is, by the derivative of the formula (A^3.), taken with respect to p: the upper sign being taken in each expression when the distance r is increasing, and the lower sign when that distance is diminishing, and the quantity p being treated as constant in calculating the two definite integrals. It results from the foregoing remarks, that this quantity p is constant also in the sense of being independent of the time, so as not to vary in the course of the motion; and that the condition (B³.), connecting this constant with r, r_0, ϑ, a, is the equation of the plane relative orbit; which is therefore (as it has long been known to be) an ellipse, hyperbola, or parabola, according as the constant a is positive, negative, or zero, the origin of r being always a focus of the curve, and p being the semiparameter. It results also, that the time of motion may be thus expressed:

$$t=\frac{\delta V_{,}}{\delta H_{,}}=\frac{2\mathrm{a}^2}{m_1 m_2}\frac{\delta V_{,}}{\delta \mathrm{a}}, \tag{C³.}$$

and therefore thus:

$$t=\int_{r_0}\frac{\pm\, dr}{\sqrt{\frac{2\mu}{r}-\frac{\mu}{\mathrm{a}}-\frac{\mu p}{r^2}}}, \tag{D³.}$$

which latter is a known expression. Confining ourselves at present to the case $\mathrm{a}>0$, and introducing the known auxiliary quantities called excentricity and excentric anomaly, namely,

$$e=\sqrt{1-\frac{p}{\mathrm{a}}}, \tag{103.}$$

and

$$v=\cos^{-1}\left(\frac{\mathrm{a}-r}{\mathrm{a}e}\right), \tag{104.}$$

which give

$$\pm\sqrt{2\mathrm{a}r-r^2-p\mathrm{a}}=\mathrm{a}e\sin v, \tag{105.}$$

v being considered as continually increasing with the time; and therefore, as is well known,

$$\begin{gathered} r=\mathrm{a}\,(1-e\cos v), \quad r_0=\mathrm{a}\,(1-e\cos v_0), \\ \vartheta=2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}}\tan\frac{v}{2}\right\}-2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}}\tan\frac{v_0}{2}\right\}, \end{gathered} \tag{106.}$$

and

$$t=\sqrt{\frac{\mathrm{a}^{\circ}}{\mu}}\,.\,(v-v_0-e\sin v+e\sin v_0); \tag{107.}$$

we find that this expression for the characteristic function of relative motion,

$$V_{,}=\frac{m_1 m_2}{\sqrt{\mu}}\int_{r_0}^{r}\frac{\pm\left(\frac{2}{r}-\frac{1}{\mathrm{a}}\right)dr}{\sqrt{\frac{2}{r}-\frac{1}{\mathrm{a}}-\frac{p}{r^2}}}, \tag{E³.}$$

deduced from (A³.) and (B³.), may be transformed as follows:

$$V.=m_1 m_2\sqrt{\frac{\mathrm{a}}{\mu}}(v-v_0+e\sin v-e\sin v_0): \tag{F³.}$$

in which the excentricity e, and the final and initial excentric anomalies v, v_0, are to be considered as functions of the final and initial radii r, r_0, and of the included angle ϑ, determined by the equations (106.). The expression (F³.) may be thus written:

$$V_{,}=2m_1 m_2\sqrt{\frac{\mathrm{a}}{\mu}}(v_{,}+e_{,}\sin v_{,}), \tag{G³.}$$

if we put, for abridgement,

$$v_{,} - \frac{}{2}, \quad e_{,} = e \cos \frac{v+v_0}{2}. \tag{108.}$$

for the complete determination of the characteristic function of the present relative motion, it remains therefore to determine the two variables $v_{,}$ and $e_{,}$, as functions of r, r_0, ϑ, or of some other set of quantities which mark the shape and size of the plane triangle bounded by the final and initial elliptic radii vectores and by the elliptic chord.

For this purpose it is convenient to introduce this elliptic chord itself, which we shall call $\pm\tau$, so that

$$\tau^2 = r^2 + r_0^2 - 2rr_0 \cos\vartheta; \tag{109.}$$

because this chord may be expressed as a function of the two variables $v_{,}$, $e_{,}$, (involving also the mean distance a,) as follows. The value (106.) for the angle ϑ, that is, by (95.), for $\theta - \theta_0$, gives

$$\theta - 2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}} \tan\frac{v}{2}\right\} = \theta_0 - 2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}} \tan\frac{v_0}{2}\right\} = \varpi, \tag{110.}$$

ϖ being a new constant independent of the time, namely, one of the values of the polar angle θ, which correspond to the minimum of radius vector; and therefore, by (106.),

$$\left.\begin{aligned} &r\cos(\theta-\varpi) = \mathrm{a}(\cos v - e), \quad r\sin(\theta-\varpi) = \mathrm{a}\sqrt{1-e^2}\sin v, \\ &r_0\cos(\theta_0-\varpi) = \mathrm{a}(\cos v_0 - e), \quad r_0\sin(\theta_0-\varpi) = \mathrm{a}\sqrt{1-e^2}\sin v_0; \end{aligned}\right\} \tag{111.}$$

expressions which give the following value for the square of the elliptic chord:

$$\left.\begin{aligned} \tau^2 &= \{r\cos(\theta-\varpi) - r_0\cos(\theta_0-\varpi)\}^2 + \{r\sin(\theta-\varpi) - r_0\sin(\theta_0-\varpi)\} \\ &= \mathrm{a}^2\{(\cos v - \cos v_0)^2 + (1-e^2)(\sin v - \sin v_0)^2\} \\ &= 4\mathrm{a}^2 \sin v_{,}^2 \left\{\left(\sin\frac{v+v_0}{2}\right)^2 + (1-e^2)\left(\cos\frac{v+v_0}{2}\right)\right. \\ &= 4\mathrm{a}^2(1-e_{,}^2)\sin v_{,}^2: \end{aligned}\right\} \tag{112.}$$

we may also consider τ as having the same sign with $\sin v_{,}$, if we consider it as alternately positive and negative, in the successive elliptic periods or revolutions, beginning with the initial position.

Besides, if we denote by σ the sum of the two elliptic radii vectores, final and initial, so that

$$\sigma = r + r_0, \tag{113.}$$

we shall have, with our present abridgements,

$$\sigma = 2\mathrm{a}(1 - e_{,}\cos v_{,}); \tag{114.}$$

the variables $v_{,}$, $e_{,}$ are therefore functions of σ, τ, a, and consequently the characteristic function $V_{,}$ is itself a function of those three quantities. We may therefore put

$$V_{,} = \frac{m_1 m_2 w}{m_1 + m_2}, \tag{H^3.}$$

w being a function of σ, τ, a, of which the form is to be determined by eliminating $v_{,}$, $e_{,}$ between the three equations,

$$\left.\begin{aligned} w &= 2\sqrt{\mu\mathrm{a}}\,(v_{,} + e_{,}\sin v_{,}), \\ \sigma &= 2\mathrm{a}(1 - e_{,}\cos v_{,}), \\ \tau &= 2\mathrm{a}(1 - e_{,}^2)^{\frac{1}{2}}\sin v_{,}; \end{aligned}\right\} \tag{I^3.}$$

and we may consider this new function w as itself a characteristic function of elliptic motion; the law of its variation being expressed as follows, in the notation of the present essay*:

$$\delta w = \xi'\delta\xi - \alpha'\delta\alpha + \eta'\delta\eta - \beta'\delta\beta + \zeta'\delta\zeta - \gamma'\delta\gamma + \frac{t\mu\,\delta\mathrm{a}}{2\mathrm{a}^2} \tag{K^3.}$$

In this expression, ξ, η, ζ are the relative coordinates of the point m_1, at the time t, referred to the other attracting point m_2 as an origin, and to any three rectangular axes; ξ', η', ζ' are their rates of increase, or the three rectangular components of final relative velocity; α, β, γ, α', β', γ' are the initial values, or values at the time zero, of these relative coordinates and components of relative velocity; a is a quantity independent of the time, namely, the mean distance of the two points m_1, m_2; and μ is the sum of their masses. And all the properties of the undisturbed elliptic motion of a planet or comet about the sun may be deduced in a new way, from the simplified characteristic function w, by comparing its variation (K^3.) with the following other form,

$$\delta w = \frac{\delta w}{\delta\sigma}\delta\sigma + \frac{\delta w}{\delta\tau}\delta\tau + \frac{\delta w}{\delta\mathrm{a}}\delta\mathrm{a}; \tag{L^3.}$$

in which we are to observe that

$$\left.\begin{aligned} \sigma &= \sqrt{\xi^2+\eta^2+\zeta^2} + \sqrt{\alpha^2+\beta^2+\gamma^2}, \\ \tau &= \pm\sqrt{(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2}. \end{aligned}\right\} \tag{M^3.}$$

By this comparison we are brought back to the general integral equations of the relative motion of a binary system, (89.) and (90.); but we have now the following particular values for the coefficients A, B, C:

$$A = \frac{1}{r}\frac{\delta w}{\delta\sigma} + \frac{1}{\tau}\frac{\delta w}{\delta\tau}, \quad B = \frac{1}{\tau}\frac{\delta w}{\delta\tau}, \quad C = \frac{1}{r_0}\frac{\delta w}{\delta\sigma} + \frac{1}{\tau}\frac{\delta w}{\delta\tau}; \tag{N^3.}$$

and with respect to the three partial differential coefficients, $\frac{\delta w}{\delta\sigma}$, $\frac{\delta w}{\delta\tau}$, $\frac{\delta w}{\delta\mathrm{a}}$, we have the following relation between them:

$$\mathrm{a}\frac{\delta w}{\delta\mathrm{a}} + \sigma\frac{\delta w}{\delta\sigma} + \tau\frac{\delta w}{\delta\tau} = \frac{w}{2}, \tag{O^3.}$$

the function w being homogeneous of the dimension $\frac{1}{2}$ with respect to the three quantities a, σ, τ; we have also, by (I^3.),

$$\frac{\delta w}{\delta\sigma} = \sqrt{\frac{\mu}{\mathrm{a}}}\cdot\frac{\sin v_{,}}{e_{,} - \cos v_{,}}, \quad \frac{\delta w}{\delta\tau} = \sqrt{\frac{\mu}{\mathrm{a}}}\cdot\frac{\sqrt{1-e_{,}^2}}{\cos v_{,} - e_{,}}, \tag{P^3.}$$

and therefore

$$\frac{\delta w}{\delta\sigma}\frac{\delta w}{\delta\tau} = \frac{-2\mu\tau}{\sigma^2-\tau^2}, \quad \left(\frac{\delta w}{\delta\sigma}\right)^2 + \left(\frac{\delta w}{\delta\tau}\right)^2 + \frac{\mu}{\mathrm{a}} = \frac{4\mu\sigma}{\sigma^2-\tau^2}, \tag{Q^3.}$$

from which may be deduced the following remarkable expressions:

$$\left.\begin{aligned} \left(\frac{\delta w}{\delta\sigma} + \frac{\delta w}{\delta\tau}\right)^2 &= \frac{4\mu}{\sigma+\tau} - \frac{\mu}{\mathrm{a}}, \\ \left(\frac{\delta w}{\delta\sigma} - \frac{\delta w}{\delta\tau}\right)^2 &= \frac{4\mu}{\sigma-\tau} - \frac{\mu}{\mathrm{a}}. \end{aligned}\right\} \tag{R^3.}$$

These expressions will be found to be important in the application of the present method to the theory of elliptic motion.

---

* $\left[\delta H, = \frac{m_1 m_2 \delta\mathrm{a}}{2\mathrm{a}^2}.\right]$

16. We shall not enter, on this occasion, into any details of such application; but we may remark, that the circumstance of the characteristic function involving only the elliptic chord and the sum of the extreme radii, (besides the mean distance and the sum of the masses,) affords, by our general method, a new proof of the well-known theorem that the elliptic time also depends on the same chord and sum of radii; and gives a new expression for the law of this dependence, namely,*

$$t=\frac{2\mathrm{a}^2}{\mu}\frac{\delta w}{\delta \mathrm{a}}. \qquad (\mathrm{S}^3.)$$

We may remark also, that the same form of the characteristic function of elliptic motion conducts, by our general method, to the following curious, but not novel property, of the ellipse, that if any two tangents be drawn to such a curve, from any common point outside, these tangents subtend equal angles at one focus; they subtend also equal angles at the other. Reciprocally, if any plane curve possess this property, when referred to a fixed point in its own plane, which may be taken as the origin of polar coordinates r, θ, the curve must satisfy the following equation in mixed differences:

$$\operatorname{cotan}\left(\frac{\Delta\theta}{2}\right).\Delta\frac{1}{r}=(\Delta+2)\frac{d}{d\theta}\frac{1}{r}, \qquad (115.)$$

which may be brought to the following form,

$$\left(\frac{d}{d\theta}+\frac{d^3}{d\theta^3}\right)\frac{1}{r}=0, \qquad (116.)$$

and therefore gives, by integration,

$$r=\frac{p}{1+e\cos(\theta-\varpi)}, \qquad (117.)$$

the curve is, consequently, a conic section, and the fixed point is one of its foci.

The properties of parabolic are included as limiting cases in those of elliptic motion, and may be deduced from them by making

$$H_{,}=0, \quad \text{or} \quad \mathrm{a}=\infty; \qquad (118.)$$

and therefore the characteristic function w and the time t, in parabolic as well as in elliptic motion, are functions of the chord and of the sum of the radii. By thus making a infinite in the foregoing expressions, we find, for parabolic motion, the partial differential equations

$$\left(\frac{\delta w}{\delta\sigma}+\frac{\delta w}{\delta\tau}\right)^2=\frac{4\mu}{\sigma+\tau}, \quad \left(\frac{\delta w}{\delta\sigma}-\frac{\delta w}{\delta\tau}\right)^2=\frac{4\mu}{\sigma-\tau}; \qquad (\mathrm{T}^3.)$$

and in fact the parabolic form of the simplified characteristic function w may easily be shown to be

$$w=2\sqrt{\mu}\,(\sqrt{\sigma+\tau}\mp\sqrt{\sigma-\tau}), \qquad (\mathrm{U}^3.)$$

τ being, as before, the chord, and σ the sum of the radii; while the analogous limit of the expression (S³.), for the time, is

$$t=\frac{}{6\sqrt{\mu}}\{(\sigma+\tau)^{\frac{3}{2}}\mp(\sigma-\tau)^{\frac{3}{2}}\}: \qquad (\mathrm{V}^3.)$$

which latter is a known expression.†

---

* [Equation (S³.) is easily reducible to the well-known form of Lambert's theorem. Cf. Whittaker, *Analytical Dynamics* (1927), p. 92.]

† [This special case of Lambert's theorem was given by Euler in 1742. Euler, *Miscell. Berolin.* Tome VII.]

The formulæ (K^3.) and (L^3.), to the comparison of which we have reduced the study of elliptic motion, extend to hyperbolic motion also; and in any binary system, with Newton's law of attraction, the simplified characteristic function w may be expressed by the definite integral*

$$w = \int_{-\tau}^{\tau} \sqrt{\frac{\mu}{\sigma+\tau} - \frac{\mu}{4\mathrm{a}}} \, . \, d\tau, \qquad (W^3.)$$

this function w being still connected with the relative action V, by the equation (H^3.); while the time t, which may always be deduced from this function, by the law of varying action, is represented by this other connected integral,

$$t = \frac{1}{4} \int_{-\tau}^{\tau} \left(\frac{\mu}{\sigma+\tau} \quad \frac{\mu}{4\mathrm{a}} \right)^{-\frac{1}{2}} d\tau: \qquad (X^3.)$$

provided that, within the extent of these integrations, the radical does not vanish nor become infinite. When this condition is not satisfied, we may still express the simplified characteristic function w, and the time t, by the following analogous integrals:

$$w = \int_{\tau_{,}}^{\sigma_{,}} \pm \sqrt{\frac{2\mu}{\sigma_{,}} - \frac{\mu}{\mathrm{a}}} \, d\sigma_{,}, \qquad (Y^3.)$$

and

$$= \int_{\tau_{,}}^{\sigma_{,}} \pm \left(\frac{2\mu}{\sigma_{,}} - \frac{\mu}{\mathrm{a}} \right)^{-\frac{1}{2}} d\sigma_{,}, \qquad (Z^3.)$$

in which we have put for abridgement

$$\sigma_{,} = \frac{\sigma+\tau}{2}, \quad \tau_{,} = \frac{\sigma-\tau}{2}, \qquad (119.)$$

and in which it is easy to determine the signs of the radicals. But to treat fully of these various transformations would carry us too far at present, for it is time to consider the properties of systems with more points than two.

On Systems of three Points, in general; and on their Characteristic Functions.

17. For any system of three points, the known differential equations of motion of the 2nd order are included in the following formula:

$$\left. \begin{aligned} m_1 (x_1'' \delta x_1 + y_1'' \delta y_1 + z_1'' \delta z_1) + m_2 (x_2'' \delta x_2 + y_2'' \delta y_2 + z_2'' \delta z_2) \\ + m_3 (x_3'' \delta x_3 + y_3'' \delta y_3 + z_3'' \delta z_3) = \delta U, \end{aligned} \right\} \qquad (120.)$$

the known force-function U having the form

$$U = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)}, \qquad (121.)$$

in which $f^{(1,2)}$, $f^{(1,3)}$, $f^{(2,3)}$ are functions respectively of the three following mutual distances of the points of the system:

$$\left. \begin{aligned} r^{(1,2)} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \\ r^{(1,3)} &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2}, \\ r^{(2,3)} &= \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2}: \end{aligned} \right\} \qquad (122.)$$

---

* [Cf. p. 48.]

the known differential equations of motion are therefore, separately, for the point m_1,

$$\left.\begin{aligned} x_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta x_1} + m_3 \frac{\delta f^{(1,3)}}{\delta x_1}, \\ y_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta y_1} + m_3 \frac{\delta f^{(1,3)}}{\delta y_1}, \\ z_1'' &= m_2 \frac{\delta f^{(1,2)}}{\delta z_1} + m_3 \frac{\delta f^{(1,3)}}{\delta z_1} \end{aligned}\right\} \qquad (123.)$$

with six other analogous equations for the points m_2 and m_3; x_1'', &c., denoting the component accelerations of the three points m_1, m_2, m_3, or the second differential coefficients of their coordinates, taken with respect to the time. To integrate these equations is to assign, by their means, nine relations between the time t, the three masses m_1, m_2, m_3, the nine varying coordinates $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$, and their nine initial values and nine initial rates of increase, which may be thus denoted, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_1', b_1', c_1', a_2', b_2', c_2', a_3', b_3', c_3'. The known intermediate integral containing the law of living force, namely,

$$\left.\begin{aligned} &\tfrac{1}{2}m_1(x_1'^2+y_1'^2+z_1'^2)+\tfrac{1}{2}m_2(x_2'^2+y_2'^2+z_2'^2)+\tfrac{1}{2}m_3(x_3'^2+y_3'^2+z_3'^2) \\ &\qquad = m_1m_2f^{(1,2)}+m_1m_3f^{(1,3)}+m_2m_3f^{(2,3)}+H, \end{aligned}\right\} \qquad (124.)$$

gives the following initial relation:

$$\left.\begin{aligned} &\tfrac{1}{2}m_1(a_1'^2+b_1'^2+c_1'^2)+\tfrac{1}{2}m_2(a_2'^2+b_2'^2+c_2'^2)+\tfrac{1}{2}m_3(a_3'^2+b_3'^2+c_3'^2) \\ &\qquad = m_1m_2f_0^{(1,2)}+m_1m_3f_0^{(1,3)}+m_2m_3f_0^{(2,3)}+H, \end{aligned}\right\} \qquad (125.)$$

in which $f_0^{(1,2)}$, $f_0^{(1,3)}$, $f_0^{(2,3)}$ are composed of the initial coordinates, in the same manner as $f^{(1,2)}$, $f^{(1,3)}$, $f^{(2,3)}$ are composed of the final coordinates. If then we knew the nine final integrals of the equations of motion of this ternary system, and combined them with the initial form (125.) of the law of living force, we should have ten relations to determine the ten quantities t, a_1', b_1', c_1', a_2', b_2', c_2', a_3', b_3', c_3', namely, the time and the nine initial components of the velocities of the three points, as functions of the nine final and nine initial coordinates, and of the quantity H, involving also the masses; we could therefore determine whatever else depends on the manner and time of motion of the system, from its initial to its final position, as a function of the same extreme coordinates, and of H. In particular, we could determine the action V, or the accumulated living force of the system, namely,

$$V = m_1\int_0^t (x_1'^2+y_1'^2+z_1'^2)\,dt + m_2\int_0^t (x_2'^2+y_2'^2+z_2'^2)\,dt + m_3\int_0^t (x_3'^2+y_3'^2+z_3'^2)\,dt, \qquad (\mathrm{A}^4.)$$

as a function of these nineteen quantities, x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, H; and might then calculate the variation of this function,

$$\left.\begin{aligned} \delta V = {} & \frac{\delta V}{\delta x_1}\delta x_1 + \frac{\delta V}{\delta y_1}\delta y_1 + \frac{\delta V}{\delta z_1}\delta z_1 + \frac{\delta V}{\delta a_1}\delta a_1 + \frac{\delta V}{\delta b_1}\delta b_1 + \frac{\delta V}{\delta c_1}\delta c_1 \\ & + \frac{\delta V}{\delta x_2}\delta x_2 + \frac{\delta V}{\delta y_2}\delta y_2 + \frac{\delta V}{\delta z_2}\delta z_2 + \frac{\delta V}{\delta a_2}\delta a_2 + \frac{\delta V}{\delta b_2}\delta b_2 + \frac{\delta V}{\delta c_2}\delta c_2 \\ & + \frac{\delta V}{\delta x_3}\delta x_3 + \frac{\delta V}{\delta y_3}\delta y_3 + \frac{\delta V}{\delta z_3}\delta z_3 + \frac{\delta V}{\delta a_3}\delta a_3 + \frac{\delta V}{\delta b_3}\delta b_3 + \frac{\delta V}{\delta c_3}\delta c_3 \\ & + \frac{\delta V}{\delta H}\delta H. \end{aligned}\right\} \qquad (\mathrm{B}^4.)$$

But the law of varying action gives, *previously*, the following expression for this variation:

$$\left.\begin{aligned}\delta V = m_1(x_1'\delta x_1 - a_1'\delta a_1 + y_1'\delta y_1 - b_1'\delta b_1 + z_1'\delta z_1 - c_1'\delta c_1)&\\ + m_2(x_2'\delta x_2 - a_2'\delta a_2 + y_2'\delta y_2 - b_2'\delta b_2 + z_2'\delta z_2 - c_2'\delta c_2)&\\ + m_3(x_3'\delta x_3 - a_3'\delta a_3 + y_3'\delta y_3 - b_3'\delta b_3 + z_3'\delta z_3 - c_3'\delta c_3)&\\ + t\,\delta H;&\end{aligned}\right\} \qquad (\mathrm{C}^4.)$$

and shows, therefore, that the research of all the intermediate and all the final integral equations, of motion of the system, may be reduced, reciprocally, to the search and differentiation of this one characteristic function V; because if we knew this one function, we should have the nine intermediate integrals of the known differential equations, under the forms

$$\begin{aligned}&\frac{\delta V}{\delta x_1} = m_1 x_1', \quad \frac{\delta V}{\delta y_1} = m_1 y_1', \quad \frac{\delta V}{\delta z_1} = m_1 z_1',\\ &\frac{\delta V}{\delta x_2} = m_2 x_2', \quad \frac{\delta V}{\delta y_2} = m_2 y_2', \quad \frac{\delta V}{\delta z_2} = m_2 z_2',\\ &\frac{\delta V}{\delta x_3} = m_3 x_3', \quad \frac{\delta V}{\delta y_3} = m_3 y_3', \quad \frac{\delta V}{\delta z_3} = m_3 z_3',\end{aligned} \qquad (\mathrm{D}^4.)$$

and the nine final integrals under the forms

$$\begin{aligned}&\frac{\delta V}{\delta a_1} = -m_1 a_1', \quad \frac{\delta V}{\delta b_1} = -m_1 b_1', \quad \frac{\delta V}{\delta c_1} = -m_1 c_1,\\ &\frac{\delta V}{\delta a_2} = -m_2 a_2', \quad \frac{\delta V}{\delta b_2} = -m_2 b_2', \quad \frac{\delta V}{\delta c_2} = -m_2 c_2,\\ &\frac{\delta V}{\delta a_3} = -m_3 a_3', \quad \frac{\delta V}{\delta b_3} = -m_3 b_3', \quad \frac{\delta V}{\delta c_3} = -m_3 c_3',\end{aligned} \qquad (\mathrm{E}^4.)$$

the auxiliary constant H being to be eliminated, and the time t introduced, by this other equation, which has often occurred in this essay,

$$t = \frac{\delta V}{\delta H}. \qquad (\mathrm{E.})$$

The same law of varying action suggests also a method of investigating the form of this characteristic function V, not requiring the previous integration of the known equations of motion; namely, the integration of a pair of partial differential equations connected with the law of living force; which are

$$\left.\begin{aligned}\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta x_1}\right)^2 + \left(\frac{\delta V}{\delta y_1}\right)^2 + \left(\frac{\delta V}{\delta z_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta x_2}\right)^2 + \left(\frac{\delta V}{\delta y_2}\right)^2 + \left(\frac{\delta V}{\delta z_2}\right)^2\right\}&\\ + \frac{1}{2m_3}\left\{\left(\frac{\delta V}{\delta x_3}\right)^2 + \left(\frac{\delta V}{\delta y_3}\right)^2 + \left(\frac{\delta V}{\delta z_3}\right)^2\right\} = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H,&\end{aligned}\right\} \qquad (\mathrm{F}^4.)$$

and

$$\left.\begin{aligned}\frac{1}{2m_1}\left\{\left(\frac{\delta V}{\delta a_1}\right)^2 + \left(\frac{\delta V}{\delta b_1}\right)^2 + \left(\frac{\delta V}{\delta c_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V}{\delta a_2}\right)^2 + \left(\frac{\delta V}{\delta b_2}\right)^2 + \left(\frac{\delta V}{\delta c_2}\right)^2\right\}&\\ + \frac{1}{2m_3}\left\{\left(\frac{\delta V}{\delta a_3}\right)^2 + \left(\frac{\delta V}{\delta b_3}\right)^2 + \left(\frac{\delta V}{\delta c_3}\right)^2\right\} = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H.&\end{aligned}\right\} \qquad (\mathrm{G}^4.)$$

And to diminish the difficulty of thus determining the function V, which depends on 18 coordinates, we may separate it, by principles already explained, into a part $V_{,,}$ depending only on the motion of the centre of gravity of the system, and determined by the formula (H¹.), and another part $V_{,}$, depending only on the relative motions of the points of the system about this internal centre, and equal to the accumulated living force, connected with this relative motion only. In this manner the difficulty is reduced to determining the relative action $V_{,}$; and if we introduce the relative coordinates

$$\left.\begin{array}{lll} \xi_1 = x_1 - x_3, & \eta_1 = y_1 - y_3, & \zeta_1 = z_1 - z_3, \\ \xi_2 = x_2 - x_3, & \eta_2 = y_2 - y_3, & \zeta_2 = z_2 - z_3, \end{array}\right\} \tag{126.}$$

and

$$\left.\begin{array}{lll} \alpha_1 = a_1 - a_3, & \beta_1 = b_1 - b_3, & \gamma_1 = c_1 - c_3, \\ \alpha_2 = a_2 - a_3, & \beta_2 = b_2 - b_3, & \gamma_2 = c_2 - c_3, \end{array}\right\} \tag{127.}$$

we easily find, by the principles of the tenth and following numbers, that the function $V_{,}$ may be considered as depending only on these relative coordinates, and on a quantity $H_{,}$ analogous to H (besides the masses of the system); and that it must satisfy two partial differential equations, analogous to (F⁴.) and (G⁴.), namely,

$$\left.\begin{array}{l} \frac{1}{2m_1}\left\{\left(\frac{\delta V_{,}}{\delta \xi_1}\right)^2 + \left(\frac{\delta V_{,}}{\delta \eta_1}\right)^2 + \left(\frac{\delta V_{,}}{\delta \zeta_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V_{,}}{\delta \xi_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \eta_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \zeta_2}\right)^2\right\} \\ \quad + \frac{1}{2m_3}\left\{\left(\frac{\delta V_{,}}{\delta \xi_1} + \frac{\delta V_{,}}{\delta \xi_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \eta_1} + \frac{\delta V_{,}}{\delta \eta_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \zeta_1} + \frac{\delta V_{,}}{\delta \zeta_2}\right)^2\right\} \\ = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H_{,}; \end{array}\right\} \tag{H⁴.}$$

and

$$\left.\begin{array}{l} \frac{1}{2m_1}\left\{\left(\frac{\delta V_{,}}{\delta \alpha_1}\right)^2 + \left(\frac{\delta V_{,}}{\delta \beta_1}\right)^2 + \left(\frac{\delta V_{,}}{\delta \gamma_1}\right)^2\right\} + \frac{1}{2m_2}\left\{\left(\frac{\delta V_{,}}{\delta \alpha_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \beta_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \gamma_2}\right)^2\right\} \\ \quad + \frac{1}{2m_3}\left\{\left(\frac{\delta V_{,}}{\delta \alpha_1} + \frac{\delta V_{,}}{\delta \alpha_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \beta_1} + \frac{\delta V_{,}}{\delta \beta_2}\right)^2 + \left(\frac{\delta V_{,}}{\delta \gamma_1} + \frac{\delta V_{,}}{\delta \gamma_2}\right)^2\right\} \\ = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H_{,}: \end{array}\right\} \tag{I⁴.}$$

the law of the variation of this function being, by (Z¹.),

$$\left.\begin{array}{l} \delta V_{,} = t\,\delta H_{,} + m_1 (\xi_1' \delta\xi_1 - \alpha_1' \delta\alpha_1 + \eta_1' \delta\eta_1 - \beta_1' \delta\beta_1 + \zeta_1' \delta\zeta_1 - \gamma_1' \delta\gamma_1) \\ \qquad + m_2 (\xi_2' \delta\xi_2 - \alpha_2' \delta\alpha_2 + \eta_2' \delta\eta_2 - \beta_2' \delta\beta_2 + \zeta_2' \delta\zeta_2 - \gamma_2' \delta\gamma_2) \\ - \frac{1}{m_1 + m_2 + m_3}\left\{\begin{array}{l} (m_1\xi_1' + m_2\xi_2')(m_1\delta\xi_1 + m_2\delta\xi_2) - (m_1\alpha_1' + m_2\alpha_2')(m_1\delta\alpha_1 + m_2\delta\alpha_2) \\ + (m_1\eta_1' + m_2\eta_2')(m_1\delta\eta_1 + m_2\delta\eta_2) - (m_1\beta_1' + m_2\beta_2')(m_1\delta\beta_1 + m_2\delta\beta_2) \\ + (m_1\zeta_1' + m_2\zeta_2')(m_1\delta\zeta_1 + m_2\delta\zeta_2) - (m_1\gamma_1' + m_2\gamma_2')(m_1\delta\gamma_1 + m_2\delta\gamma_2) \end{array}\right\} \end{array}\right\} \tag{K⁴.}$$

which resolves itself in the same manner as before into the six intermediate and six final integrals of relative motion, namely, into the following equations:

$$\left.\begin{array}{ll} \frac{1}{m_1}\frac{\delta V_{,}}{\delta \xi_1} = \xi_1' - \frac{m_1\xi_1' + m_2\xi_2'}{m_1 + m_2 + m_3}; & \frac{1}{m_2}\frac{\delta V_{,}}{\delta \xi_2} = \xi_2' - \frac{m_1\xi_1' + m_2\xi_2'}{m_1 + m_2 + m_3}. \\ \frac{1}{m_1}\frac{\delta V_{,}}{\delta \eta_1} = \eta_1' - \frac{m_1\eta_1' + m_2\eta_2'}{m_1 + m_2 + m_3}; & \frac{1}{m_2}\frac{\delta V_{,}}{\delta \eta_2} = \eta_2' - \frac{m_1\eta_1' + m_2\eta_2'}{m_1 + m_2 + m_3}. \\ \frac{1}{m_1}\frac{\delta V_{,}}{\delta \zeta_1} = \zeta_1' - \frac{m_1\zeta_1' + m_2\zeta_2'}{m_1 + m_2 + m_3}; & \frac{1}{m_2}\frac{\delta V_{,}}{\delta \zeta_2} = \zeta_2' - \frac{m_1\zeta_1' + m_2\zeta_2'}{m_1 + m_2 + m_3}. \end{array}\right\} \tag{L⁴.}$$

and

$$\left.\begin{array}{ll}\dfrac{-1}{m_1}\dfrac{\delta V_{,}}{\delta \alpha_1}=\alpha'_1-\dfrac{m_1\alpha'_1+m_2\alpha'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{,}}{\delta \alpha_2}=\alpha'_2-\dfrac{m_1\alpha'_1+m_2\alpha'_2}{m_1+m_2+m_3};\\ \dfrac{-1}{m_1}\dfrac{\delta V_{,}}{\delta \beta_1}=\beta'_1-\dfrac{m_1\beta'_1+m_2\beta'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{,}}{\delta \beta_2}=\beta'_2-\dfrac{m_1\beta'_1+m_2\beta'_2}{m_1+m_2+m_3};\\ \dfrac{-1}{m_1}\dfrac{\delta V_{,}}{\delta \gamma_1}=\gamma'_1-\dfrac{m_1\gamma'_1+m_2\gamma'_2}{m_1+m_2+m_3}; & \dfrac{-1}{m_2}\dfrac{\delta V_{,}}{\delta \gamma_2}=\gamma'_2-\dfrac{m_1\gamma'_1+m_2\gamma'_2}{m_1+m_2+m_3};\end{array}\right\} \qquad (\text{M}^4.)$$

which must be combined with our old formula,

$$\frac{\delta V_{,}}{\delta H}=t. \qquad (\text{O}^1.)$$

18. The quantity $H_{,}$ in $V_{,}$, and the analogous quantity $H_{,,}$ in $V_{,,}$, are indeed independent of the time, and do not vary in the course of the motion; but it is required by the spirit of our method, that in deducing the absolute action or original characteristic function V from the two parts $V_{,}$ and $V_{,,}$, we should consider these two parts $H_{,}$ and $H_{,,}$ of the original quantity H, as functions involving each the nine initial and nine final coordinates of the points of the ternary system; the forms of these two functions, of the eighteen coordinates and of H, being determined by the two conditions,

$$\frac{\delta V_{,}}{\delta H_{,}}=\frac{\delta V_{,,}}{\delta H_{,,}}, \quad H_{,}+H_{,,}=H. \qquad (\text{N}^4.)$$

However, it results from these conditions, that in taking the variation of the whole original function V, of the first order, with respect to the eighteen coordinates, we may treat the two auxiliary quantities $H_{,}$ and $H_{,,}$ as constant; and therefore that we have the following expressions for the partial differential coefficients of the first order of V, taken with respect to the coordinates parallel to x,

$$\left.\begin{array}{ll}\dfrac{\delta V}{\delta x_1}=\dfrac{\delta V_{,}}{\delta \xi_1}+\dfrac{m_1}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta x_{,,}}, & \dfrac{\delta V}{\delta a_1}=\dfrac{\delta V_{,}}{\delta \alpha_1}+\dfrac{m_1}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta a_{,,}},\\ \dfrac{\delta V}{\delta x_2}=\dfrac{\delta V_{,}}{\delta \xi_2}+\dfrac{m_2}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta x_{,,}}, & \dfrac{\delta V}{\delta a_2}=\dfrac{\delta V_{,}}{\delta \alpha_2}+\dfrac{m_2}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta a_{,,}},\\ \dfrac{\delta V}{\delta x_3}=-\dfrac{\delta V_{,}}{\delta \xi_1}-\dfrac{\delta V_{,}}{\delta \xi_2}+\dfrac{m_3}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta x_{,,}}, & \dfrac{\delta V}{\delta a_3}=-\dfrac{\delta V_{,}}{\delta \alpha_1}-\dfrac{\delta V_{,}}{\delta \alpha_2}+\dfrac{m_3}{m_1+m_2+m_3}\dfrac{\delta V_{,,}}{\delta a_{,,}},\end{array}\right\} \qquad (\text{O}^4.)$$

together with analogous expressions for the partial differential coefficients of the same order, taken with respect to the other coordinates. Substituting these expressions in the equations of the form (O.), namely, in the following,

$$\left.\begin{array}{l}\dfrac{\delta V}{\delta x_1}+\dfrac{\delta V}{\delta x_2}+\dfrac{\delta V}{\delta x_3}+\dfrac{\delta V}{\delta a_1}+\dfrac{\delta V}{\delta a_2}+\dfrac{\delta V}{\delta a_3}=0,\\ \dfrac{\delta V}{\delta y_1}+\dfrac{\delta V}{\delta y_2}+\dfrac{\delta V}{\delta y_3}+\dfrac{\delta V}{\delta b_1}+\dfrac{\delta V}{\delta b_2}+\dfrac{\delta V}{\delta b_3}=0,\\ \dfrac{\delta V}{\delta z_1}+\dfrac{\delta V}{\delta z_2}+\dfrac{\delta V}{\delta z_3}+\dfrac{\delta V}{\delta c_1}+\dfrac{\delta V}{\delta c_2}+\dfrac{\delta V}{\delta c_3}=0,\end{array}\right. \qquad (\text{P}^4.)$$

we find that these equations become identical, because

$$\frac{\delta V_{,,}}{\delta x_{,,}}+\frac{\delta V_{,,}}{\delta a_{,,}}=0, \quad \frac{\delta V_{,,}}{\delta y_{,,}}+\frac{\delta V_{,,}}{\delta b_{,,}}=0, \quad \frac{\delta V_{,,}}{\delta z_{,,}}+\frac{\delta V_{,,}}{\delta c_{,,}}=0. \tag{Q^4.}$$

But substituting, in like manner, the expressions (O^4.) in the equations of the form (P.), of which the first is, for a ternary system,

$$\left.\begin{aligned} & x_1\frac{\delta V}{\delta y_1}-y_1\frac{\delta V}{\delta x_1}+x_2\frac{\delta V}{\delta y_2}-y_2\frac{\delta V}{\delta x_2}+x_3\frac{\delta V}{\delta y_3}-y_3\frac{\delta V}{\delta x_3} \\ & +a_1\frac{\delta V}{\delta b_1}-b_1\frac{\delta V}{\delta a_1}+a_2\frac{\delta V}{\delta b_2}-b_2\frac{\delta V}{\delta a_2}+a_3\frac{\delta V}{\delta b_3}-b_3\frac{\delta V}{\delta a_3}; \end{aligned}\right. \tag{R^4.}$$

and observing that we have

$$x_{,,}\frac{\delta V_{,,}}{\delta y_{,,}}-y_{,,}\frac{\delta V_{,,}}{\delta x_{,,}}+a_{,,}\frac{\delta V_{,,}}{\delta b_{,,}}-b_{,,}\frac{\delta V_{,,}}{\delta a_{,,}}=0, \tag{S^4.}$$

along with two other analogous conditions, we find that the part $V_{,}$, or the characteristic function of relative motion of the ternary system, must satisfy the three following conditions, involving its partial differential coefficients of the first order and in the first degree,

$$\left.\begin{aligned} 0&=\xi_1\frac{\delta V_{,}}{\delta\eta_1}-\eta_1\frac{\delta V_{,}}{\delta\xi_1}+\xi_2\frac{\delta V_{,}}{\delta\eta_2}-\eta_2\frac{\delta V_{,}}{\delta\xi_2}+\alpha_1\frac{\delta V_{,}}{\delta\beta_1}-\beta_1\frac{\delta V_{,}}{\delta\alpha_1}+\alpha_2\frac{\delta V_{,}}{\delta\beta_2}-\beta_2\frac{\delta V_{,}}{\delta\alpha_2}, \\ 0&=\eta_1\frac{\delta V_{,}}{\delta\zeta_1}-\zeta_1\frac{\delta V_{,}}{\delta\eta_1}+\eta_2\frac{\delta V_{,}}{\delta\zeta_2}-\zeta_2\frac{\delta V_{,}}{\delta\eta_2}+\beta_1\frac{\delta V_{,}}{\delta\gamma_1}-\gamma_1\frac{\delta V_{,}}{\delta\beta_1}+\beta_2\frac{\delta V_{,}}{\delta\gamma_2}-\gamma_2\frac{\delta V_{,}}{\delta\beta_2}, \\ 0&=\zeta_1\frac{\delta V_{,}}{\delta\xi_1}-\xi_1\frac{\delta V_{,}}{\delta\zeta_1}+\zeta_2\frac{\delta V_{,}}{\delta\xi_2}-\xi_2\frac{\delta V_{,}}{\delta\zeta_2}+\gamma_1\frac{\delta V_{,}}{\delta\alpha_1}-\alpha_1\frac{\delta V_{,}}{\delta\gamma_1}+\gamma_2\frac{\delta V_{,}}{\delta\alpha_2}-\alpha_2\frac{\delta V_{,}}{\delta\gamma_2}, \end{aligned}\right\} \tag{T^4.}$$

which show that this function can depend only on the shape and size of a pentagon, not generally plane, formed by the point m_3 considered as fixed, and by the initial and final positions of the other two points m_1 and m_2; for example, the pentagon, of which the corners are, in order, $m_3, (m_1), (m_2), m_2, m_1$; (m_1) and (m_2) denoting the initial positions of the points m_1 and m_2, referred to m_3 as a fixed origin. The shape and size of this pentagon may be determined by the ten mutual distances of its five points, that is, by the five sides and five diagonals, which may be thus denoted:

$$\left.\begin{aligned} & m_3(m_1)=\sqrt{s_1},\ (m_1)(m_2)=\sqrt{s_2},\ (m_2)m_2=\sqrt{s_3},\ m_2m_1=\sqrt{s_4},\ m_1m_3=\sqrt{s_5}, \\ & m_3(m_2)=\sqrt{d_1},\ (m_1)m_2=\sqrt{d_2},\ (m_2)m_1=\sqrt{d_3},\ m_2m_3=\sqrt{d_4},\ m_1(m_1)=\sqrt{d_5}; \end{aligned}\right\} \tag{128.}$$

the values of $s_1, \ldots d_5$ as functions of the twelve relative coordinates being

$$\left.\begin{aligned} s_1&=\alpha_1^2+\beta_1^2+\gamma_1^2, & s_2&=(\alpha_2-\alpha_1)^2+(\beta_2-\beta_1)^2+(\gamma_2-\gamma_1)^2, \\ & & s_3&=(\xi_2-\alpha_2)^2+(\eta_2-\beta_2)^2+(\zeta_2-\gamma_2)^2, \\ s_5&=\xi_1^2+\eta_1^2+\zeta_1^2, & s_4&=(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2, \\ d_1&=\alpha_2^2+\beta_2^2+\gamma_2^2, & d_2&=(\xi_2-\alpha_1)^2+(\eta_2-\beta_1)^2+(\zeta_2-\gamma_1)^2, \\ & & d_3&=(\xi_1-\alpha_2)^2+(\eta_1-\beta_2)^2+(\zeta_1-\gamma_2)^2, \\ d_4&=\xi_2^2+\eta_2^2+\zeta_2^2, & d_5&=(\xi_1-\alpha_1)^2+(\eta_1-\beta_1)^2+(\zeta_1-\gamma_1)^2. \end{aligned}\right\} \tag{129.}$$

These ten distances $\sqrt{s_1}$, &c., are not, however, all independent, but are connected by one equation of condition, namely,*

$$\left.\begin{array}{lllll}
0=s_1^2s_3^2 & +s_2^2s_4^2 & +s_3^2s_5^2 & +s_4^2s_1^2 & +s_5^2s_2^2 \\
+s_1^2d_3^2 & +s_2^2d_4^2 & +s_3^2d_5^2 & +s_4^2d_1^2 & +s_5^2d_2^2 \\
+d_1^2d_2^2 & +d_2^2d_3^2 & +d_3^2d_4^2 & +d_4^2d_5^2 & +d_5^2d_1^2 \\
-2s_1^2s_3s_4 & -2s_2^2s_4s_5 & -2s_3^2s_5s_1 & -2s_4^2s_1s_2 & -2s_5^2s_2s_3 \\
-2s_1^2s_3d_3 & -2s_2^2s_4d_4 & -2s_3^2s_5d_5 & -2s_4^2s_1d_1 & -2s_5^2s_2d_2 \\
-2s_1^2s_4d_3 & -2s_2^2s_5d_4 & -2s_3^2s_1d_5 & -2s_4^2s_2d_1 & -2s_5^2s_3d_2 \\
-2s_1d_2d_3^2 & -2s_2d_3d_4^2 & -2s_3d_4d_5^2 & -2s_4d_5d_1^2 & -2s_5d_1d_2^2 \\
-2s_1d_3^2d_4 & -2s_2d_4^2d_5 & -2s_3d_5^2d_1 & -2s_4d_1^2d_2 & -2s_5d_2^2d_3 \\
-2d_1d_2^2d_3 & -2d_2d_3^2d_4 & -2d_3d_4^2d_5 & -2d_4d_5^2d_1 & -2d_5d_1^2d_2 \\
-4s_1s_3s_4d_3 & -4s_2s_4s_5d_4 & -4s_3s_5s_1d_5 & -4s_4s_1s_2d_1 & -4s_5s_2s_3d_2 \\
-4s_1d_2d_3d_4 & -4s_2d_3d_4d_5 & -4s_3d_4d_5d_1 & -4s_4d_5d_1d_2 & -4s_5d_1d_2d_3 \\
-2s_1s_2s_3d_4 & -2s_2s_3s_4d_5 & -2s_3s_4s_5d_1 & -2s_4s_5s_1d_2 & -2s_5s_1s_2d_3 \\
-2s_1s_3d_1d_2 & -2s_2s_4d_2d_3 & -2s_3s_5d_3d_4 & -2s_4s_1d_4d_5 & -2s_5s_2d_5d_1 \\
-2s_1d_1d_3d_5 & -2s_2d_2d_4d_1 & -2s_3d_3d_5d_2 & -2s_4d_4d_1d_3 & -2s_5d_5d_2d_4 \\
+2s_1s_2s_3s_4 & +2s_2s_3s_4s_5 & +2s_3s_4s_5s_1 & +2s_4s_5s_1s_2 & +2s_5s_1s_2s_3 \\
+2s_1s_2s_4d_3 & +2s_2s_3s_5d_4 & +2s_3s_4s_1d_5 & +2s_4s_5s_2d_1 & +2s_5s_1s_3d_2 \\
+2s_1s_3s_4d_1 & +2s_2s_4s_5d_2 & +2s_3s_5s_1d_3 & +2s_4s_1s_2d_4 & +2s_5s_2s_3d_5 \\
+2s_1s_2d_3d_4 & +2s_2s_3d_4d_5 & +2s_3s_4d_5d_1 & +2s_4s_5d_1d_2 & +2s_5s_1d_2d_3 \\
+2s_1s_3d_2d_3 & +2s_2s_4d_3d_4 & +2s_3s_5d_4d_5 & +2s_4s_1d_5d_1 & +2s_5s_2d_1d_2 \\
+2s_1s_4d_1d_2 & +2s_2s_5d_2d_3 & +2s_3s_1d_3d_4 & +2s_4s_2d_4d_5 & +2s_5s_3d_5d_1 \\
+2s_1s_4d_1d_3 & +2s_2s_5d_2d_4 & +2s_3s_1d_3d_5 & +2s_4s_2d_4d_1 & +2s_5s_3d_5d_2 \\
+2s_1s_4d_2d_3 & +2s_2s_5d_3d_4 & +2s_3s_1d_4d_5 & +2s_4s_2d_5d_1 & +2s_5s_3d_1d_2 \\
+2s_1s_4d_3d_4 & +2s_2s_5d_4d_5 & +2s_3s_1d_5d_1 & +2s_4s_2d_1d_2 & +2s_5s_3d_2d_3 \\
+2s_1d_1d_2d_3 & +2s_2d_2d_3d_4 & +2s_3d_3d_4d_5 & +2s_4d_4d_5d_1 & +2s_5d_5d_1d_2 \\
+2s_1d_3d_4d_5 & +2s_2d_4d_5d_1 & +2s_3d_5d_1d_2 & +2s_4d_1d_2d_3 & +2s_5d_2d_3d_4 \\
+2d_1d_2d_3d_4 & +2d_2d_3d_4d_5 & +2d_3d_4d_5d_1 & +2d_4d_5d_1d_2 & +2d_5d_1d_2d_3;
\end{array}\right\} \quad (130.)$$

they may therefore be expressed as functions of nine independent quantities; for example, of four lines and five angles, $r^{(1)}$, $r_0^{(1)}$, $r^{(2)}$, $r_0^{(2)}$, $\theta^{(1)}$, $\theta_0^{(1)}$, $\theta^{(2)}$, $\theta_0^{(2)}$, ι, on which they depend as follows:

---

* [This is the relation connecting the distances of five points in space. It can be written more symmetrically in determinant form (cf. Salmon, *Geometry of three Dimensions* (1914), p. 47).

$$\begin{matrix}
0, & 1, & 1, & 1, & 1, & 1 \\
1, & 0, & s_2, & d_5, & d_2, & s_1 \\
1, & s_2, & 0, & d_3, & s_3, & d_1 \\
1, & d_5, & d_3, & 0, & s_4, & s_5 \\
1, & d_2, & s_3, & s_4, & 0, & d_4 \\
1, & s_1, & d_1, & s_5, & d_4, & 0
\end{matrix} \quad .]$$

$$\left.\begin{aligned}
s_2 &= r_0^{(1)2} + r_0^{(2)2} - 2r_0^{(1)}r_0^{(2)}(\cos\theta_0^{(1)}\cos\theta_0^{(2)} + \sin\theta_0^{(1)}\sin\theta_0^{(2)}\cos\iota),\\
s_3 &= r^{(2)2} + r_0^{(2)2} - 2r^{(2)}r_0^{(2)}\cos(\theta^{(2)} - \theta_0^{(2)}),\\
s_4 &= r^{(2)2} + r^{(1)2} - 2r^{(2)}r^{(1)}(\cos\theta^{(1)}\cos\theta^{(2)} + \sin\theta^{(1)}\sin\theta^{(2)}\cos\iota),\\
s_5 &= r^{(1)2},\\
d_1 &= r_0^{(2)2},\\
d_2 &= r^{(2)2} + r_0^{(1)2} - 2r^{(2)}r_0^{(1)}(\cos\theta^{(2)}\cos\theta_0^{(1)} + \sin\theta^{(2)}\sin\theta_0^{(1)}\cos\iota),\\
d_3 &= r_0^{(2)2} + r^{(1)2} - 2r_0^{(2)}r^{(1)}(\cos\theta_0^{(2)}\cos\theta^{(1)} + \sin\theta_0^{(2)}\sin\theta^{(1)}\cos\iota),\\
d_4 &= r^{(2)2},\\
d_5 &= r^{(1)2} + r_0^{(1)2} - 2r^{(1)}r_0^{(1)}\cos(\theta^{(1)} - \theta_0^{(1)}),
\end{aligned}\right\} \qquad (131).$$

the two line-symbols $r^{(1)}$, $r^{(2)}$ denoting, for abridgement, the same two final radii vectores which were before denoted by $r^{(1,3)}$, $r^{(2,3)}$, and $r_0^{(1)}$, $r_0^{(2)}$ representing the initial values of these radii; while $\theta^{(1)}$, $\theta^{(2)}$, $\theta_0^{(1)}$, $\theta_0^{(2)}$ are angles made by these four radii, with the line of intersection of the two planes $r_0^{(1)}r^{(1)}$, $r_0^{(2)}r^{(2)}$; and ι is the inclination of those two planes to each other. We may therefore consider the characteristic function $V_{\prime}$ of relative motion, for any ternary system, as depending only on these latter lines and angles, along with the quantity $H_{\prime}$.

The reasoning which it has been thought useful to develope here, for any system of three points, attracting or repelling one another according to any functions of their distances, was alluded to, under a more general form, in the twelfth number of this essay; and shows, for example, that the characteristic function of relative motion in a system of four such points, depends on the shape and size of a heptagon, and therefore only on the mutual distances of its seven corners, which are in number $\left(\frac{7\times 6}{2} = \right)21$, but are connected by six equations of condition, leaving only fifteen independent. It is easy to extend these remarks to any multiple system.

General method of improving an approximate expression for the Characteristic Function of motion of a System in any Problem of Dynamics.

19. The partial differential equation (F.), which the characteristic function V must satisfy, in every dynamical question, may receive some useful general transformations, by the separation of this function V into any two parts

$$V_1 + V_2 = V. \qquad (\mathrm{U}^4.)$$

For if we establish, for abridgement, the two following equations of definition,

$$\left.\begin{aligned}
T_1 &= \Sigma\,.\,\frac{1}{2m}\left\{\left(\frac{\delta V_1}{\delta x}\right)^2 + \left(\frac{\delta V_1}{\delta y}\right)^2 + \left(\frac{\delta V_1}{\delta z}\right)^2\right\},\\
T_2 &= \Sigma\,.\,\frac{1}{2m}\left\{\left(\frac{\delta V_2}{\delta x}\right)^2 + \left(\frac{\delta V_2}{\delta y}\right)^2 + \left(\frac{\delta V_2}{\delta z}\right)^2\right\},
\end{aligned}\right\} \qquad (\mathrm{V}^4.)$$

analogous to the relation

$$T = \Sigma\,.\,\frac{1}{2m}\left\{\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right\}, \qquad (\mathrm{W}^4.)$$

which served to transform the law of living force into the partial differential equation (F.); we shall have, by (U^4.),

$$T = T_1 + T_2 + \Sigma \,.\, \frac{1}{m}\left(\frac{\delta V_1}{\delta x}\frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y}\frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z}\frac{\delta V_2}{\delta z}\right); \qquad (X^4.)$$

and this expression may be further transformed by the help of the formula (C.), or by the law of varying action. For that law gives the following symbolic equation,

$$\Sigma \,.\, \frac{1}{m}\left(\frac{\delta V}{\delta x}\frac{\delta}{\delta x} + \frac{\delta V}{\delta y}\frac{\delta}{\delta y} + \frac{\delta V}{\delta z}\frac{\delta}{\delta z}\right) = \frac{d}{dt}, \qquad (Y^4.)$$

the symbols in both members being prefixed to any one function of the varying coordinates of a system, not expressly involving the time; it gives therefore by (U^4.), (V^4.),

$$\Sigma \,.\, \frac{1}{m}\left(\frac{\delta V_1}{\delta x}\frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y}\frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z}\frac{\delta V_2}{\delta z}\right) = \frac{dV_2}{dt} - 2T_2. \qquad (Z^4.)$$

In this manner we find the following general and rigorous transformation of the equation (F.),

$$\frac{dV_2}{dt} = T - T_1 + T_2; \qquad (A^5.)$$

T being here retained for the sake of symmetry and conciseness, instead of the equal expression $U + H$. And if we suppose, as we may, that the part V_1, like the whole function V, is chosen so as to vanish with the time, then the other part V_2 will also have that property, and may be expressed by the definite integral,

$$V_2 = \int_0^t (T - T_1 + T_2)\, dt. \qquad (B^5.)$$

More generally, if we employ the principles of the seventh number, and introduce any $3n$ marks $\eta_1, \eta_2, \ldots \eta_{3n}$, of the varying positions of the n points of any system, (whether they be the rectangular coordinates themselves, or any functions of them,) we shall have

$$T = F\left(\frac{\delta V}{\delta \eta_1}, \frac{\delta V}{\delta \eta_2}, \ldots \frac{\delta V}{\delta \eta_{3n}}\right), \qquad (C^5.)$$

and may establish by analogy the two following equations of definition,

$$\left.\begin{aligned} T_1 &= F\left(\frac{\delta V_1}{\delta \eta_1}, \frac{\delta V_1}{\delta \eta_2}, \ldots \frac{\delta V_1}{\delta \eta_{3n}}\right), \\ T_2 &= F\left(\frac{\delta V_2}{\delta \eta_1}, \frac{\delta V_2}{\delta \eta_2}, \ldots \frac{\delta V_2}{\delta \eta_{3n}}\right), \end{aligned}\right\} \qquad (D^5.)$$

the function F being always rational and integer, and homogeneous of the second dimension; and being therefore such that (besides other properties)

$$T = T_1 + T_2 + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}}\frac{\delta V_2}{\delta \eta_1} + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_2}}\frac{\delta V_2}{\delta \eta} + \ldots + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}}\frac{\delta V_2}{\delta \eta_{3n}}, \qquad (E^5.)$$

$$\frac{\delta T}{\delta \frac{\delta V}{\delta \eta_1}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_1}}, \ldots \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_{3n}}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_{3n}}}, \qquad (F^5.)$$

and

$$\frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_1}}\frac{\delta V_2}{\delta \eta_1} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_2}}\frac{\delta V_2}{\delta \eta_2} + \ldots + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_{3n}}}\frac{\delta V_2}{\delta \eta_{3n}} = 2T_2. \qquad (G^5.)$$

By the principles of the eighth number, we have also,

$$\frac{\delta T}{\delta \dfrac{\delta V}{\delta \eta_1}} = \eta'_1, \quad \frac{\delta T}{\delta \dfrac{\delta V}{\delta \eta_2}} = \eta'_2, \qquad \frac{\delta T}{\delta \dfrac{\delta V}{\delta \eta_{3n}}} = \eta'_{3n}; \qquad (\mathrm{H}^5.)$$

and since the meanings of $\eta'_1, \dots \eta'_{3n}$ give evidently the symbolical equation,

$$\eta'_1 \frac{\delta}{\delta \eta_1} + \eta'_2 \frac{\delta}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta}{\delta \eta_{3n}} = \frac{d}{dt}, \qquad (\mathrm{I}^5$$

we see that the equation (A^5.) still holds with the present more general marks of position of a moving system, and gives still the expression (B^5.), supposing only, as before, that the two parts of the whole characteristic function are chosen so as to vanish with the time.

It may not at first sight appear, that this rigorous transformation (B^5.), of the partial differential equation (F.), or of the analogous equation (T.) with coordinates not rectangular, is likely to assist much in discovering the form of the part V_2 of the characteristic function V, (the other part V_1 being supposed to have been previously assumed;) because it involves under the sign of integration, in the term T_2, the partial differential coefficients of the sought part V_2. But if we observe that these unknown coefficients enter only by their squares and products, we shall perceive that it offers a general method of improving an approximation in any problem of dynamics. For if the first part V_1 be an approximate value of the whole sought function V, the second part V_2 will be small, and the term T_2 will not only be also small, but will be in general of a higher order of smallness; we shall therefore in general improve an approximate value V_1 of the characteristic function V, by adding to it the definite integral,

$$V_2 = \int_0^t (T - T_1)\, dt; \qquad (\mathrm{K}^5.)$$

though this is not, like (B^5.), a perfectly rigorous expression for the remaining part of the function. And in calculating this integral (K^5.), for the improvement of an approximation V_1, we may employ the following analogous approximations to the rigorous formulæ (D.) and (E.),

$$\begin{aligned}
&\frac{\delta V_1}{\delta a_1} = -m_1 a'_1; \quad \frac{\delta V_1}{\delta a_2} = -m_2 a'_2; \qquad \frac{\delta V_1}{\delta a_n} = -m_n a'_n;\\
&\frac{\delta V_1}{\delta b_1} = -m_1 b'_1; \quad \frac{\delta V_1}{\delta b_2} = -m_2 b'_2; \qquad \frac{\delta V_1}{\delta b_n} = -m_n \qquad (\mathrm{L}^5.)\\
&\frac{\delta V_1}{\delta c_1} = -m_1 c'_1; \quad \frac{\delta V_1}{\delta c_2} = -m_2 c'_2; \qquad \frac{\delta V_1}{\delta c_n} = -m_n c'_n;
\end{aligned}$$

and

$$\frac{\delta V_1}{\delta H} = t; \qquad (\mathrm{M}^5.)$$

or with any other marks of final and initial position, (instead of rectangular coordinates,) the following approximate forms of the rigorous equations (S.),

$$\frac{\delta V_1}{\delta e_1} = -\frac{\delta T_0}{\delta e'_1}, \quad \frac{\delta V_1}{\delta e_2} = -\frac{\delta T_0}{\delta e'_2}, \dots \frac{\delta V_1}{\delta e_{3n}} = -\frac{\delta T_0}{\delta e'_{3n}}, \qquad (\mathrm{N}^5.)$$

together with the formula (M^5.); by which new formulæ the manner of motion of the system is approximately though not rigorously expressed.

It is easy to extend these remarks to problems of relative motion, and to show that in such problems we have the rigorous transformation

$$V_{,2}=\int_0^t (T_{,}-T_{,1}+T_{,2})\,dt, \tag{O^5.}$$

and the approximate expression

$$V_{,2}=\int_0^t (T_{,}-T_{,1})\,dt, \tag{P^5.}$$

$V_{,1}$ being any approximate value of the function $V_{,}$ of relative motion, and $V_{,2}$ being the correction of this value; and $T_{,1}$, $T_{,2}$, being homogeneous functions of the second dimension, composed of the partial differential coefficients of these two parts $V_{,1}$, $V_{,2}$, in the same way as $T_{,}$ is composed of the coefficients of the whole function $V_{,}$. These general remarks may usefully be illustrated by a particular but extensive application.

Application of the foregoing method to the case of a Ternary or Multiple System, with any laws of attraction or repulsion, and with one predominant mass.

20. The value (68.), for the relative living force $2T_{,}$ of a system, reduces itself successively to the following parts, $2T_{,}^{(1)}$, $2T_{,}^{(2)}$, ... $2T_{,}^{(n-1)}$, when we suppose that all the $n-1$ first masses vanish, with the exception of each successively; namely, to the part

$$2T_{,}^{(1)}=\frac{m_1 m_n}{m_1+m_n}\,(\xi_1'^2+\eta_1'^2+\zeta_1'^2), \tag{132.}$$

when only m_1, m_n, do not vanish; the part

$$2T_{,}^{(2)}=\frac{m_2 m_n}{m_2+m_n}\,(\xi_2'^2+\eta_2'^2+\zeta_2'^2), \tag{133.}$$

when all but m_2, m_n, vanish; and so on, as far as the part

$$2T_{,}^{(n-1)}=\frac{m_{n-1} m_n}{m_{n-1}+m_n}\,(\xi_{n-1}'^2+\eta_{n-1}'^2+\zeta_{n-1}'^2), \tag{134.}$$

which remains, when only the two last masses are retained. The sum of these $n-1$ parts is not, in general, equal to the whole relative living force $2T_{,}$ of the system, with all the n masses retained; but it differs little from that whole when the first $n-1$ masses are small in comparison with the last mass m_n; for the rigorous value of this difference is, by (68.), and by (132.) (133.) (134.),*

$$\begin{aligned}2T_{,}-2T_{,}^{(1)}-2T_{,}^{(2)}-\dots-2T_{,}^{(n-1)}=\frac{2m_1}{m_n}(T_{,}^{(1)}-T_{,})+\frac{2m_2}{m_n}(T_{,}^{(2)}-T_{,})+\dots+\frac{2m_{n-1}}{m_n}(T_{,}^{(n-1)}-T_{,})\\ +\frac{1}{m_n}\Sigma_{,}\,.\,m_i m_k\{(\xi_i'-\xi_k')^2+(\eta_i'-\eta_k')^2+(\zeta_i'-\zeta_k')^2\}:\end{aligned} \tag{135.}$$

an expression which is small of the second order when the $n-1$ first masses are small of the first order. If, then, we denote by $V_{,}^{(1)}$, $V_{,}^{(2)}$, ... $V_{,}^{(n-1)}$, the relative actions, or accumulated relative

$$\Big[2\Sigma_{,}\,{}^{m_r+m_n}\,T_{,}^{(r)}=\Sigma_{,}m\,(\dot\xi^2+\dot\eta^2+\dot\zeta^2)=\frac{2\Sigma m}{m_n}\,T_{,}-\frac{1}{m_n}\,\Sigma_{,}m_i m_k\,[(\dot\xi_i-\dot\xi_k)^2+\dots+\dots],$$

$$\therefore\ \frac{2}{m_n}\,\Sigma_{,}m_r T_{,}^{(r)}+2\Sigma_{,}T_{,}^{(r)}=\frac{2}{m_n}\,(\Sigma_{,}m_r)\,T_{,}+2T_{,}-\frac{1}{m_n}\,\Sigma_{,}m_i m_k\,[(\dot\xi_i-\dot\xi_k)^2+\dots+\dots].\Big]$$

living forces, such as they would be in the $n-1$ binary systems, $(m_1\, m_n)$, $(m_2\, m_n)$, … $(m_{n-1}\, m_n)$, without the perturbations of the other small masses of the entire multiple system of n points; so that

$$V_{,}^{(1)} = \int_0^t 2T_{,}^{(1)}\, dt, \quad V_{,}^{(2)} = \int_0^t 2T_{,}^{(2)}\, dt, \quad \ldots \quad V_{,}^{(n-1)} = \int_0^t 2T_{,}^{(n-1)}\, dt, \tag{Q^5.}$$

the perturbations being neglected in calculating these $n-1$ definite integrals; we shall have, as an approximate value for the whole relative action $V_{,}$ of the system, the sum $V_{,1}$ of its values for these separate binary systems,

$$V_{,1} = V_{,}^{(1)} + V_{,}^{(2)} + \ldots + V_{,}^{(n-1)}. \tag{R^5.}$$

This sum, by our theory of binary systems, may be otherwise expressed as follows:

$$V_{,1} = \frac{m_1 m_n w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n w^{(2)}}{m_2 + m_n} + \ldots + \frac{m_{n-1} m_n w^{(n-1)}}{m_{n-1} + m_n} \tag{S^5.}$$

if we put for abridgement

$$\begin{aligned} w^{(1)} &= h^{(1)} \vartheta^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} r'^{(1)}\, dr^{(1)}, \\ w^{(2)} &= h^{(2)} \vartheta^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} r'^{(2)}\, dr^{(2)}, \\ &\ldots\ldots \\ w^{(n-1)} &= h^{(n-1)} \vartheta^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1}} r'^{(n-1)}\, dr^{(n-1)}. \end{aligned} \tag{T^5.}$$

In this expression,

$$\begin{aligned} r'^{(1)} &= \pm \sqrt{2\,(m_1 + m_n) f^{(1)} + 2g^{(1)} - \frac{h^{(1)2}}{r^{(1)2}}}, \\ r'^{(n-1)} &= \pm \sqrt{2\,(m_{n-1} + m_n) f^{(n-1)} + 2g^{(n-1)} - \frac{h^{(n-1)2}}{r^{(n-1)2}}} \end{aligned} \tag{U^5.}$$

$r^{(1)}$, … $r^{(n-1)}$ being abridged expressions for the distances $r^{(1,n)}$, … $r^{(n-1,n)}$, and $f^{(1)}$, … $f^{(n-1)}$ being abridgements for the functions $f^{(1,n)}$, … $f^{(n-1,n)}$, of these distances, of which the derivatives, according as they are negative or positive, express the laws of attraction or repulsion: we have also introduced $2n-2$ auxiliary quantities $h^{(1)}, g^{(1)}, \ldots h^{(n-1)}, g^{(n-1)}$, to be eliminated or determined by the following equations of condition:*

$$\begin{aligned} 0 &= \vartheta^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} \frac{\delta r'^{(1)}}{\delta h^{(1)}}\, dr^{(1)}, \\ 0 &= \vartheta^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} \frac{\delta r'^{(2)}}{\delta h^{(2)}}\, dr^{(2)}, \\ 0 &= \vartheta^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{\delta r'^{(n-1)}}{\delta h^{(n-1)}}\, dr^{(n-1)}, \end{aligned} \tag{V^5.}$$

---

* [The equations (V^5.), (X^5.) are obtained by equating to zero the partial derivatives of V with respect to the arbitrary constants $h^{(i)}$, $g^{(i)}$. For this method of obtaining the characteristic function see Appendix, Note 2, p. 613.]

and

$$\int_{r_0^{(1)}}^{r^{(1)}} \frac{dr^{(1)}}{r'^{(1)}} = \int_{r_0^{(2)}}^{r^{(2)}} \frac{dr^{(2)}}{r'^{(2)}} = \ldots = \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{dr^{(n-1)}}{r'^{(n-1)}}, \tag{W^5.}$$

or

$$\frac{\delta w^{(1)}}{\delta g^{(1)}} = \frac{\delta w^{(2)}}{\delta g^{(2)}} = \qquad = \frac{\delta w^{(n-1)}}{\delta g^{(n-1)}}, \tag{X^5.}$$

along with this last condition,

$$\frac{m_1 g^{(1)}}{m_1 + m_n} + \frac{m_2 g^{(2)}}{m_2 + m_n} + \frac{m_3 g^{(3)}}{m_3 + m_n} + \ldots + \frac{m_{n-1} g^{(n-1)}}{m_{n-1} + m_n} - \frac{H_{,}}{m_n}. \tag{Y^5.}$$

and we have denoted by $\vartheta^{(1)}, \ldots \vartheta^{(n-1)}$, the angles which the final distances $r^{(1)}, \ldots r^{(n-1)}$, of the first $n-1$ points from the last or nth point of the system, make respectively with the initial distances corresponding, namely, $r_0^{(1)}, \ldots r_0^{(n-1)}$. The variation of the sum $V_{,1}$ is, by (S^5.),

$$\delta V_{,1} = \frac{m_1 m_n \delta w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n \delta w^{(2)}}{m_2 + m_n} + \ldots + \frac{m_{n-1} m_n \delta w^{(n-1)}}{m_{n-1} + m_n}; \tag{Z^5.}$$

in which, by the equations of condition, we may treat all the auxiliary quantities $h^{(1)}, g^{(1)}, \ldots h^{(n-1)}, g^{(n-1)}$ as constant, if $H_{,}$ be considered as given: so that the part of this variation $\delta V_{,1}$, which depends on the variations of the final relative coordinates, may be put under the form,

$$\left.\begin{aligned} \delta_{\xi,\eta,\zeta} V_{,1} = {} & \frac{m_1 m_n}{m_1 + m_n} \left(\frac{\delta w^{(1)}}{\delta \xi_1} \delta \xi_1 + \frac{\delta w^{(1)}}{\delta \eta_1} \delta \eta_1 + \frac{\delta w^{(1)}}{\delta \zeta_1} \delta \zeta_1 \right) \\ & + \frac{m_2 m_n}{m_2 + m_n} \left(\frac{\delta w^{(2)}}{\delta \xi_2} \delta \xi_2 + \frac{\delta w^{(2)}}{\delta \eta_2} \delta \eta_2 + \frac{\delta w^{(2)}}{\delta \zeta_2} \delta \zeta_2 \right) + \ldots \\ & + \frac{m_{n-1} m_n}{m_{n-1} + m_n} \left(\frac{\delta w^{(n-1)}}{\delta \xi_{n-1}} \delta \xi_{n-1} + \frac{\delta w^{(n-1)}}{\delta \eta_{n-1}} \delta \eta_{n-1} + \frac{\delta w^{(n-1)}}{\delta \zeta_{n-1}} \delta \zeta_{n-1} \right). \end{aligned}\right\} \tag{A^6.}$$

By the equations (T^5.), (U^5.), or by the theory of binary systems, we have, rigorously,

$$\left.\begin{aligned} & \left(\frac{\delta w^{(1)}}{\delta \xi_1}\right)^2 + \left(\frac{\delta w^{(1)}}{\delta \eta_1}\right)^2 + \left(\frac{\delta w^{(1)}}{\delta \zeta_1}\right)^2 = 2\,(m_1 + m_n) f^{(1)} + 2g^{(1)}; \\ & \left(\frac{\delta w^{(2)}}{\delta \xi_2}\right)^2 + \left(\frac{\delta w^{(2)}}{\delta \eta_2}\right)^2 + \left(\frac{\delta w^{(2)}}{\delta \zeta_2}\right)^2 = 2\,(m_2 + m_n) f^{(2)} + 2g^{(2)}; \\ & \ldots\ldots \\ & \left(\frac{\delta w^{(n-1)}}{\delta \xi_{n-1}}\right)^2 + \left(\frac{\delta w^{(n-1)}}{\delta \eta_{n-1}}\right)^2 + \left(\frac{\delta w^{(n-1)}}{\delta \zeta_{n-1}}\right)^2 = 2\,(m_{n-1} + m_n) f^{(n-1)} + 2g^{(n-1)}; \end{aligned}\right\} \tag{B^6.}$$

and the rigorous law of relative living force for the whole multiple system is

$$T_{,} = U + H_{,}, \tag{50.}$$

in which

$$U = m_n\,(m_1 f^{(1)} + m_2 f^{(2)} + \ldots + m_{n-1} f^{(n-1)}) + \Sigma_{,}\,.\,m_i m_k f^{(i,k)}, \tag{C^6.}$$

and*

$$\begin{aligned}T_{,}=&\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_n}\right)\left\{\left(\frac{\delta V_{,}}{\delta\xi_1}\right)^2+\left(\frac{\delta V_{,}}{\delta\eta_1}\right)^2+\left(\frac{\delta V_{,}}{\delta\zeta_1}\right)^2\right\}\\ &+\frac{1}{2}\left(\frac{1}{m_2}+\frac{1}{m_n}\right)\left\{\left(\frac{\delta V_{,}}{\delta\xi_2}\right)^2+\left(\frac{\delta V_{,}}{\delta\eta_2}\right)^2+\left(\frac{\delta V_{,}}{\delta\zeta_2}\right)^2\right\}+\dots\\ &+\frac{1}{2}\left(\frac{1}{m_{n-1}}+\frac{1}{m_n}\right)\left\{\left(\frac{\delta V_{,}}{\delta\xi_{n-1}}\right)^2+\left(\frac{\delta V_{,}}{\delta\eta_{n-1}}\right)^2+\left(\frac{\delta V_{,}}{\delta\zeta_{n-1}}\right)^2\right\}\\ &+\frac{1}{m_n}\Sigma_{,}\left(\frac{\delta V_{,}}{\delta\xi_i}\frac{\delta V_{,}}{\delta\xi_k}+\frac{\delta V_{,}}{\delta\eta_i}\frac{\delta V_{,}}{\delta\eta_k}+\frac{\delta V_{,}}{\delta\zeta_i}\frac{\delta V_{,}}{\delta\zeta_k}\right).\end{aligned}\qquad (D^6).$$

We have therefore, by changing in this last expression the coefficients of the characteristic function $V_{,}$ to those of its first part $V_{,1}$, and by attending to the foregoing equations,†

$$T_{,1}=m_n\Sigma_{,}\,.\,m_if^{(i)}+H_{,}+m_n\Sigma_{,}\,.\,\frac{m_i}{m_n+m_i}\frac{m_k}{m_n+m_k}\left(\frac{\delta w^{(i)}}{\delta\xi_i}\frac{\delta w^{(k)}}{\delta\xi_k}+\frac{\delta w^{(i)}}{\delta\eta_i}\frac{\delta w^{(k)}}{\delta\eta_k}+\frac{\delta w^{(i)}}{\delta\zeta_i}\frac{\delta w^{(k)}}{\delta\zeta_k}\right);\qquad (E^6.)$$

and consequently

$$T_{,}-T_{,1}=\Sigma_{,}\,.\,m_im_k\left\{f^{(i,k)}-\frac{m_n}{(m_n+m_i)(m_n+m_k)}\left(\frac{\delta w^{(i)}}{\delta\xi_i}\frac{\delta w^{(k)}}{\delta\xi_k}+\frac{\delta w^{(i)}}{\delta\eta_i}\frac{\delta w^{(k)}}{\delta\eta_k}+\frac{\delta w^{(i)}}{\delta\zeta_i}\frac{\delta w^{(k)}}{\delta\zeta_k}\right)\right\}.\qquad (F^6.)$$

The general transformation of the foregoing number gives therefore, rigorously, for the remaining part $V_{,2}$ of the characteristic function $V_{,}$ of relative motion of the multiple system, the equation

$$V_{,2}=\int_0^t T_{,2}dt+\Sigma_{,}\,.\,m_im_k\int_0^t\left\{f^{(i,k)}-\frac{\dfrac{\delta w^{(i)}}{\delta\xi_i}\dfrac{\delta w^{(k)}}{\delta\xi_k}+\dfrac{\delta w^{(i)}}{\delta\eta_i}\dfrac{\delta w^{(k)}}{\delta\eta_k}+\dfrac{\delta w^{(i)}}{\delta\zeta_i}\dfrac{\delta w^{(k)}}{\delta\zeta_k}}{\dfrac{1}{m_n}(m_n+m_i)(m_n+m_k)}\right\}dt;\qquad (G^6.)$$

---

* [From (X¹.), p. 126, we have $\Sigma_{,}m\xi'=\frac{\Sigma m}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta\xi}$ and hence $\xi_i'=\frac{1}{m_i}\frac{\delta V_{,}}{\delta\xi_i}+\frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta\xi}$. Consequently the expression (69) for $T_{,}$ gives

$$T_{,}=\tfrac{1}{2}\Sigma_{,}\frac{1}{m_i}\left(\frac{\delta V_{,}}{\delta\xi_i}\right)^2+\frac{1}{2m_n}\left(\Sigma_{,}\frac{\delta V_{,}}{\delta\xi}\right)^2,\text{ etc.]}$$

† [$T_{,1}$ is formed from (D⁶.) by replacing $V_{,}$ by $V_{,1}$. For the binary system composed of the points m_i, m_n we have

$$2T_{,}^{(i)}=\frac{m_im_n}{m_i+m_n}(\xi_i'^2+\eta_i'^2+\zeta_i'^2),\qquad V_{,}^{(i)}=\frac{m_im_n}{m_i+m_n}w^{(i)}.$$

From equations (X¹.), p. 126, $\quad\frac{\partial V_{,}^{(i)}}{\partial\xi_i}=\frac{m_im_n}{m_i+m_n}\xi_i',\quad \xi_i'=\frac{\partial w^{(i)}}{\partial\xi_i};$

$$\therefore\ 2T_{,}^{(i)}=\frac{m_i+m_n}{m_im_n}\left\{\left(\frac{\partial V_{,}^{(i)}}{\partial\xi_i}\right)^2+\left(\frac{\partial V_{,}^{(i)}}{\partial\eta_i}\right)^2+\left(\frac{\partial V_{,}^{(i)}}{\partial\zeta_i}\right)^2\right\}=\frac{m_im_n}{m_i+m_n}\left\{\left(\frac{\partial w^{(i)}}{\partial\xi_i}\right)^2+\left(\frac{\partial w^{(i)}}{\partial\eta_i}\right)^2+\left(\frac{\partial w^{(i)}}{\partial\zeta_i}\right)^2\right\}=2m_im_nf^{(i)}+2H_{,}^{(i)},$$

where $\frac{m_i+m_n}{m_im_n}H_{,}^{(i)}=g^{(i)}$,

$$\begin{aligned}T_{,1}&=\tfrac{1}{2}\Sigma_{,}\frac{m_i+m_n}{m_im_n}\left[\left(\frac{\partial V_{,}^{(i)}}{\partial\xi_i}\right)^2+\left(\frac{\partial V_{,}^{(i)}}{\partial\eta_i}\right)^2+\left(\frac{\partial V_{,}^{(i)}}{\partial\zeta_i}\right)^2\right]+\frac{1}{m_n}\Sigma_{,}\left\{\frac{\partial V_{,}^{(i)}}{\partial\xi_i}\frac{\partial V_{,}^{(k)}}{\partial\xi_k}+\frac{\partial V_{,}^{(i)}}{\partial\eta_i}\frac{\partial V_{,}^{(k)}}{\partial\eta_k}+\frac{\partial V_{,}^{(i)}}{\partial\zeta_i}\frac{\partial V_{,}^{(k)}}{\partial\zeta_k}\right\}\\ &=\tfrac{1}{2}\Sigma_{,}\frac{m_im_n}{m_i+m_n}\left\{\left(\frac{\partial w^{(i)}}{\partial\xi_i}\right)^2+\left(\frac{\partial w^{(i)}}{\partial\eta_i}\right)^2+\left(\frac{\partial w^{(i)}}{\partial\zeta_i}\right)^2\right\}+m_n\Sigma_{,}\frac{m_im_k}{(m_n+m_i)(m_n+m_k)}\left\{\frac{\partial w^{(i)}}{\partial\xi_i}\frac{\partial w^{(k)}}{\partial\xi_k}+\frac{\partial w^{(i)}}{\partial\eta_i}\frac{\partial w^{(k)}}{\partial\eta_k}+\frac{\partial w^{(i)}}{\partial\zeta_i}\frac{\partial w^{(k)}}{\partial\zeta_k}\right\}\\ &=m_n\Sigma_{,}m_if^{(i)}+H_{,}+\dots,\qquad\qquad T_{,}=\Sigma m_im_kf^{(i,k)}+H_{,},\text{ etc.]}\end{aligned}$$

and, approximately, the expression

$$V_{,2} = \Sigma_{,} \,.\, m_i m_k \int_0^t \left\{ f^{(i,k)} - \frac{1}{m_n} (\xi_i' \xi_k' + \eta_i' \eta_k' + \zeta_i' \zeta_k') \right\} dt: \tag{H^6.}$$

with which last expression we may combine the following approximate formulæ belonging in rigour to binary systems only,

$$\xi_i' = \frac{\delta w^{(i)}}{\delta \xi_i}, \quad \eta_i' = \frac{\delta w^{(i)}}{\delta \eta_i}, \quad \zeta_i' = \frac{\delta w^{(i)}}{\delta \zeta_i}, \tag{I^6.}$$

$$\alpha_i' = -\frac{\delta w^{(i)}}{\delta \alpha_i}, \quad \beta_i' = -\frac{\delta w^{(i)}}{\delta \beta_i}, \quad \gamma_i' = \quad \frac{\delta w^{(i)}}{\delta \gamma_i}, \tag{K^6.}$$

and

$$t = \frac{\delta w^{(i)}}{\delta g^{(i)}}. \tag{L^6.}$$

We have also, rigorously, for binary systems, the following differential equations of motion of the second order,

$$\xi_i'' = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \xi_i}; \quad \eta_i'' = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \eta_i}; \quad \zeta_i'' = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \zeta_i}; \tag{M^6.}$$

which enable us to transform in various ways the approximate expression (H^6.). Thus, in the case of a ternary system, with any laws of attraction or repulsion, but with one predominant mass m_3, the *disturbing part* $V_{,2}$ of the characteristic function $V_{,}$ of relative motion, may be put under the form

$$V_{,2} = m_1 m_2 W, \tag{N^6.}$$

in which the coefficient W may approximately be expressed as follows:

$$W = \int_0^t \left\{ f^{(1,2)} - \frac{1}{m_3} (\xi_1' \xi_2' + \eta_1' \eta_2' + \zeta_1' \zeta_2') \right\} dt, \tag{O^6.}$$

or thus:*

$$\begin{aligned} W = {} & \int_0^t \left(f^{(1,2)} + \xi_2 \frac{\delta f^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta f^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta f^{(1)}}{\delta \zeta_1} \right) dt \\ & - \frac{1}{m} \left(\xi_2 \frac{\delta w^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta w^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta w^{(1)}}{\delta \zeta_1} + \alpha_2 \frac{\delta w^{(1)}}{\delta \alpha_1} + \beta_2 \frac{\delta w^{(1)}}{\delta \beta_1} + \gamma_2 \frac{\delta w^{(1)}}{\delta \gamma_1} \right), \end{aligned} \tag{P^6.}$$

or finally,

$$\begin{aligned} W = {} & \int_0^t \left(f^{(1,2)} + \xi_1 \frac{\delta f^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta f^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta f^{(2)}}{\delta \zeta_2} \right) dt \\ & - \frac{1}{m_3} \left(\xi_1 \frac{\delta w^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta w^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta w^{(2)}}{\delta \zeta_2} + \alpha_1 \frac{\delta w^{(2)}}{\delta \alpha_2} + \beta_1 \frac{\delta w^{(2)}}{\delta \beta_2} + \gamma_1 \frac{\delta w^{(2)}}{\delta \gamma_2} \right). \end{aligned} \tag{Q^6.}$$

In general, for a multiple system, we may put

$$V_{,2} = \Sigma_{,} \,.\, m_i m_k W^{(i,k)}; \tag{R^6.}$$

and approximately,

$$\begin{aligned} W^{(i,k)} = {} & \int_0^t \left(f^{(i,k)} + \xi_k \frac{\delta f^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta f^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta f^{(i)}}{\delta \zeta_i} \right) dt \\ & - \frac{1}{m_n} \left(\xi_k \frac{\delta w^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta w^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta w^{(i)}}{\delta \zeta_i} + \alpha_k \frac{\delta w^{(i)}}{\delta \alpha_i} + \beta_k \frac{\delta w^{(i)}}{\delta \beta_i} + \gamma_k \frac{\delta w^{(i)}}{\delta \gamma_i} \right), \end{aligned} \tag{S^6.}$$

or

$$\begin{aligned} W^{(i,k)} = {} & \int_0^t \left(f^{(i,k)} + \xi_i \frac{\delta f^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta \zeta_k} \right) dt \\ & - \frac{1}{m_n} \left(\xi_i \frac{\delta w^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta w^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta w^{(k)}}{\delta \zeta_k} + \alpha_i \frac{\delta w^{(k)}}{\delta \alpha_k} + \beta_i \frac{\delta w^{(k)}}{\delta \beta_k} + \gamma_i \frac{\delta w^{(k)}}{\delta \gamma_k} \right). \,] \end{aligned} \tag{T^6.}$$

---

* [Integrating by parts.]

Rigorous transition from the theory of Binary to that of Multiple Systems, by means of the disturbing part of the whole Characteristic Function; and approximate expressions for the perturbations.

21. The three equations (K^6.) when the auxiliary constant $g^{(i)}$ is eliminated by the formula (L^6.) are rigorously (by our theory) the three final integrals of the three known equations of the second order (M^6.), for the relative motion of the binary system $(m_i\, m_n)$; and give, for such a system, the three varying relative coordinates ξ_i, η_i, ζ_i, as functions of their initial values and initial rates of increase α_i, β_i, γ_i, α'_i, β'_i, γ'_i, and of the time t. In like manner the three equations (I^6.), when $g^{(i)}$ is eliminated by (L^6.), are rigorously the three intermediate integrals of the same known differential equations of motion of the same binary system. These integrals, however, cease to be rigorous when we introduce the perturbations of the relative motion of this partial or binary system $(m_i\, m_n)$, arising from the attractions or repulsions of the other points m_k, of the whole proposed multiple system; but they may be corrected and rendered rigorous by employing the remaining part $V_{,2}$ of the whole characteristic function of relative motion $V_{,}$, along with the principal part of approximate value $V_{,1}$.

The equations (X^1.), (Y^1.) of the twelfth number give rigorously

$$\xi'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \xi_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \xi_i}, \quad \eta'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \eta_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \eta_i}, \quad \zeta'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \zeta_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \zeta_i}, \tag{U^6.}$$

and

$$-\alpha'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \alpha_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \alpha_i}, \quad -\beta'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \beta_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \beta_i}, \quad -\gamma'_i = \frac{1}{m_i}\frac{\delta V_{,}}{\delta \gamma_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,}}{\delta \gamma_i}, \tag{V^6.}$$

and therefore, by (A^6.),

$$\left.\begin{aligned}
\frac{\delta w^{(i)}}{\delta \xi_i} &= \xi'_i - \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \xi_k} - \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \xi_i} - \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \xi_i},\\
\frac{\delta w^{(i)}}{\delta \eta_i} &= \eta'_i - \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \eta_k} - \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \eta_i} - \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \eta_i},\\
\frac{\delta w^{(i)}}{\delta \zeta_i} &= \zeta'_i - \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \zeta_k} - \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \zeta_i} - \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \zeta_i},
\end{aligned}\right\} \tag{W^6.}$$

and similarly

$$\left.\begin{aligned}
-\frac{\delta w^{(i)}}{\delta \alpha_i} &= \alpha'_i + \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \alpha_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \alpha_i},\\
-\frac{\delta w^{(i)}}{\delta \beta_i} &= \beta'_i + \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \beta_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \beta_i},\\
-\frac{\delta w^{(i)}}{\delta \gamma_i} &= \gamma'_i + \Sigma_{,,} \cdot \frac{m_k}{m_k+m_n}\frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i}\frac{\delta V_{,2}}{\delta \gamma_i} + \frac{1}{m_n}\Sigma_{,}\frac{\delta V_{,2}}{\delta \gamma_i},
\end{aligned}\right\} \tag{X^6.}$$

the sign of summation $\Sigma_{,,}$ referring only to the disturbing masses m_k, to the exclusion of m_i and m_n; and these equations (W^6.), (X^6.) are the rigorous formulæ, corresponding to the approximate relations (I^6.), (K^6.). In like manner, the formula (L^6.) for the time of motion in a binary system, which is only an approximation when the system is considered as multiple, may be rigorously corrected for perturbation by adding to it an analogous term deduced from the disturbing part $V_{,2}$ of the whole characteristic function; that is, by changing it to the following:

$$t = \frac{\delta w^{(i)}}{\delta g^{(i)}} + \frac{\delta V_{,2}}{\delta H}, \tag{Y^6.}$$

which gives, for this other coefficient of $w^{(i)}$, the corrected and rigorous expression

$$\frac{\delta w^{(i)}}{\delta g^{(i)}} = t - \frac{\delta V_{,2}}{\delta H}: \tag{Z^6.}$$

$V_{,2}$ being here supposed so chosen as to be rigorously the correction of $V_{,1}$. If therefore by the theory of binary systems, or by eliminating $g^{(i)}$ between the four equations (K^6.) (L^6.), we have deduced expressions for the three varying relative coordinates ξ_i, η_i, ζ_i as functions of the time t, and of the six initial quantities α_i, β_i, γ_i, α'_i, β'_i, γ'_i, which may be thus denoted,

$$\left.\begin{aligned} \xi_i &= \phi_1(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \eta_i &= \phi_2(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \zeta_i &= \phi_3(\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t); \end{aligned}\right\} \tag{A^7.}$$

we shall know that the following relations are rigorously and *identically* true,*

$$\left.\begin{aligned} \xi_i &= \phi_1\left(\alpha_i, \beta_i, \gamma_i, \ -\frac{\delta w^{(i)}}{\delta \alpha_i}, \ -\frac{\delta w^{(i)}}{\delta \beta_i}, \ -\frac{\delta w^{(i)}}{\delta \gamma_i}, \ \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \\ \eta_i &= \phi_2\left(\alpha_i, \beta_i, \gamma_i, \ -\frac{\delta w^{(i)}}{\delta \alpha_i}, \ -\frac{\delta w^{(i)}}{\delta \beta_i}, \ -\frac{\delta w^{(i)}}{\delta \gamma_i}, \ \frac{\delta w^{(i)}}{\delta g^{(i)}}\right. \\ \zeta_i &= \phi_3\left(\alpha_i, \beta_i, \gamma_i, \ \frac{\delta w^{(i)}}{\delta \alpha_i}, \ -\frac{\delta w^{(i)}}{\delta \beta_i} \ -\frac{\delta w^{(i)}}{\delta \gamma_i}, \ \frac{\delta w^{(i)}}{\delta g^{(i)}}\right. \end{aligned}\right\} \tag{B^7.}$$

and consequently that these relations will still be rigorously true when we substitute for the four coefficients of $w^{(i)}$ their rigorous values (X^6.) and (Z^6.) for the case of a multiple system. We may thus retain in rigour for any multiple system the final integrals (A^7.) of the motion of a binary system, if only we add to the initial components α'_i, β'_i, γ'_i of relative velocity, and to the time t, the following perturbational terms:

$$\left.\begin{aligned} \Delta\alpha'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \alpha_i} + \frac{1}{m_n} \Sigma_{\prime} \frac{\delta V_{,2}}{\delta \alpha_i}, \\ \Delta\beta'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \beta_i} + \frac{1}{m_n} \Sigma_{\prime} \frac{\delta V_{,2}}{\delta \beta_i}, \\ \Delta\gamma'_i &= \Sigma_{\prime\prime} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \gamma_i} + \frac{1}{m_n} \Sigma_{\prime} \frac{\delta V_{,2}}{\delta \gamma_i}, \end{aligned}\right\} \tag{C^7.}$$

and

$$\Delta t = -\frac{\delta V_{,2}}{\delta H}. \tag{D^7.}$$

---

* [Otherwise we would get one or more equations connecting the initial and final points. Of course when the rigorous values are substituted, ξ_i, η_i, ζ_i denote the actual coordinates at the time t.]

In the same way, if the theory of binary systems, or the elimination of $g^{(i)}$ between the four equations (I⁶.) (L⁶.), has given three intermediate integrals, of the form

$$\left.\begin{aligned} \xi'_i &= \psi_1(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \\ \eta'_i &= \psi_2(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \\ \zeta'_i &= \psi_3(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, t), \end{aligned}\right\} \qquad (\mathrm{E}^7.)$$

we can conclude that the following equations are rigorous and identical,

$$\left.\begin{aligned} \frac{\delta w^{(i)}}{\delta \xi_i} &= \psi_1\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \\ \frac{\delta w^{(i)}}{\delta \eta_i} &= \psi_2\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \\ \frac{\delta w^{(i)}}{\delta \zeta_i} &= \psi_3\left(\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i, \gamma_i, \frac{\delta w^{(i)}}{\delta g^{(i)}}\right), \end{aligned}\right\} \qquad (\mathrm{F}^7.)$$

and must therefore be still true, when, in passing to a multiple system, we change the coefficients of $w^{(i)}$ to their rigorous values (W⁶.) (Z⁶.). The three intermediate integrals (E⁷.) of the motion of a binary system may therefore be adapted rigorously to the case of a multiple system, by first adding to the time t the perturbational term (D⁷.), and afterwards adding to the resulting values of the final components of relative velocity the terms

$$\left.\begin{aligned} \Delta\xi'_i &= \Sigma_{,,} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \xi_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \xi_i} + \frac{1}{m_n} \Sigma_{,} \frac{\delta V_{,}}{\delta \xi_i}, \\ \Delta\eta'_i &= \Sigma_{,,} \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \eta_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \eta_i} + \frac{1}{m_n} \Sigma_{,} \frac{\delta V_{,2}}{\delta \eta_i}, \\ \Delta\zeta'_i &= \Sigma_{,,} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \zeta_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta \zeta_i} + \frac{1}{m_n} \Sigma_{,} \frac{\delta V_{,2}}{\delta \zeta_i}. \end{aligned}\right. \qquad (\mathrm{G}^7.)$$

22. To derive now, from these rigorous results, some useful approximate expressions, we shall neglect, in the perturbations, the terms which are of the second order, with respect to the small masses of the system, and with respect to the constant $2H_{,}$ of relative living force, which is easily seen to be small of the same order as the masses: and then the perturbations of the coordinates, deduced by the method that has been explained, become

$$\left.\begin{aligned} \Delta\xi_i &= \frac{\delta\xi_i}{\delta\alpha'_i} \Delta\alpha'_i + \frac{\delta\xi_i}{\delta\beta'_i} \Delta\beta'_i + \frac{\delta\xi_i}{\delta\gamma'_i} \Delta\gamma'_i + \frac{\delta\xi_i}{\delta t} \Delta t, \\ \Delta\eta_i &= \frac{\delta\eta_i}{\delta\alpha'_i} \Delta\alpha'_i + \frac{\delta\eta_i}{\delta\beta'_i} \Delta\beta'_i + \frac{\delta\eta_i}{\delta\gamma'_i} \Delta\gamma'_i + \frac{\delta\eta_i}{\delta t} \Delta t, \\ \Delta\zeta_i &= \frac{\delta\zeta_i}{\delta\alpha'_i} \Delta\alpha'_i + \frac{\delta\zeta_i}{\delta\beta'_i} \Delta\beta'_i + \frac{\delta\zeta_i}{\delta\gamma'_i} \Delta\gamma'_i + \frac{\delta\zeta_i}{\delta t} \Delta t, \end{aligned}\right. \qquad (\mathrm{H}^7.)$$

in which we may employ, instead of the rigorous values (C⁷.) for $\Delta\alpha'_i$, $\Delta\beta'_i$, $\Delta\gamma'_i$, the following approximate values:

$$\left.\begin{aligned} \Delta\alpha'_i &= \Sigma_{,,} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta\alpha_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta\alpha_i}, \\ {}'_i &= \Sigma_{,,} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta\beta_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta\beta_i}, \\ \Delta\gamma'_i &= \Sigma_{,,} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta\gamma_k} + \frac{1}{m_i} \frac{\delta V_{,2}}{\delta\gamma_i}. \end{aligned}\right\} \qquad (\mathrm{I}^7.)$$

To calculate the four coefficients

$$\frac{\delta V_{,2}}{\delta \alpha_i}, \quad \frac{\delta V_{,2}}{\delta \beta_i}, \quad \frac{\delta V_{,2}}{\delta \gamma_i}, \quad \frac{\delta V_{,2}}{\delta H_{,}},$$

which enter into the values (I^7.) (D^7.), we may consider $V_{,2}$, by (R^6.) (T^6.), and by the theory of binary systems, as a function of the initial and final relative coordinates, and initial components of relative velocities, involving also expressly the time t, and the $n-2$ auxiliary quantities of the form $g^{(k)}$; and then we are to consider those initial components and auxiliary quantities and the time, as depending themselves on the initial and final coordinates, and on $H_{,}$. But it is not difficult to prove, by the foregoing principles, that when t and $g^{(k)}$ are thus considered, their variations are, in the present order of approximation,*

$$\delta t = \frac{\Sigma_{,}\,.\,m\left(\frac{\delta^2 w}{\delta g^2}\right)^{-1}\delta_{,}\frac{\delta w}{\delta g} + \delta H_{,}}{\Sigma_{,}\,.\,m\left(\frac{\delta^2 w}{\delta g^2}\right)^{-1}} \tag{K^7.}$$

and

$$\delta g^{(k)} = \left(\frac{\delta^2 w^{(k)}}{\delta g^{(k)2}}\right)^{-1}\left(\delta t - \delta_{,}\frac{\delta w^{(k)}}{\delta g^{(k)}}\right), \tag{L^7.}$$

the sign of variation $\delta_{,}$ referring only to the initial and final coordinates; and also that

$$\frac{\delta^2 w^{(i)}}{\delta g^{(i)2}}\frac{\delta \xi_i}{\delta t} = \frac{\delta^2 w^{(i)}}{\delta \alpha_i\,\delta g^{(i)}}\frac{\delta \xi_i}{\delta \alpha'_i} + \frac{\delta^2 w^{(i)}}{\delta \beta_i\,\delta g^{(i)}}\frac{\delta \xi_i}{\delta \beta'_i} + \frac{\delta^2 w^{(i)}}{\delta \gamma_i\,\delta g^{(i)}}\frac{\delta \xi_i}{\delta \gamma'_i}, \tag{M^7.}$$

along with two other analogous relations between the coefficients of the two other coordinates η_i, ζ_i; from which it follows that t and $g^{(k)}$, and therefore α'_k, β'_k, γ'_k, may be treated as constant, in taking the variation of the disturbing part $V_{,2}$, for the purpose of calculating the perturbations (H^7.): and that the terms involving Δt are destroyed by other terms. We may therefore put simply†

$$\begin{aligned}
\Delta \xi_i &= \frac{\delta \xi_i}{\delta \alpha'_i}\Delta \alpha'_i + \frac{\delta \xi_i}{\delta \beta'_i}\Delta \beta'_i + \frac{\delta \xi_i}{\delta \gamma'_i}\Delta \gamma'_i, \\
\Delta \eta_i &= \frac{\delta \eta_i}{\delta \alpha'_i}\Delta \alpha'_i + \frac{\delta \eta_i}{\delta \beta'_i}\Delta \beta'_i + \frac{\delta \eta_i}{\delta \gamma'_i}\Delta \gamma'_i, \\
\Delta \zeta_i &= \frac{\delta \zeta_i}{\delta \alpha'_i}\Delta \alpha'_i + \frac{\delta \zeta_i}{\delta \beta'_i}\Delta \beta'_i + \frac{\delta \zeta_i}{\delta \gamma'_i}\Delta \gamma'_i,
\end{aligned} \tag{N^7.}$$

---

* [We have $\frac{\delta w^{(k)}}{\delta g^{(k)}} = t$. But $w^{(k)}$ depends on the initial and final coordinates and also on $g^{(k)}$. Thus

$$\frac{\delta^2 w^{(k)}}{\delta g^{(k)2}}\,\delta g^{(k)} + \delta_{,}\frac{\delta w^{(k)}}{\delta g^{(k)}} = \delta t.$$

(L^7.) follows immediately and then (K^7.) by summation. Equation (M^7.) is obtained by differentiating (B^7.) partially with respect to $g^{(k)}$.]

† [To obtain (N^7.): $V_{,2} = \Sigma_{,} m_i m_k W^{(i,k)}$, where $W^{(i,k)}$ is given by (T^6.). Three groups of terms come from

$$\frac{1}{m_i}\frac{\delta V_{,2}}{\delta \alpha_i}.$$

(i) From the integrated part of $W^{(i,k)}$ we get $-\frac{m_k}{m_n}\frac{\delta w^{(k)}}{\delta \alpha_k}$. This cancels against the first term of $\Delta\alpha'_i$ in (I^7.).

(ii) From t which occurs "expressly" (explicitly) in the upper limit of the integral. From (K^7.)

$$\frac{\delta t}{\delta \alpha_i} = m_i\left(\frac{\delta^2 w^{(i)}}{\delta g^{(i)2}}\right)^{-1}\frac{\delta^2 w^{(i)}}{\delta \alpha_i\,\delta g^{(i)}}\Big/\Sigma_{,} m\left(\frac{\delta^2 w}{\delta g^2}\right)^{-1} \text{ and } \frac{\delta t}{\delta H_{,}} = 1/\Sigma_{,} m\left(\frac{\delta^2 w}{\delta g^2}\right)^{-1}.$$

Since $\Delta t = -\frac{\delta V_{,2}}{\delta H_{,}}$, we see, by (M^7.), that this group of terms is cancelled by the terms involving Δt.

(iii) The remaining terms which appear in (O^7.).]

employing for $\Delta\alpha_i'$ the following new expression,

$$\begin{aligned}\Delta\alpha_i' = \Sigma_{''}\,.\,m_k \Bigg\{ \int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i}\,dt + \frac{\delta\alpha_i'}{\delta\alpha_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i'}\,dt \\ + \frac{\delta\beta_i'}{\delta\alpha_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\beta_i'}\,dt + \frac{\delta\gamma_i'}{\delta\alpha_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma_i'}\,dt \Bigg\}\end{aligned} \qquad (\mathrm{O}^7.)$$

together with analogous expressions for $\Delta\beta_i'$, $\Delta\gamma_i'$, in which the sign of summation $\Sigma_{''}$ refers to the disturbing masses, and in which the quantity

$$R^{(i,k)} = f^{(i,k)} + \xi_i \frac{\delta f^{(k)}}{\delta\xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta\eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta\zeta_k} \qquad (\mathrm{P}^7.)$$

is considered as depending on $\alpha_i, \beta_i, \gamma_i, \alpha_i', \beta_i', \gamma_i', \alpha_k, \beta_k, \gamma_k, \alpha_k', \beta_k', \gamma_k', t$, by the theory of binary systems, while α_i', β_i', γ_i' are considered as depending, by the same rules, on $\alpha_i, \beta_i, \gamma_i, \xi_i, \eta_i, \zeta_i$ and t.

It may also be easily shown, that*

$$\frac{\delta\xi_i}{\delta\alpha_i'}\frac{\delta\alpha_i'}{\delta\alpha_i} + \frac{\delta\xi_i}{\delta\beta_i'}\frac{\delta\alpha_i'}{\delta\beta_i} + \frac{\delta\xi_i}{\delta\gamma_i'}\frac{\delta\alpha_i'}{\delta\gamma_i} = -\frac{\delta\xi_i}{\delta\alpha_i'}; \qquad (\mathrm{Q}^7.)$$

with other analogous equations: the perturbation of the coordinates ξ_i may therefore be thus expressed,

$$\begin{aligned}\Delta\xi_i = \Sigma_{''}\,.\,m_k \Bigg\{ \frac{\delta\xi_i}{\delta\alpha_i'}\int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i}\,dt - \frac{\delta\xi_i}{\delta\alpha_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i'}\,dt \\ + \frac{\delta\xi_i}{\delta\beta_i'}\int_0^t \frac{\delta R^{(i,k)}}{\delta\beta_i}\,dt - \frac{\delta\xi_i}{\delta\beta_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\beta_i'} \\ + \frac{\delta\xi_i}{\delta\gamma_i'}\int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma_i}\,dt - \frac{\delta\xi_i}{\delta\gamma_i}\int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma_i'}\,dt \Bigg\},\end{aligned} \qquad (\mathrm{R}^7.)$$

and the perturbations of the two other coordinates may be expressed in an analogous manner.

It results from the same principles, that in taking the first differentials of these perturbations (R⁷.), the integrals may be treated as constant; and therefore that we may either represent the change of place of the disturbed point m_i, in its relative orbit about m_n, by altering a little the initial components of velocity without altering the initial position, and then employing the rules of binary systems; or calculate at once the perturbations of place and of velocity, by employing the same rules, and altering at once the initial position and initial velocity. If we adopt the former of these two methods, we are to employ the expressions (O⁷.), which may be thus summed up,

$$\begin{aligned}\Delta\alpha_i' &= \Sigma_{''}\,.\,m_k \frac{\delta}{\delta\alpha_i}\int_0^t R^{(i,k)}\,dt, \\ \Delta\beta_i' &= \Sigma_{''}\,.\,m_k \frac{\delta}{\delta\beta_i}\int_0^t R^{(i,k)}\,dt, \\ \Delta\gamma_i' &= \Sigma_{''}\,.\,m_k \frac{\ }{\delta\gamma_i}\int_0^t R^{(i,k)}\,dt;\end{aligned} \qquad (\mathrm{S}^7.)$$

$$\left[\frac{\partial\alpha_i'}{\partial\beta_i} = -\frac{\partial^2 w^{(i)}}{\partial\alpha_i\,\partial\beta_i} = \frac{\partial\beta_i'}{\partial\alpha_i}, \quad \frac{\partial\alpha_i'}{\partial\gamma_i} = -\frac{\partial^2 w^{(i)}}{\partial\gamma_i\,\partial\alpha_i} = \frac{\partial\gamma_i'}{\partial\alpha_i}.\right]$$

and if we adopt the latter method, we are to make,

$$\left.\begin{aligned}\Delta\alpha'_i &= \Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i}\,dt, \quad \Delta\alpha_i = -\Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha'_i}\,dt,\\ \Delta\beta'_i &= \Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\beta_i}\,dt, \quad \Delta\beta_i = -\Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\beta'_i}\,dt,\\ \Delta\gamma'_i &= \Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma_i}\,dt, \quad \Delta\gamma_i = -\Sigma_{\prime\prime}\,.\,m_k\int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma'_i}\,dt.\end{aligned}\right\} \qquad (\mathrm{T}^7.)$$

The latter was the method of Lagrange: the former is suggested more immediately by the principles of the present essay.*

General introduction of the Time into the expression of the Characteristic Function in any dynamical problem.

23. Before we conclude this sketch of our general method in dynamics, it will be proper to notice briefly a transformation of the characteristic function, which may be used in all applications. This transformation consists in putting, generally,†

$$V = tH + S, \qquad (\mathrm{U}^7.)$$

and considering the part S, namely, the definite integral

$$S = \int_0^t (T+U)\,dt, \qquad (\mathrm{V}^7.)$$

as a function of the initial and final coordinates and of the time, of which the variation is, by our law of varying action,

$$\delta S = -H\,\delta t + \Sigma\,.\,m\,(x'\delta x - a'\delta a + y'\delta y - b'\delta b + z'\delta z - c'\delta c). \qquad (\mathrm{W}^7.)$$

The partial differential coefficients of the first order of this auxiliary‡ function S are hence,

$$\frac{\delta S}{\delta t} = -H; \qquad (\mathrm{X}^7.)$$

$$\frac{\delta S}{\delta x_i} = m_i x'_i, \quad \frac{\delta S}{\delta y_i} = m_i y'_i, \quad \frac{\delta S}{\delta z_i} = m_i z'_i; \qquad (\mathrm{Y}^7.)$$

---

* [If we give variations $\Delta\alpha_i$, $\Delta\beta_i$, $\Delta\gamma_i$, the new variations of α'_i, β'_i, γ'_i will be

$$\Delta\alpha'_i + \frac{\delta\alpha'_i}{\delta\alpha_i}\,\Delta\alpha_i + \frac{\delta\alpha'_i}{\delta\beta_i}\,\Delta\beta_i + \frac{\delta\alpha'_i}{\delta\gamma_i}\Delta\gamma_i \quad \text{or} \quad \Delta\alpha'_i + \frac{\delta\alpha'_i}{\delta\alpha_i}\,\Delta\alpha_i + \frac{\delta\beta'_i}{\delta\alpha_i}\,\Delta\beta_i + \frac{\delta\gamma'_i}{\delta\alpha_i}\,\Delta\gamma_i,$$

since

$$\frac{\delta\beta'_i}{\delta\alpha_i} = \frac{\delta\alpha'_i}{\delta\beta_i} = -\frac{\delta^2 w^{(i)}}{\delta\alpha_i\,\delta\beta_i},$$

where $\Delta\alpha'_i$ etc. mean the values given in (O^7.). By giving the values of $\Delta\alpha_i$, $\Delta\beta_i$, $\Delta\gamma_i$ in (T^7.) we get the values of $\Delta\alpha'_i$ in the same set of equations. See Appendix, Note 3, p. 622. Cf. Lagrange, *Mécanique Analytique*, Tome I, 2nd Part, Section v. Lagrange, "Sur la théorie générale de la variation des constantes arbitraires," *Mémoires de l'Institut* (1808), p. 257. For a comparison of the methods of Lagrange, Poisson and Hamilton in the Theory of Perturbations, see Lovett, *Quarterly Journal of Mathematics*, xxx (1899), pp. 47–149.]

† [Hamilton perceived at this stage that it was inconvenient to have H appearing in the characteristic function; so he transformed to another function which contained t as a variable instead of H. The transformation to adopt is obvious from the following equations:

$$\delta V = \Sigma m\,(x'\delta x - a'\delta a + y'\delta y - b'\delta b + z'\delta z - c'\delta c) + t\,\delta H,$$

$$\therefore\ \delta\,(V - Ht) = \Sigma m\,(x'dx - a'da + \ldots) - H\delta t.]$$

‡ [Hamilton changes the name "auxiliary" to "principal" in the Second Essay.]

and
$$\frac{\delta S}{\delta a_i} = -m_i a'_i, \quad \frac{\delta S}{\delta b_i} = -m_i \upsilon_i, \quad \frac{\delta S}{\delta c_i} = -m_i c'_i. \tag{Z^7.}$$

These last expressions (Z^7.) are forms for the final integrals of motion of any system, corresponding to the result of elimination of H between the equations (D.) and (E.); and the expressions (Y^7.) are forms for the intermediate integrals, more convenient in many respects than the forms already employed.

24. The limits of the present essay do not permit us here to develope the consequences of these new expressions. We can only observe, that the auxiliary function S must satisfy the two following equations, in partial differentials of the first order, analogous to, and deduced from, the equations (F.) and (G.):

$$\frac{\delta S}{\delta t} + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta S}{\delta x}\right)^2 + \left(\frac{\delta S}{\delta y}\right)^2 + \left(\frac{\delta S}{\delta z}\right)^2 \right\} = U, \tag{A^8.}$$

and

$$\frac{\delta S}{\delta t} + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta S}{\delta a}\right)^2 + \left(\frac{\delta S}{\delta b}\right)^2 + \left(\frac{\delta S}{\delta c}\right)^2 \right\} = U_0; \tag{B^8.}$$

and that to correct an approximate value S_1 of S, in the integration of these equations, or to find the remaining part S_2, if

$$S = S_1 + S_2, \tag{C^8.}$$

we may employ the symbolic equation

$$\frac{d}{dt} = \frac{\delta}{\delta t} + \Sigma \cdot \frac{1}{m} \left(\frac{\delta S}{\delta x}\frac{\delta}{\delta x} + \frac{\delta S}{\delta y}\frac{\delta}{\delta y} + \frac{\delta S}{\delta z}\frac{\delta}{\delta z} \right); \tag{D^8.}$$

which gives, rigorously,

$$\frac{dS_2}{dt} = U - U_1 + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta S_2}{\delta x}\right)^2 + \left(\frac{\delta S_2}{\delta y}\right)^2 + \left(\frac{\delta S_2}{\delta z}\right) \right. \tag{E^8.}$$

if we establish by analogy the definition

$$U_1 = \frac{\delta S_1}{\delta t} + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta S_1}{\delta x}\right)^2 + \left(\frac{\delta S_1}{\delta y}\right)^2 + \left(\frac{\delta S_1}{\delta z}\right)^2 \right\}; \tag{F^8.}$$

and therefore approximately

$$S_2 = \int_0^t (U - U_1)\, dt, \tag{G^8.}$$

the parts S_1, S_2 being chosen so as to vanish with the time. These remarks may all be extended easily, so as to embrace relative and polar coordinates, and other marks of position, and offer a new and better way of investigating the orbits and perturbations of a system, by a new and better form of the function and method of this Essay.

March 29, 1834.

III.

SECOND ESSAY ON A GENERAL METHOD IN DYNAMICS.*

Received October 29, 1834,—Read January 15, 1835.

[*Philosophical Transactions of the Royal Society*, Part I for 1835, pp. 95–144.]

CONTENTS

PAGE
Introductory Remarks. 163
Transformations of the Differential Equations of Motion of an Attracting or Repelling System. 164
Integration of the Equations of Motion, by means of one Principal Function. 166
Investigation of a Pair of Partial Differential Equations of the first Order, which the Principal Function must satisfy. 168
General Method of improving an approximate Expression for the Principal Function in any Problem of Dynamics. 169
Rigorous Theory of Perturbations, founded on the Properties of the Disturbing Part of the whole Principal Function. 170
Approximate expressions deduced from the foregoing rigorous Theory. 172
Other Rigorous Theory of Perturbation, founded on the properties of the disturbing part of the constant of living force, and giving formulae for the Variation of Elements more analogous to those already known. 174
Simplification of the differential equations which determine these gradually varying elements, in any problem of Perturbation; and Integration of the simplified equations by means of certain Functions of Elements. 177
Analogous formulae for the motion of a Single Point. 181
Mathematical Example, suggested by the motion of Projectiles. 186
Remarks on the foregoing Example. 197
Attracting Systems resumed: Differential Equations of internal or Relative Motion; Integration by the Principal Function. 197
Second Example: Case of a Ternary or Multiple System with one Predominant Mass; Equations of the undisturbed motions of the other masses about this, in their several Binary Systems; Differentials of all their Elements, expressed by the coefficients of one Disturbing Function. 200
Simplification of the Differential Expressions by a suitable choice of the Elements. . . 202
On the essential distinction between the Systems of Varying Elements considered in this Essay and those hitherto employed by mathematicians. 207
Integration of the Simplified Equations, which determine the new varying Elements. . 209

---

* [A short account of this essay is to be found on pp. 212–216.]

Introductory Remarks.

The former Essay[*] contained a general method for reducing all the most important problems of dynamics to the study of one characteristic function, one central or radical relation. It was remarked at the close of that Essay, that many eliminations required by this method in its first conception might be avoided by a general transformation, introducing the time explicitly into a part S of the whole characteristic function V; and it is now proposed to fix the attention chiefly on this part S, and to call it the *Principal Function.* The properties of this part or function S, which were noticed briefly in the former Essay,[†] are now more fully set forth; and especially its uses in questions of perturbation, in which it dispenses with many laborious and circuitous processes, and enables us to express accurately the disturbed configuration of a system by the rules of undisturbed motion, if only the initial components of velocities be changed in a suitable manner. Another manner of extending rigorously to disturbed motion the rules of undisturbed, by the gradual variation of elements, in number double the number of the coordinates or other marks of position of the system, which was first invented by Lagrange,[‡] and was afterwards improved by Poisson, is considered in this Second Essay under a form perhaps a little more general; and the general method of calculation which has already been applied to other analogous questions in optics and in dynamics by the author of the present Essay, is now applied to the integration of the equations which determine these elements. This general method is founded chiefly on a combination of the principles of variations with those of partial differentials, and may furnish, when it shall be matured by the labours of other analysts, a separate branch of algebra, which may be called perhaps the *Calculus of Principal Functions*;[§] because, in all the chief applications of algebra to physics, and in a very extensive class of purely mathematical questions, it reduces the determination of many mutually connected functions to the search and study of one principal or central relation. When applied to the integration of the equations of varying elements, it suggests, as is now shown, the consideration of a certain *Function of Elements,* which may be variously chosen, and may either be rigorously determined, or at least approached to, with an indefinite accuracy, by a corollary of the general method. And to illustrate all these new general processes, but especially those which are connected with problems of perturbation, they are applied in this Essay to a very simple example, suggested by the motions of projectiles, the parabolic path being treated as the undisturbed. As a more important example, the problem of determining the motions of a ternary or multiple system, with any laws of attraction or repulsion, and with one predominant mass, which was touched upon in the former Essay, is here resumed in a new way, by forming and integrating the differential equations of a new set of varying elements, entirely distinct in theory (though little differing in practice) from the elements conceived by Lagrange,[||] and having this advantage, that the differentials of all the new elements for *both* the disturbed and disturbing masses may be expressed by the coefficients of *one* disturbing function.

---

[*] [P. 103.]

[†] [Pp. 160, 161.]

[‡] [Lagrange, "Sur la théorie générale de la variation des constantes arbitraires," *Mémoires de l'Institut* (1808), p. 257; *Méc. Anal.* 3rd ed. Tome I, pp. 299–320; Poisson, "Sur la variation des constantes arbitraires dans les questions de mécanique," *Journal de l'École Polyt.* Tome VIII (1809), p. 266.]

[§] [For the development of the Calculus of Principal Functions see pp. 297–410 of this volume. A brief account of the theory is given on pp. 408–410.]

[||] [For Lagrange's elements see *Méc. Anal.* 3rd ed. Tome II, Chap. II, § 2.]

Transformations of the Differential Equations of Motion of an Attracting or Repelling System.

1. It is well known to mathematicians, that the differential equations of motion of any system of free points, attracting or repelling one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula:

$$\Sigma \,.\, m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \delta U: \tag{1.}$$

the sign of summation Σ extending to all the points of the system; m being, for any one such point, the constant called its mass, and x, y, z being its rectangular coordinates; while x'', y'', z'' are the accelerations, or second differential coefficients taken with respect to the time, and δx, δy, δz are any arbitrary infinitesimal variations of those coordinates, and U is a certain *force-function,* introduced into dynamics by Lagrange, and involving the masses and mutual distances of the several points of the system. If the number of those points be n, the formula (1.) may be decomposed into $3n$ ordinary differential equations of the second order, between the coordinates and the time,

$$m_i x''_i = \frac{\delta U}{\delta x_i}; \quad m_i y''_i = \frac{\delta U}{\delta y_i}; \quad m_i z''_i = \frac{\delta U}{\delta z_i}: \tag{2.}$$

and to integrate these differential equations of motion of an attracting or repelling system, or some transformations of these, is the chief and perhaps ultimately the only problem of mathematical dynamics.

2. To facilitate and generalize the solution of this problem, it is useful to express previously the $3n$ rectangular coordinates x, y, z as functions of $3n$ other and more general marks of position $\eta_1, \eta_2, \ldots \eta_{3n}$; and then the differential equations of motion take this more general form, discovered by Lagrange,*

$$\frac{d}{dt}\frac{\delta T}{\delta \eta'_i} - \frac{\delta T}{\delta \eta_i} = \frac{\delta U}{\delta \eta_i}, \tag{3.}$$

in which

$$T = \tfrac{1}{2}\Sigma \,.\, m\,(x'^2 + y'^2 + z'^2). \tag{4.}$$

For, from the equations (2.) or (1.),

$$\begin{aligned} \frac{\delta U}{\delta \eta_i} &= \Sigma \,.\, m\left(x''\frac{\delta x}{\delta \eta_i} + y''\frac{\delta y}{\delta \eta_i} + z''\frac{\delta z}{\delta \eta_i}\right) \\ &= \frac{d}{dt}\Sigma \,.\, m\,\Big(x'\frac{\delta x}{\delta \eta_i} + y'\frac{\delta y}{\delta \eta_i} + z'\frac{\delta z}{\delta \eta} \\ &\quad - \Sigma \,.\, m\left(x'\frac{d}{dt}\frac{\delta x}{\delta \eta_i} + y'\frac{d}{dt}\frac{\delta y}{\delta \eta_i} + z'\frac{d}{dt}\frac{\delta z}{\delta \eta_i}\right); \end{aligned} \tag{5.}$$

in which

$$\begin{aligned} &\Sigma \,.\, m\,\Big(x'\frac{\delta x}{\delta \eta_i} + y'\frac{\delta y}{\delta \eta_i} + z'\frac{\delta z}{\delta \eta_i} \\ &= \Sigma \,.\, m\left(x'\frac{\delta x'}{\delta \eta'_i} + y'\frac{\delta y'}{\delta \eta'_i} + z'\frac{\delta z'}{\delta \eta'_i}\right) = \frac{\delta T}{\delta \eta'_i}, \end{aligned} \tag{6.}$$

---

* [*Méc. Anal.* 3rd ed. Tome I, pp. 290–292.]

and
$$\Sigma \,.\, m \left(x' \frac{d}{dt} \frac{\delta x}{\delta \eta_i} + y' \frac{d}{dt} \frac{\delta y}{\delta \eta_i} + z' \frac{d}{dt} \frac{\delta z}{\delta \eta_i} \right)$$
$$= \Sigma \,.\, m \left(x' \frac{\delta x'}{\delta \eta_i} + y' \frac{\delta y'}{\delta \eta_i} + z' \frac{\delta z'}{\delta \eta_i} \right) = \frac{\delta T}{\delta \eta_i}, \tag{7.}$$

T being here considered as a function of the $6n$ quantities of the forms η' and η, obtained by introducing into its definition (4.) the values

$$x' = \eta'_1 \frac{\delta x}{\delta \eta_1} + \eta'_2 \frac{\delta x}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta x}{\delta \eta_{3n}}, \ \&\text{c.} \tag{8.}$$

A different proof of this important transformation (3.) is given in the *Mécanique Analytique.*

3. The function T, being homogeneous of the second dimension with respect to the quantities η', must satisfy the condition

$$2T = \Sigma \,.\, \eta' \frac{\delta T}{\delta \eta'}; \tag{9.}$$

and since the variation of the same function T may evidently be expressed as follows,

$$\delta T = \Sigma \left(\frac{\delta T}{\delta \eta'} \delta \eta' + \frac{\delta T}{\delta \eta} \delta \eta \right), \tag{10.}$$

we see that this variation may be expressed in this other way,

$$\delta T = \Sigma \left(\eta' \delta \frac{\delta T}{\delta \eta'} - \frac{\delta T}{\delta \eta} \delta \eta \right). \tag{11.}$$

If then we put, for abridgement,

$$\frac{\delta T}{\delta \eta'_1} = \varpi_1, \quad \dots \quad \frac{\delta T}{\delta \eta'_{3n}} = \varpi_{3n}, \tag{12.}$$

and consider T (as we may) as a function of the following form,

$$T = F(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}), \tag{13.}$$

we see that

$$\frac{\delta F}{\delta \varpi_1} = \eta'_1, \quad \dots \quad \frac{\delta F}{\delta \varpi_{3n}} \quad \eta'_{3n}, \tag{14.}$$

and

$$\frac{\delta F}{\delta \eta_1} = -\frac{\delta T}{\delta \eta_1}, \quad \dots \quad \frac{\delta F}{\delta \eta_{3n}} - -\frac{\delta T}{\delta \eta_{3n}}, \tag{15.}$$

and therefore that the general equation (3.) may receive this new transformation,

$$\frac{d\varpi_i}{dt} = \frac{\delta (U - F)}{\delta \eta_i}. \tag{16.}$$

If then we introduce, for abridgement, the following expression H,

$$H = F - U = F(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U(\eta_1, \eta_2, \dots \eta_{3n}), \tag{17.}$$

we are conducted to this new manner of presenting the differential equations of motion of a system of n points, attracting or repelling one another:*

$$\left.\begin{array}{ll} \dfrac{d\eta_1}{dt} = \dfrac{\delta H}{\delta\varpi_1}; & \dfrac{d\varpi_1}{dt} = -\dfrac{\delta H}{\delta\eta_1}; \\[2ex] \dfrac{d\eta_2}{dt} = \dfrac{\delta H}{\delta\varpi_2}; & \dfrac{d\varpi_2}{dt} = -\dfrac{\delta H}{\delta\eta_2}; \\[2ex] \dfrac{d\eta_{3n}}{dt} = \dfrac{\delta H}{\delta\varpi_{3n}}; & \dfrac{d\varpi_{3n}}{dt} = -\dfrac{\delta H}{\delta\eta_{3n}}. \end{array}\right\} \qquad \text{(A.)}$$

In this view, the problem of mathematical dynamics, for a system of n points, is to integrate a system (A.) of $6n$ ordinary differential equations of the first order, between the $6n$ variables η_i, ϖ_i and the time t; and the solution of the problem must consist in assigning these $6n$ variables as functions of the time, and of their own initial values, which we may call e_i, p_i. And all these $6n$ functions, or $6n$ relations to determine them, may be expressed, with perfect generality and rigour, by the method of the former Essay, or by the following simplified process.

Integration of the Equations of Motion, by means of one Principal Function.

4. If we take the variation of the definite integral

$$S = \int_0^t \left(\Sigma\,.\,\varpi\frac{\delta H}{\delta\varpi} - H\right) dt \qquad (18.)$$

without varying t or dt, we find, by the Calculus of Variations,

$$\delta S = \int_0^t \delta S'\,.\,dt, \qquad (19.)$$

in which

$$S' = \Sigma\,.\,\varpi\frac{\delta H}{\delta\varpi} - H, \qquad (20.)$$

and therefore†

$$\delta S' = \Sigma\left(\varpi\delta\frac{\delta H}{\delta\varpi} - \frac{\delta H}{\delta\eta}\,\delta\eta\right), \qquad (21.)$$

that is, by the equations of motion (A.),

$$\delta S' = \Sigma\left(\varpi\delta\frac{d\eta}{dt} + \frac{d\varpi}{dt}\,\delta\eta\right) = \frac{d}{dt}\Sigma\,.\,\varpi\delta\eta; \qquad (22.)$$

the variation of the integral S is therefore

$$\delta S = \Sigma\,(\varpi\,\delta\eta - p\,\delta e), \qquad (23.)$$

---

* [The canonical form (A.) had already been introduced by Lagrange in the particular case of the equations for the variations of the elements in his perturbation theory. Cf. *Méc. Anal.* 3rd ed. Tome I, p. 310; also Whittaker, *Analytical Dynamics* (1927), p. 264.]

† $\left[\delta S' = \Sigma\left(\frac{\partial H}{\partial\varpi}\,\delta\varpi + \varpi\delta\frac{\partial H}{\partial\varpi}\right) - \Sigma\frac{\partial H}{\partial\varpi}\,\delta\varpi - \Sigma\frac{\partial H}{\partial\eta}\,\delta\eta.\right]$

(p and e being still initial values,) and it decomposes itself into the following $6n$ expressions, when S is considered as a function of the $6n$ quantities η_i, e_i, (involving also the time,)

$$\left.\begin{array}{ll} \varpi_1 = \frac{\delta S}{\delta \eta_1}; & p_1 = -\frac{\delta S}{\delta e_1}, \\ \varpi_2 = \frac{\delta S}{\delta \eta_2} & p_2 = -\frac{\delta S}{\delta e_2}; \\ & \\ \varpi_{3n} = \frac{\delta S}{\delta \eta_{3n}}, & p_{3n} = \frac{\delta S}{\delta e_{3n}}, \end{array}\right\} \tag{B.}$$

which are evidently forms for the sought integrals of the $6n$ differential equations of motion (A.), containing only one unknown function S. The difficulty of mathematical dynamics is therefore reduced to the search and study of this one function S, which may for that reason be called the PRINCIPAL FUNCTION of motion of a system.

This function S was introduced in the first Essay under the form*

$$S = \int_0^t (T + U)\, dt,$$

the symbols T and U having in this form their recent meanings; and it is worth observing, that when S is expressed by this definite integral, the conditions for its variation vanishing (if the final and initial coordinates and the time be given) are precisely the differential equations of motion (3.), under the forms assigned by Lagrange. The variation of this definite integral S has therefore the double property, of giving the differential equations of motion for any transformed coordinates when the extreme positions are regarded as fixed, and of giving the integrals of those differential equations when the extreme positions are treated as varying.†

5. Although the function S seems to deserve the name here given it of *Principal Function*, as serving to express, in what appears the simplest way, the integrals of the equations of motion, and the differential equations themselves; yet the same analysis conducts to other functions, which also may be used to express the integrals of the same equations. Thus, if we put

$$Q = \int_0^t \left(-\Sigma \,.\, \eta \frac{\delta H}{\delta \eta} + H \right) dt, \tag{24.}$$

and take the variation of this integral Q without varying t or dt, we find, by a similar process,

$$\delta Q = \Sigma\, (\eta\, \delta \varpi - e\, \delta p); \tag{25.}$$

so that if we consider Q as a function of the $6n$ quantities ϖ_i, p_i and of the time, we shall have $6n$ expressions

$$\eta_i = +\frac{\delta Q}{\delta \varpi_i}, \quad e_i = -\frac{\delta Q}{\delta p_i}, \tag{26.}$$

which are other forms for the integrals of the equations of motion (A.), involving the function Q instead of S. We might also employ the integral

$$r = \int_0^t \Sigma \,.\, \varpi \frac{\delta H}{\delta \varpi}\, dt = \Sigma \int_e^\eta \varpi\, d\eta, \tag{27.}$$

---

* [P. 160.] $\left[\delta S = \left[\Sigma \frac{\partial T}{\partial \eta'} \delta \eta\right]_0^t - \int_0^t \left[\Sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \eta'} - \frac{\partial T}{\partial \eta} - \frac{\partial U}{\partial \eta}\right) \delta \eta\right] dt.\right]$

which was called the *Characteristic Function* in the former Essay, and of which, when considered as a function of the $6n+1$ quantities η_i, e_i, H, the variation is

$$\delta V = \Sigma\,(\varpi\,\delta\eta - p\,\delta e) + t\,\delta H. \tag{28.}$$

And all these functions S, Q, V, are connected in such a way, that the forms and properties of any one may be deduced from those of any other.[*]

Investigation of a Pair of Partial Differential Equations of the first Order, which the Principal Function must satisfy.

6. In forming the variation (23.), or the partial differential coefficients (B.), of the Principal Function S, the variation of the time was omitted; but it is easy to calculate the coefficient $\dfrac{\delta S}{\delta t}$ corresponding to this variation, since the evident equation

$$\frac{dS}{dt} = \frac{\delta S}{\delta t} + \Sigma\,\frac{\delta S}{\delta \eta}\,\frac{d\eta}{dt} \tag{29.}$$

gives, by (20.), and by (A.), (B.),

$$\frac{\delta S}{\delta t} = S' - \Sigma\,.\,\varpi\,\frac{\delta H}{\delta \varpi} = -H. \tag{30.}$$

It is evident also that this coefficient, or the quantity $-H$, is constant, so as not to alter during the motion of the system; because the differential equations of motion (A.) give

$$\frac{dH}{dt} = \Sigma\left(\frac{\delta H}{\delta \eta}\,\frac{d\eta}{dt} + \frac{\delta H}{\delta \varpi}\,\frac{d\varpi}{dt}\right) = 0. \tag{31.}$$

If, therefore, we attend to the equation (17.), and observe that the function F is necessarily rational and integer and homogeneous of the second dimension with respect to the quantities ϖ_i, we shall perceive that the principal function S must satisfy the two following equations between its partial differential coefficients of the first order, which offer the chief means of discovering its form:[†]

$$\left.\begin{aligned}
&\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \ldots \frac{\delta S}{\delta \eta_{3n}}, \eta_1, \eta_2, \ldots \eta_{3n}\right) = U\,(\eta_1, \eta_2, \ldots \eta_{3n}),\\
&\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \ldots \frac{\delta S}{\delta e_{3n}}, e_1, e_2, \ldots e_{3n}\right) = U\,(e_1, e_2, \ldots e_{3n}).
\end{aligned}\right\} \tag{C.}$$

---

[*] [$S = V - Ht$, $Q = -S + \Sigma\,(\eta\varpi - ep)$. The process is analogous to that on pp. 174, 175, *Mathematical Papers*, Vol. I, in which the functions W and T are introduced. See also Routh, *Advanced Rigid Dynamics* (1905), Art. 487.]

[†] [It is now more usual to write $S = \displaystyle\int_{t_0}^{t}\left(\Sigma\varpi\frac{\partial H}{\partial \varpi} - H\right)dt$, and we have, analogous to (29) and (30),

$$\frac{dS}{dt_0} = \frac{\partial S}{\partial t_0} + \Sigma\,\frac{\partial S}{\partial e}\,\frac{de}{dt_0} \text{ and } \frac{\partial S}{\partial t_0} = -\left(\Sigma p\,\frac{\partial H_0}{\partial p} - H_0\right) + \Sigma p\,\frac{\partial H_0}{\partial p} = H_0.$$

The two partial differential equations are therefore

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial \eta_1}, \frac{\partial S}{\partial \eta_2}, \ldots \frac{\partial S}{\partial \eta_{3n}}, \eta_1, \eta_2, \ldots \eta_{3n}, t\right) = 0,$$

$$\frac{\partial S}{\partial t_0} - H\left(-\frac{\partial S}{\partial e_1}, -\frac{\partial S}{\partial e_2}, \ldots -\frac{\partial S}{\partial e_{3n}}, e_1, e_2, \ldots e_{3n}, t_0\right) = 0,$$

even when H is not constant. It is easy to see that these are equivalent to (C.) when H is constant. Jacobi (*Crelle*, XXVII, pp. 97–162) showed that equations (C.) still hold when U contains t explicitly, and the corresponding equations for the function V are also given there.]

Reciprocally, if the form of S be known, the forms of these equations (C.) can be deduced from it, by elimination of the quantities e or η between the expressions of its partial differential coefficients; and thus we can return from the principal function S to the functions F and U, and consequently to the expression H, and the equations of motion (A.).

Analogous remarks apply to the functions Q and V, which must satisfy the partial differential equations,

$$\left.\begin{aligned} -\frac{\delta Q}{\delta t}+F\left(\varpi_1,\varpi_2,\dots\varpi_{3n},\frac{\delta Q}{\delta\varpi_1},\frac{\delta Q}{\delta\varpi_2},\dots\frac{\delta Q}{\delta\varpi_{3n}}\right)&=U\left(\frac{\delta Q}{\delta\varpi_1},\frac{\delta Q}{\delta\varpi_2},\dots\frac{\delta Q}{\delta\varpi_{3n}}\right),\\ -\frac{\delta Q}{\delta t}+F\left(p_1,p_2,\dots p_{3n},-\frac{\delta Q}{\delta p_1},-\frac{\delta Q}{\delta p_2},\dots-\frac{\delta Q}{\delta p_{3n}}\right)&=U\left(-\frac{\delta Q}{\delta p_1},-\frac{\delta Q}{\delta p_2},\dots-\frac{\delta Q}{\delta p_{3n}}\right),\end{aligned}\right\}\quad(32.)$$

and

$$\left.\begin{aligned} F\left(\frac{\delta V}{\delta\eta_1},\frac{\delta V}{\delta\eta_2},\dots\frac{\delta V}{\delta\eta_{3n}},\eta_1,\eta_2,\dots\eta_{3n}\right)&=H+U\left(\eta_1,\eta_2,\dots\eta_{3n}\right),\\ F\left(\frac{\delta V}{\delta e_1},\frac{\delta V}{\delta e_2},\dots\frac{\delta V}{\delta e_{3n}},e_1,e_2,\dots e_{3n}\right)&=H+U\left(e_1,e_2,\dots e_{3n}\right).\end{aligned}\right\}\quad(33.)$$

General Method of improving an approximate Expression for the Principal Function in any Problem of Dynamics.

7. If we separate the principal function S into any two parts,

$$S_1+S_2=S,\qquad(34.)$$

and substitute their sum for S in the first equation (C.), the function F, from its rational and integer and homogeneous form and dimension, may be expressed in this new way,

$$\left.\begin{aligned} F\left(\frac{\delta S}{\delta\eta_1},\dots\frac{\delta S}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right)&=F\left(\frac{\delta S_1}{\delta\eta_1},\dots\frac{\delta S_1}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right)\\ &\quad+F'\left(\frac{\delta S_1}{\delta\eta_1}\right)\frac{\delta S_2}{\delta\eta_1}+\dots+F'\left(\frac{\delta S_1}{\delta\eta_{3n}}\right)\frac{\delta S_2}{\delta\eta_{3n}}+F\left(\frac{\delta S_2}{\delta\eta_1},\dots\frac{\delta S_2}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right)\\ &=F\left(\frac{\delta S_1}{\delta\eta_1},\dots\frac{\delta S_1}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right)-F\left(\frac{\delta S_2}{\delta\eta_1},\dots\frac{\delta S_2}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right.\\ &\quad+F'\left(\frac{\delta S}{\delta\eta_1}\right)\frac{\delta S_2}{\delta\eta_1}+\dots+F'\left(\frac{\delta S}{\delta\eta_{3n}}\right)\frac{\delta S_2}{\delta\eta_{3n}},\end{aligned}\right\}\quad(35.)$$

because*

$$F'\left(\frac{\delta S_1}{\delta\eta_i}\right)=F'\left(\frac{\delta S}{\delta\eta_i}\right)-F'\left(\frac{\delta S_2}{\delta\eta_i}\right),\qquad(36.)$$

and

$$\Sigma\,.\,F'\left(\frac{\delta S_2}{\delta\eta}\right)\frac{\delta S_2}{\delta\eta}=2F\left(\frac{\delta S_2}{\delta\eta_1},\dots\frac{\delta S_2}{\delta\eta_{3n}},\eta_1,\dots\eta_{3n}\right);\qquad(37.)$$

and since, by (A.) and (B.),

$$F'\left(\frac{\delta S}{\delta\eta_i}\right)=F'(\varpi_i)=\frac{\delta H}{\delta\varpi_i}=\frac{d\eta_i}{dt},\qquad(38.)$$

[By $F'\left(\frac{\delta S}{\delta\eta_i}\right)$ is meant the partial derivative of F with respect to $\frac{\delta S}{\delta\eta_i}$.]

we easily transform the first equation (C.) to the following,

$$\frac{dS_2}{dt} = -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) + F\left(\frac{\delta S_2}{\delta \eta_1}, \dots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right), \tag{D.}$$

which gives rigorously

$$\left.\begin{aligned} S_2 = \int_0^t \left\{ -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right)\right\} dt \\ + \int_0^t F\left(\frac{\delta S_2}{\delta \eta_1}, \dots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right) dt, \end{aligned}\right\} \tag{E.}$$

supposing only that the two parts S_1, S_2, like the whole principal function S, are chosen so as to vanish with the time.

This general and rigorous transformation offers a general method of improving an approximate expression for the principal function S, in any problem of dynamics. For if the part S_1 be such an approximate expression, then the remaining part S_2 will be small; and the homogeneous function F involving the squares and products of the coefficients of this small part, in the second definite integral (E.), will be in general also small, and of a higher order of smallness; we may therefore in general neglect this second definite integral, in passing to a second approximation, and may in general improve a first approximate expression S_1 by adding to it the following correction,

$$\Delta S_1 = \int_0^t \left\{ -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right)\right\} dt; \tag{F.}$$

in calculating which definite integral we may employ the following approximate forms for the integrals of the equations of motion,

$$p_1 = -\frac{\delta S_1}{\delta e_1}, \quad p_2 = -\frac{\delta S_1}{\delta e_2}, \qquad p_{3n} = -\frac{\delta S_1}{\delta e_{3n}}, \tag{39.}$$

expressing first, by these, the variables η_i as functions of the time and of the $6n$ constants e_i, p_i, and then eliminating, after the integration, the $3n$ quantities p_i, by the same approximate forms. And when an improved expression, or second approximate value $S_1 + \Delta S_1$, for the principal function S, has been thus obtained, it may be substituted in like manner for the first approximate value S_1, so as to obtain a still closer approximation, and the process may be repeated indefinitely.

An analogous process applies to the indefinite improvement of a first approximate expression for the function Q or V.

Rigorous Theory of Perturbations, founded on the Properties of the Disturbing Part of the whole Principal Function.

8. If we separate the expression H (17.) into any two parts of the same kind,

$$H_1 + H_2 = H, \tag{40.}$$

in which

$$H_1 = F_1(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U_1(\eta_1, \eta_2, \dots \eta_{3n}), \tag{41.}$$

and

$$H_2 = F_2(\varpi_1, \varpi_2, \dots \varpi_{3n}, \eta_1, \eta_2, \dots \eta_{3n}) - U_2(\eta_1, \eta_2, \dots \eta_{3n}), \tag{42.}$$

the functions F_1, F_2, U_1, U_2 being such that

$$F_1 + F_2 = F, \quad U_1 + U_2 = U; \tag{43.}$$

the differential equations of motion (A.) will take this form,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i} + \frac{\delta H_2}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i} - \frac{\delta H_2}{\delta \eta_i}, \tag{G.}$$

and if the part H_2 and its coefficients be small, they will not differ much from these other differential equations,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i}; \tag{H.}$$

so that the rigorous integrals of the latter system will be approximate integrals of the former. Whenever then, by a proper choice of the predominant term H_1, a system of $6n$ equations such as (H.) has been formed and rigorously integrated, giving expressions for the $6n$ variables η_i, ϖ_i as functions of the time t, and of their own initial values e_i, p_i, which may be thus denoted:

$$\eta_i = \phi_i\,(t, e_1, e_2, \ldots e_{3n}, p_1, p_2, \ldots p_{3n}), \tag{44.}$$

and

$$\varpi_i = \psi_i\,(t, e_1, e_2, \ldots e_{3n}, p_1, p_2, \ldots p_{3n}); \tag{45.}$$

the simpler motion thus defined by the rigorous integrals of (H.) may be called the *undisturbed motion* of the proposed system of n points, and the more complex motion expressed by the rigorous integrals of (G.) may be called by contrast the *disturbed motion* of that system; and to pass from the one to the other, may be called a *Problem of Perturbation*.

9. To accomplish this passage, let us observe that the differential equations of undisturbed motion (H.), being of the same form as the original equations (A.), may have their integrals similarly expressed, that is, as follows:

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i}; \tag{I.}$$

S_1 being here the *principal function of undisturbed motion*, or the definite integral

$$S_1 = \int_0^t \left(\Sigma \,.\, \varpi \frac{\delta H_1}{\delta \varpi} - H_1\right) dt, \tag{46.}$$

considered as a function of the time and of the quantities η_i, e_i. In like manner if we represent by $S_1 + S_2$ the whole principal function of disturbed motion, the rigorous integrals of (G.) may be expressed by (B.), as follows:

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i} + \frac{\delta S_2}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i} - \frac{\delta S_2}{\delta e_i}. \tag{K.}$$

Comparing the forms (44.) with the second set of equations (I.) for the integrals of undisturbed motion, we find that the following relations between the functions ϕ_i, S_1 must be rigorously and *identically* true:

$$\eta_i = \phi_i \left(t, e_1, e_2, \ldots e_{3n}, -\frac{\delta S_1}{\delta e_1}, -\frac{\delta S_1}{\delta e_2}, \ldots -\frac{\delta S_1}{\delta e_{3n}}\right); \tag{47.}$$

and therefore, by (K.), that the integrals of disturbed motion may be put under the following forms,

$$\eta_i = \phi_i \left(t, e_1, e_2, \ldots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \ldots p_{3n} + \frac{\delta S_2}{\delta e_{3n}}\right. \tag{L.}$$

We may therefore calculate rigorously the disturbed variables η_i by the rules of undisturbed motion (44.), if without altering the time t, or the initial values e_i of those variables, which determine the initial configuration, we alter (in general) the initial velocities and directions, by adding to the elements p_i the following perturbational terms,

$$\Delta p_1 = \frac{\delta S_2}{\delta e_1}, \quad \Delta p_2 = \frac{\delta S_2}{\delta e_2}, \qquad\qquad \Delta p_{3n} = \frac{\delta S_2}{\delta e_{3n}} \tag{M.}$$

a remarkable result, which includes the whole theory of perturbation. We might deduce from it the differential coefficients η_i', or the connected quantities ϖ_i, which determine the disturbed directions and velocities of motion at any time t; but a similar reasoning gives at once the general expression,

$$\varpi_i = \frac{\delta S_2}{\delta \eta_i} + \psi_1 \left(t, e_1, e_2, \ldots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \ldots p_{3n} + \frac{\delta S_2}{\delta e_{3n}} \right. \tag{N.}$$

implying, that after altering the initial velocities and directions or the elements p_i as before, by the perturbational terms (M.), we may then employ the rules of undisturbed motion (45.) to calculate the velocities and directions at the time t, or the varying quantities ϖ_i, if we finally apply to these quantities thus calculated the following new corrections for perturbation:

$$\Delta \varpi_1 = \frac{\delta S_2}{\delta \eta_1}, \quad \Delta \varpi_2 = \frac{\delta S_2}{\delta \eta_2}, \qquad\qquad \Delta \varpi_{3n} = \frac{\delta S_2}{\delta \eta_{3n}}. \tag{O.}$$

Approximate expressions deduced from the foregoing rigorous Theory.

10. The foregoing theory gives indeed rigorous expressions for the perturbations, in passing from the simpler motion (H.) or (I.) to the more complex motion (G.) or (K.): but it may seem that these expressions are of little use, because they involve an unknown *disturbing function* S_2, (namely, the perturbational part of the whole principal function S,) and also unknown or disturbed coordinates or marks of position η_i. However, it was lately shown that whenever a first approximate form for the principal function S, such as here the principal function S_1 of undisturbed motion, has been found, the correction S_2 can in general be assigned, with an indefinitely increasing accuracy; and since the perturbations (M.) and (O.) involve the disturbed coordinates η_i only as they enter into the coefficients of this small disturbing function S_2, it is evidently permitted to substitute for these coordinates, at first, their undisturbed values, and then to correct the results by substituting more accurate expressions.

11. The function S_1 of undisturbed motion must satisfy rigorously two partial differential equations of the form (C.), namely,

$$\left.\begin{aligned} \frac{\delta S_1}{\delta t} + F_1\left(\frac{\delta S_1}{\delta \eta_1}, \ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) &= U_1(\eta_1, \ldots \eta_{3n}), \\ \frac{\delta S_1}{\delta t} + F_1\left(\frac{\delta S_1}{\delta e_1}, \ldots \frac{\delta S_1}{\delta e_{3n}}, e_1, \ldots e_{3n}\right) &= U_1(e_1, \ldots e_{3n}); \end{aligned}\right| \tag{P.}$$

and therefore, by (D.), the disturbing function S_2 must satisfy rigorously the following other condition:

$$\frac{dS_2}{dt} = U_2(\eta_1, \ldots \eta_{3n}) - F_2\left(\frac{\delta S_1}{\delta \eta_1}, \ldots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right) + F\left(\frac{\delta S_2}{\delta \eta_1}, \ldots \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \ldots \eta_{3n}\right), \tag{Q.}$$

and may, on account of the homogeneity and dimension of F, be approximately expressed as follows:

$$S_2 = \int_0^t \left\{ U_2(\eta_1, \dots \eta_{3n}) - F_2\left(\frac{\delta S_1}{\delta \eta_1}, \dots \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots \eta_{3n}\right)\right\} dt, \tag{R.}$$

or thus, by (I.),

$$S_2 = \int_0^t \left\{ U_2(\eta_1, \dots \eta_{3n}) - F_2(\varpi_1, \dots \varpi_{3n}, \eta_1, \dots \eta_{3n})\right\} dt, \tag{S.}$$

that is, by (42.)

$$S_2 = -\int_0^t H_2 dt. \tag{T.}$$

In this expression, H_2 is given immediately as a function of the varying quantities η_i, ϖ_i, but it may be considered in the same order of approximation as a known function of their initial values e_i, p_i and of the time t, obtained by substituting for η_i, ϖ_i their undisturbed values (44.), (45.) as functions of those quantities; its variation may therefore be expressed in either of the two following ways:

$$\delta H_2 = \Sigma\left(\frac{\delta H_2}{\delta \eta}\delta\eta + \frac{\delta H_2}{\delta \varpi}\delta\varpi\right), \tag{48.}$$

or

$$\delta H_2 = \Sigma\left(\frac{\delta H_2}{\delta e}\delta e + \frac{\delta H_2}{\delta p}\delta p\right) + \frac{\delta H_2}{\delta t}\delta t. \tag{49.}$$

Adopting the latter view, and effecting the integration (T.) with respect to the time, by treating the elements e_i, p_i as constant, we are afterwards to substitute for the quantities p_i their undisturbed expressions (39.) or (I.), and then we find for the variation of the disturbing function S_2 the expression*

$$\delta S_2 = -H_2\delta t + \Sigma\left(-\delta e \,.\int_0^t \frac{\delta H_2}{\delta e}dt + \delta\frac{\delta S_1}{\delta e}\,.\int_0^t \frac{\delta H_2}{\delta p}dt\right), \tag{50.}$$

which enables us to transform the perturbational terms (M.), (O.) into the following approximate forms:

$$\Delta p_i = -\int_0^t \frac{\delta H_2}{\delta e_i}dt + \Sigma\,.\frac{\delta^2 S_1}{\delta e\,\delta e_i}\int_0^t \frac{\delta H_2}{\delta p}dt, \tag{U.}$$

and

$$\Delta \varpi_i = \Sigma\,.\frac{\delta^2 S_1}{\delta e\,\delta \eta_i}\int_0^t \frac{\delta H_2}{\delta p}dt, \tag{V.}$$

containing only functions and quantities which may be regarded as given, by the theory of undisturbed motion.

12. In the same order of approximation, if the variation of the expression (44.) for an undisturbed coordinate η_i be thus denoted,

$$\delta\eta_i = \frac{\delta\eta_i}{\delta t}\delta t + \Sigma\left(\frac{\delta\eta_i}{\delta e}\delta e + \frac{\delta\eta_i}{\delta p}\delta p\right), \tag{51.}$$

the perturbation of that coordinate may be expressed as follows:

$$\Delta\eta_i = \Sigma\,.\frac{\delta\eta_i}{\delta p}\Delta p; \tag{W.}$$

---

* [The object is to express S_2 in terms of η and e, or δS_2 in terms of η, $\delta\eta$, e, δe.]

that is, by (U.),

$$\begin{aligned}\Delta\eta_i = & -\frac{\delta\eta_i}{\delta p_1}\int_0^t \frac{\delta H_2}{\delta e_1}\,dt - \frac{\delta\eta_i}{\delta p_2}\int_0^t \frac{\delta H_2}{\delta e_2}\,dt - \dots - \frac{\delta\eta_i}{\delta p_{3n}}\int_0^t \frac{\delta H_2}{\delta e_{3n}}\,dt \\ & + \left(\frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_1^2} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_1\,\delta e_2} + \dots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_1\,\delta e_{3n}}\right)\int_0^t \frac{\delta H_2}{\delta p_1}\,dt \\ & + . \\ & + \left(\frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_{3n}\,\delta e_1} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_{3n}\,\delta e_2} + \dots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_{3n}^2}\right. \quad {}_0^t\frac{\delta H_2}{\delta p_{3n}}\,dt.\end{aligned} \tag{52.}$$

Besides, the identical equation (47.) gives*

$$\frac{\delta\eta_i}{\delta e_k} = \frac{\delta\eta_i}{\delta p_1}\frac{\delta^2 S_1}{\delta e_k\,\delta e_1} + \frac{\delta\eta_i}{\delta p_2}\frac{\delta^2 S_1}{\delta e_k\,\delta e_2} + \dots + \frac{\delta\eta_i}{\delta p_{3n}}\frac{\delta^2 S_1}{\delta e_k\,\delta e_{3n}}; \tag{53.}$$

the expression (52.) may therefore be thus abridged,

$$\begin{aligned}\Delta\eta_i = & -\frac{\delta\eta_i}{\delta p_1}\int_0^t \frac{\delta H_2}{\delta e_1}\,dt - \dots - \frac{\delta\eta_i}{\delta p_{3n}}\int_0^t \frac{\delta H_2}{\delta e_{3n}}\,dt \\ & + \frac{\delta\eta_i}{\delta e_1}\int_0^t \frac{\delta H_2}{\delta p_1}\,dt + \dots + \frac{\delta\eta_i}{\delta e_{3n}}\int_0^t \frac{\delta H_2}{\delta p_{3n}}\,dt,\end{aligned} \tag{X.}$$

and shows that instead of the rigorous perturbational terms (M.) we may approximately employ the following,

$$\Delta p_i = -\int_0^t \frac{\delta H_2}{\delta e_i}\,dt, \tag{Y.}$$

in order to calculate the disturbed configuration at any time t by the rules of undisturbed motion, provided that besides thus altering the initial velocities and directions we alter also the initial configuration, by the formula

$$\Delta e_i = \int_0^t \frac{\delta H_2}{\delta p_i}\,dt. \tag{Z.}$$

It would not be difficult to calculate, in like manner, approximate expressions for the disturbed directions and velocities at any time t; but it is better to resume, in another way, the rigorous problem of perturbation.

Other Rigorous Theory of Perturbation, founded on the properties of the disturbing part of the constant of living force, and giving formulæ for the Variation of Elements more analogous to those already known.

13. Suppose that the theory of undisturbed motion has given the $6n$ constants e_i, p_i or any combinations of these, $\kappa_1, \kappa_2, \dots \kappa_{6n}$, as functions of the $6n$ variables η_i, ϖ_i and of the time t, which may be thus denoted:

$$\kappa_i = \chi_i(t, \eta_1, \eta_2, \dots \eta_{3n}, \varpi_1, \varpi_2, \dots \varpi_{3n}), \tag{54.}$$

and which give reciprocally expressions for the variables η_i, ϖ_i in terms of these elements and of the time, analogous to (44.) and (45.), and capable of being denoted similarly,

$$\eta_i = \phi_i(t, \kappa_1, \kappa_2, \dots \kappa_{6n}), \quad \varpi_i = \psi_i(t, \kappa_1, \kappa_2, \dots \kappa_{6n}); \tag{55.}$$

---

* [Equation (47.) is differentiated partially with respect to e_k, the η's being kept constant.]

then, the total differential coefficient of every such *element* or function κ_i, taken with respect to the time, (both as it enters explicitly and implicitly into the expressions (54.),) must vanish in the undisturbed motion; so that, by the differential equations of such motion (H.), the following general relation must be rigorously and *identically* true:

$$0=\frac{\delta\kappa_i}{\delta t}+\Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta H_1}{\delta\varpi}-\frac{\delta\kappa_i}{\delta\varpi}\frac{\delta H_1}{\delta\eta}\right). \tag{56.}$$

In passing to disturbed motion, if we retain the equation (54.) as a *definition* of the quantity κ_i, that quantity will no longer be constant, but it will continue to satisfy the inverse relations (55.), and may be called, by analogy, a *varying element* of the motion; and its total differential coefficient, taken with respect to the time, may, by the identical equation (56.), and by the differential equations of disturbed motion (G.), be rigorously expressed as follows:

$$\frac{d\kappa_i}{dt}=\Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta H_2}{\delta\varpi}-\frac{\delta\kappa_i}{\delta\varpi}\frac{\delta H_2}{\delta\eta}\right). \tag{A^1.}$$

14. This result (A^1.) contains the whole theory of the gradual variation of the elements of disturbed motion of a system; but it may receive an advantageous transformation, by the substitution of the expressions (55.) for the variables η_i, ϖ_i as functions of the time and of the elements; since it will thus conduct to a system of $6n$ rigorous and ordinary differential equations of the first order between those varying elements and the time. Expressing, therefore, the quantity H_2 as a function of these latter variables, its variation δH_2 takes this new form,

$$\delta H_2=\Sigma\,.\frac{\delta H_2}{\delta\kappa}\delta\kappa+\frac{\delta H_2}{\delta t}\delta t, \tag{57.}$$

and gives, by comparison with the form (48.), and by (54.),

$$\frac{\delta H_2}{\delta\eta_r}=\Sigma\,.\frac{\delta H_2}{\delta\kappa}\frac{\delta\kappa}{\delta\eta_r};\quad \frac{\delta H_2}{\delta\varpi_r}=\Sigma\,.\frac{\delta H_2}{\delta\kappa}\frac{\delta\kappa}{\delta\varpi_r}; \tag{58.}$$

and thus the general equation (A^1.) is transformed to the following,

$$\frac{d\kappa_i}{dt}=a_{i,1}\frac{\delta H_2}{\delta\kappa_1}+a_{i,2}\frac{\delta H_2}{\delta\kappa_2}+\dots+a_{i,6n}\frac{\delta H_2}{\delta\kappa_{6n}}, \tag{B^1.}$$

in which

$$a_{i,s}=\Sigma\left(\frac{\delta\kappa_i}{\delta\eta}\frac{\delta\kappa_s}{\delta\varpi}-\frac{\delta\kappa_i}{\delta\varpi}\frac{\delta\kappa_s}{\delta\eta}\right): \tag{C^1.}$$

so that it only remains to eliminate the variables η, ϖ from the expressions of these latter coefficients. Now it is remarkable that this elimination removes the symbol t also, and leaves the coefficients $a_{i,s}$ expressed as functions of the elements κ alone, not explicitly involving the time. This general theorem of dynamics, which is, perhaps, a little more extensive than the analogous results discovered by Lagrange and by Poisson,* since it does not limit the disturbing terms in the differential equations of motion to depend on the configuration only, may be investigated in the following way.

---

* [Lagrange, "Sur la théorie générale de la variation des constantes arbitraires," *Mémoires de l'Institut* (1808), p. 288; Poisson, "Sur la variation des constantes arbitraires dans les questions de mécanique," *Journal de l'École Polyt.* Tome VIII (1809), pp. 288, 289. See reference to Lovett, p. 160.]

15. The sign of summation Σ in (C¹.), like the same sign in those other analogous equations in which it has already occurred without an index in this Essay, refers not to the expressed indices, such as here i, s, in the quantity to be summed, but to an index which is not expressed, and which may be here called r; so that if we introduce for greater clearness this variable index and its limits, the expression (C¹.) becomes

$$a_{i,s} = \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta \kappa_i}{\delta \eta_r} \frac{\delta \kappa_s}{\delta \varpi_r} - \frac{\delta \kappa_i}{\delta \varpi_r} \frac{\delta \kappa_s}{\delta \eta_r} \right) : \tag{59.}$$

and its total differential coefficient, taken with respect to the time, may be separated into the two following parts,

$$\left.\begin{aligned} \frac{d}{dt} a_{i,s} = {} & \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta \kappa_i}{\delta \eta_r} \frac{d}{dt} \frac{\delta \kappa_s}{\delta \varpi_r} - \frac{\delta \kappa_s}{\delta \eta_r} \frac{d}{dt} \frac{\delta \kappa_i}{\delta \varpi_r} \right) \\ & + \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta \kappa_s}{\delta \varpi_r} \frac{d}{dt} \frac{\delta \kappa_i}{\delta \eta_r} - \frac{\delta \kappa_i}{\delta \varpi_r} \frac{d}{dt} \frac{\delta \kappa_s}{\delta \eta_r} \right), \end{aligned}\right\} \tag{60.}$$

which we shall proceed to calculate separately, and then to add them together. By the definition (54.), and the differential equations of disturbed motion (G.),

$$\frac{d}{dt} \frac{\delta \kappa_i}{\delta \varpi_r} = \frac{\delta^2 \kappa_i}{\delta t \, \delta \varpi_r} + \Sigma_{(u)\,1}^{\;3n} \left\{ \frac{\delta^2 \kappa_i}{\delta \eta_u \delta \varpi_r} \left(\frac{\delta H_1}{\delta \varpi_u} + \frac{\delta H_2}{\delta \varpi_u} \right. \right. - \frac{\delta^2 \kappa_i}{\delta \varpi_u \delta \varpi_r} \left(\frac{\delta H_1}{\delta \eta_u} + \frac{\delta H_2}{\delta \eta_u} \right. \tag{61.}$$

in which, by the identical equation (56.),

$$\frac{\delta^2 \kappa_i}{\delta t \, \delta \varpi_r} = - \frac{\delta}{\delta \varpi_r} \Sigma_{(u)\,1}^{\;3n} \left(\frac{\delta \kappa_i}{\delta \eta_u} \frac{\delta H_1}{\delta \varpi_u} - \frac{\delta \kappa_i}{\delta \varpi_u} \frac{\delta H_1}{\delta \eta_u} \right); \tag{62.}$$

we have therefore

$$\frac{d}{dt} \frac{\delta \kappa_i}{\delta \varpi_r} = \Sigma_{(u)\,1}^{\;3n} \left(\frac{\delta^2 \kappa_i}{\delta \eta_u \delta \varpi_r} \frac{\delta H_2}{\delta \varpi_u} - \frac{\delta^2 \kappa_i}{\delta \varpi_u \delta \varpi_r} \frac{\delta H_2}{\delta \eta_u} + \frac{\delta \kappa_i}{\delta \varpi_u} \frac{\delta^2 H_1}{\delta \eta_u \delta \varpi_r} - \frac{\delta \kappa_i}{\delta \eta_u} \frac{\delta^2 H_1}{\delta \varpi_u \delta \varpi_r} \right), \tag{63.}$$

and $\frac{d}{dt} \frac{\delta \kappa_s}{\delta \varpi_r}$ may be found from this, by merely changing i to s: so that*

$$\begin{aligned} & \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta \kappa_i}{\delta \eta_r} \frac{d}{dt} \frac{\delta \kappa_s}{\delta \varpi_r} - \frac{\delta \kappa_s}{\delta \eta_r} \frac{d}{dt} \frac{\delta \kappa_i}{\delta \varpi_r} \right) \\ & = \Sigma_{(r,u)\,1,1}^{\;3n,3n} \left\{ \left(\frac{\delta \kappa_s}{\delta \eta_r} \frac{\delta^2 \kappa_i}{\delta \varpi_u \delta \varpi_r} - \frac{\delta \kappa_i}{\delta \eta_r} \frac{\delta^2 \kappa_s}{\delta \varpi_u \delta \varpi_r} \right) \frac{\delta H_2}{\delta \eta_u} + \left(\frac{\delta \kappa_i}{\delta \eta_r} \frac{\delta^2 \kappa_s}{\delta \eta_u \delta \varpi_r} - \frac{\delta \kappa_s}{\delta \eta_r} \frac{\delta^2 \kappa_i}{\delta \eta_u \delta \varpi_r} \right) \frac{\delta H_2}{\delta \varpi_u} \right. \\ & \qquad \left. + \left(\frac{\delta \kappa_i}{\delta \eta_r} \frac{\delta \kappa_s}{\delta \varpi_u} - \frac{\delta \kappa_s}{\delta \eta_r} \frac{\delta \kappa_i}{\delta \varpi_u} \right) \frac{\delta^2 H_1}{\delta \eta_u \delta \varpi_r} + \left(\frac{\delta \kappa_s}{\delta \eta_r} \frac{\delta \kappa_i}{\delta \eta_u} - \frac{\delta \kappa_i}{\delta \eta_r} \frac{\delta \kappa_s}{\delta \eta_u} \right) \frac{\delta^2 H_1}{\delta \varpi_u \delta \varpi_r} \right\} \end{aligned} \tag{64.}$$

and similarly,

$$\begin{aligned} & \Sigma_{(r)\,1}^{\;3n} \left(\frac{\delta \kappa_s}{\delta \varpi_r} \frac{d}{dt} \frac{\delta \kappa_i}{\delta \eta_r} - \frac{\delta \kappa_i}{\delta \varpi_r} \frac{d}{dt} \frac{\delta \kappa_s}{\delta \eta_r} \right) \\ & \quad \Sigma_{(r,u)\,1,1}^{\;3n,3n} \left\{ \left(\frac{\delta \kappa_s}{\delta \varpi_r} \frac{\delta^2 \kappa_i}{\delta \eta_u \delta \eta_r} - \frac{\delta \kappa_i}{\delta \varpi_r} \frac{\delta^2 \kappa_s}{\delta \eta_u \delta \eta_r} \right) \frac{\delta H_2}{\delta \varpi_u} + \left(\frac{\delta \kappa_i}{\delta \varpi_r} \frac{\delta^2 \kappa_s}{\delta \varpi_u \delta \eta_r} - \frac{\delta \kappa_s}{\delta \varpi_r} \frac{\delta^2 \kappa_i}{\delta \varpi_u \delta \eta_r} \right) \frac{\delta H_2}{\delta \eta_u} \right. \\ & \qquad \left. + \left(\frac{\delta \kappa_i}{\delta \varpi_r} \frac{\delta \kappa_s}{\delta \eta_u} - \frac{\delta \kappa_s}{\delta \varpi_r} \frac{\delta \kappa_i}{\delta \eta_u} \right) \frac{\delta^2 H_1}{\delta \varpi_u \delta \eta_r} + \left(\frac{\delta \kappa_s}{\delta \varpi_r} \frac{\delta \kappa_i}{\delta \varpi_u} - \frac{\delta \kappa_i}{\delta \varpi_r} \frac{\delta \kappa_s}{\delta \varpi_u} \right) \frac{\delta^2 H_1}{\delta \eta_u \delta \eta_r} \right\}. \end{aligned} \tag{65.}$$

---

[*] [In (64.) and (65.) the coefficients of $\frac{\delta^2 H_1}{\delta \eta_u \delta \eta_r}$ and of $\frac{\delta^2 H_1}{\delta \varpi_u \delta \varpi_r}$ are easily seen to be zero. Also if we interchange the suffices in the coefficient of $\frac{\delta^2 H_1}{\delta \eta_u \delta \varpi_r}$ in (64.) the term cancels against the corresponding term of (65.).]

Adding, therefore, the two last expressions, and making the reductions which present themselves, we find, by (60.),*

$$\frac{d}{dt}a_{i,s}=\Sigma_{(u)\,1}^{\;3n}\left(A_{i,s}^{(u)}\frac{\delta H_2}{\delta \eta_u}+B_{i,s}^{(u)}\frac{\delta H_2}{\delta \varpi_u}\right), \tag{D^1.}$$

in which

$$\begin{aligned} A_{i,s}^{(u)}&=\Sigma_{(r)\,1}^{\;3n}\left(\frac{\delta\kappa_s}{\delta\eta_r}\frac{\delta^2\kappa_i}{\delta\varpi_u\delta\varpi_r}-\frac{\delta\kappa_i}{\delta\eta_r}\frac{\delta^2\kappa_s}{\delta\varpi_u\delta\varpi_r}+\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\varpi_u\delta\eta_r}-\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\varpi_u\delta\eta_r}\right)\\ B_{i,s}^{(u)}&=\Sigma_{(r)\,1}^{\;3n}\left(\frac{\delta\kappa_s}{\delta\varpi_r}\frac{\delta^2\kappa_i}{\delta\eta_u\delta\eta_r}-\frac{\delta\kappa_i}{\delta\varpi_r}\frac{\delta^2\kappa_s}{\delta\eta_u\delta\eta_r}+\frac{\delta\kappa_i}{\delta\eta_r}\frac{\delta^2\kappa_s}{\delta\eta_u\delta\varpi_r}-\frac{\delta\kappa_s}{\delta\eta_r}\frac{\delta^2\kappa_i}{\delta\eta_u\delta\varpi_r}\right)\end{aligned} \tag{66.}$$

and since this general form (D¹.) for $\frac{d}{dt}a_{i,s}$ contains no term independent of the disturbing quantities $\frac{\delta H_2}{\delta\eta}, \frac{\delta H_2}{\delta\varpi}$, it is easy to infer from it the important consequence already mentioned, namely, that the coefficients $a_{i,s}$, in the differentials (B¹.) of the elements, may be expressed as functions of those elements alone, not explicitly involving the time.†

It is evident also, that these coefficients $a_{i,s}$ have the property

(67.)

and

$$a_{i,i}=0; \tag{68.}$$

the term proportional to $\frac{\delta H_2}{\delta\kappa_i}$ disappears therefore from the expression (B¹.) for $\frac{d\kappa_i}{dt}$; and the term

$$\frac{\delta H_2}{\delta\kappa_i}\,.\,a_{i,s}\,.\,\frac{\delta H_2}{\delta\kappa_s}\quad\text{in}\quad\frac{\delta H_2}{\delta\kappa_i}\frac{d\kappa_i}{dt}$$

destroys the term

$$\frac{\delta H_2}{\delta\kappa_s}\,.\,a_{s,i}\,.\,\frac{\delta H_2}{\delta\kappa_i}\quad\text{in}\quad\frac{\delta H_2}{\delta\kappa_s}\frac{d\kappa_s}{dt},$$

when these terms are added together; we have, therefore,

$$\Sigma\,.\,\frac{\delta H_2}{\delta\kappa}\frac{d\kappa}{dt}=0, \tag{E^1.}$$

or

$$\frac{dH_2}{dt}=\frac{\delta H_2}{\delta t}, \tag{F^1.}$$

that is, in taking the first total differential coefficient of the disturbing expression H_2 with respect to the time, the elements may be treated as constant.

Simplification of the differential equations which determine these gradually varying elements, in any problem of Perturbation; and Integration of the simplified equations by means of certain Functions of Elements.

16. The most natural choice of these elements is that which makes them correspond, in undisturbed motion, to the initial quantities e_i, p_i. These quantities, by the differential equations (H.), may be expressed in undisturbed motion as follows,

---

* [It follows that in the undisturbed motion the a's are constant, which is Poisson's theorem (*Journal de l'École Polyt.* Tome VIII (1809), pp. 281, 282).]

† [This is most easily seen as follows. If $a_{i,s}$, when expressed in terms of the κ's, contains t explicitly, it would not remain constant during the undisturbed motion, which would contradict (D¹.).]

$$e_i = \eta_i - \int_0^t \frac{\delta H_1}{\delta \varpi_i} dt, \quad p_i = \varpi_i + \int_0^t \frac{\delta H_1}{\delta \eta_i} dt; \tag{69.}$$

and if we suppose them found, by elimination, under the forms

$$\left.\begin{aligned} e_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ p_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \end{aligned}\right\} \tag{70.}$$

it is easy to see that the following equations must be rigorously and identically true,* for all values of η_i, ϖ_i,

$$\left.\begin{aligned} 0 &= \Phi_i(0, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ 0 &= \Psi_i(0, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}). \end{aligned}\right\} \tag{71.}$$

When, therefore, in passing to disturbed motion, we establish the equations of definition,

$$\left.\begin{aligned} \kappa_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \\ \lambda_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \ldots \eta_{3n}, \varpi_1, \varpi_2, \ldots \varpi_{3n}), \end{aligned}\right\} \tag{72.}$$

introducing $6n$ varying elements κ_i, λ_i, of which the set λ_i would have been represented in our recent notation as follows:

$$\lambda_i = \kappa_{3n+i}; \tag{73.}$$

we see that all the partial differential coefficients of the forms $\frac{\delta \kappa_i}{\delta \eta_r}, \frac{\delta \kappa_i}{\delta \varpi_r}, \frac{\delta \lambda_i}{\delta \eta_r}, \frac{\delta \lambda_i}{\delta \varpi_r}$, vanish when $t = 0$, except the following:

$$\frac{\delta \kappa_i}{\delta \eta_i} = 1, \quad \frac{\delta \lambda_i}{\delta \varpi_i} = 1; \tag{74.}$$

and, therefore, that when t is made $= 0$, in the coefficients $a_{i,s}$, (59.), all those coefficients vanish, except the following:

$$a_{r,3n+r} = 1, \quad a_{3n+r,r} = -1. \tag{75.}$$

But it has been proved that these coefficients $a_{i,s}$, when expressed as functions of the elements, do not contain the time explicitly; and the supposition $t = 0$ introduces no relation between those $6n$ elements κ_i, λ_i, which still remain independent: the coefficients $a_{i,s}$, therefore, could not acquire the values 1, 0, -1, by the supposition $t = 0$, unless they had those values constantly, and independently of that supposition. The differential equations of the forms (B^1.) may therefore be expressed, for the present system of varying elements, in the following simpler way:

$$\frac{d\kappa_i}{dt} = \frac{\delta H_2}{\delta \lambda_i}; \quad \frac{d\lambda_i}{dt} = -\frac{\delta H_2}{\delta \kappa_i}; \tag{G^1.}$$

and an easy verification of these expressions is offered by the formula (E^1.), which takes now this form,

$$\Sigma \left(\frac{\delta H_2}{\delta \kappa} \frac{d\kappa}{dt} + \frac{\delta H_2}{\delta \lambda} \frac{d\lambda}{dt} \right) = 0. \tag{H^1.}$$

---

* [Putting $t = 0$ in (70.) we have $\Phi_i(0, e_1, \ldots e_{3n}, p_1, \ldots p_{3n}) = 0$ for all values of the e's and p's. Hence (71.) follows.]

17. The initial values of the varying elements κ_i, λ_i are evidently e_i, p_i, by the definitions (72.), and by the identical equations (71.); the problem of integrating rigorously the equations of disturbed motion (G.), between the variables η_i, ϖ_i and the time, or of determining these variables as functions of the time and of their own initial values e_i, p_i, is therefore rigorously transformed into the problem of integrating the equations (G^1.), or of determining the $6n$ elements κ_i, λ_i as functions of the time and of the same initial values. The chief advantage of this transformation is, that if the perturbations be small, the new variables (namely, the elements,) alter but little: and that, since the new differential equations are of the same form as the old, they may be integrated by a similar method. Considering, therefore, the definite integral

$$E=\int_0^t \left(\Sigma\,.\,\lambda\frac{\delta H_2}{\delta\lambda}-H_2\right)dt, \tag{76.}$$

as a function of the time and of the $6n$ quantities $\kappa_1, \kappa_2, \ldots \kappa_{3n}, e_1, e_2, \ldots e_{3n}$, and observing that its variation, taken with respect to the latter quantities, may be shown by a process similar to that of the fourth number of this Essay to be

$$\delta E=\Sigma\,(\lambda\,\delta\kappa-p\,\delta e), \tag{I^1.}$$

we find that the rigorous integrals of the differential equations (G^1.) may be expressed in the following manner:

$$\lambda_i=\frac{\delta E}{\delta\kappa_i},\quad p_i=-\frac{\delta E}{\delta e_i}, \tag{K^1.}$$

in which there enters only one unknown *function of elements* E, to the search and study of which single function the problem of perturbation is reduced by this new method.

We might also have put

$$C=\int_0^t \left(-\Sigma\,.\,\kappa\frac{\delta H_2}{\delta\kappa}+H_2\right)dt, \tag{77.}$$

and have considered this definite integral C as a function of the time and of the $6n$ quantities λ_i, p_i; and then we should have found the following other forms for the integrals of the differential equations of varying elements,

$$\kappa_i=+\frac{\delta C}{\delta\lambda_i},\quad e_i=-\frac{\delta C}{\delta p_i}. \tag{L^1.}$$

And each of these *functions of elements*, C and E, must satisfy a certain partial differential equation, analogous to the first equation of each pair mentioned in the sixth number of this Essay, and deduced on similar principles.*

18. Thus, it is evident, by the form of the function E, and by the equations (K^1.), (G^1.), and (76.), that the partial differential coefficient of this function, taken with respect to the time, is

$$\frac{\delta E}{\delta t}=\frac{dE}{dt}-\Sigma\,.\,\frac{\delta E}{\delta\kappa}\frac{d\kappa}{dt}=-H_2; \tag{M^1.}$$

and therefore that if we separate this function E into any two parts

$$E_1+E_2=E, \tag{N^1.}$$

---

* [C and E are the generating functions for the contact transformations from p, e to λ, κ.]

and if, for greater clearness, we put the expression H_2 under the form

$$H_2 = H_2(t, \kappa_1, \kappa_2, \dots \kappa_{3n}, \lambda_1, \lambda_2, \dots \lambda_{3n}), \tag{O^1.}$$

we shall have rigorously the partial differential equation

$$0 = \frac{\delta E_1}{\delta t} + \frac{\delta E_2}{\delta t} + H_2\left(t, \kappa_1, \dots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1} + \frac{\delta E_2}{\delta \kappa_1}, \dots \frac{\delta E_1}{\delta \kappa_{3n}} + \frac{\delta E_2}{\delta \kappa_{3n}}\right): \tag{P^1.}$$

which gives, approximately, by (G^1.) and (K^1.), when the part E_2 is small, and when we neglect the squares and products of its partial differential coefficients,*

$$0 = \frac{dE_2}{dt} + \frac{\delta E_1}{\delta t} + H_2\left(t, \kappa_1, \dots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1}, \dots \frac{\delta E_1}{\delta \kappa_{3n}}\right). \tag{Q^1.}$$

Hence, in the same order of approximation, if the part E_1, like the whole function E, be chosen so as to vanish with the time, we shall have

$$E_2 = -\int_0^t \left\{\frac{\delta E_1}{\delta t} + H_2\left(t, \kappa_1, \dots \kappa_{3n}, \frac{\delta E_1}{\delta \kappa_1}, \dots \frac{\delta E_1}{\delta \kappa_{3n}}\right)\right\} dt: \tag{R^1.}$$

and thus a first approximate expression E_1 can be successively and indefinitely corrected.

Again, by (L^1.) and (G^1.), and by the definition (77.),

$$\frac{\delta C}{\delta t} = \frac{dC}{dt} - \Sigma \,.\, \frac{\delta C}{\delta \lambda}\frac{d\lambda}{dt} = H_2; \tag{S^1.}$$

the function C must therefore satisfy rigorously the partial differential equation,

$$\frac{\delta C}{\delta t} = H_2\left(t, \frac{\delta C}{\delta \lambda_1}, \dots \frac{\delta C}{\delta \lambda_{3n}}, \lambda_1, \dots \lambda_{3n}\right): \tag{T^1.}$$

and if we put

$$C = C_1 + C_2, \tag{U^1.}$$

and suppose that the part C_2 is small, then the rigorous equation

$$\frac{\delta C_1}{\delta t} + \frac{\delta C_2}{\delta t} = H_2\left(t, \frac{\delta C_1}{\delta \lambda_1} + \frac{\delta C_2}{\delta \lambda_1}, \dots \frac{\delta C_1}{\delta \lambda_{3n}} + \frac{\delta C_2}{\delta \lambda_{3n}}, \lambda_1, \dots \lambda_{3n}\right) \tag{V^1.}$$

becomes approximately, by (G^1.) and (L^1.),

$$\frac{dC_2}{dt} = -\frac{\delta C_1}{\delta t} + H_2\left(t, \frac{\delta C_1}{\delta \lambda_1}, \dots \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \dots \lambda_{3n}\right), \tag{W^1.}$$

and gives by integration,

$$C_2 = \int_0^t \left\{-\frac{\delta C_1}{\delta t} + H_2\left(t, \frac{\delta C_1}{\delta \lambda_1}, \dots \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \dots \lambda_{3n}\right)\right\} dt, \tag{X^1.}$$

the parts C_1 and C_2 being supposed to vanish separately when $t = 0$, like the whole function of elements C.

And to obtain such a first approximation, E_1 or C_1, to either of these two functions of elements E, C, we may change, in the definitions (76.), (77.), the varying elements κ, λ, to their initial values e, p, and then eliminate one set of these initial values by the corresponding set of the following approximate equations, deduced from the formulæ ($G^1.$):

---

* [The deduction is the same as that of (D.), p. 170.]

$$\kappa_i = e_i + \int_0^t \frac{\delta H_2}{\delta p_i} dt; \qquad (Y^1.)$$

and

$$\lambda_i = p_i - \int_0^t \frac{\delta H_2}{\delta e_i} dt. \qquad (Z^1.)$$

It is easy also to see that these two functions of elements C and E are connected with each other,* and with the disturbing function S_2, so that the form of any one may be deduced from that of any other, when the function S_1 of undisturbed motion is known.

Analogous formulæ for the motion of a Single Point.

19. Our general method in dynamics, though intended chiefly for the study of attracting and repelling systems, is not confined to such, but may be used in all questions to which the law of living forces applies. And all the analysis of this Essay, but especially the theory of perturbations, may usefully be illustrated by the following analogous reasonings and results respecting the motion of a single point.

Imagine then such a point, having for its three rectangular coordinates x, y, z, and moving in an orbit determined by three ordinary differential equations of the second order of forms analogous to the equations (2.), namely,

$$x'' = \frac{\delta U}{\delta x}; \quad y'' = \frac{\delta U}{\delta y}; \quad z'' = \frac{\delta U}{\delta z}, \qquad (78.)$$

U being any given function of the coordinates not expressly involving the time: and let us establish the following definition, analogous to (4.),

$$T = \tfrac{1}{2}(x'^2 + y'^2 + z'^2), \qquad (79.)$$

x', y', z' being the first, and x'', y'', z'' being the second differential coefficients of the coordinates, considered as functions of the time t. If we express, for greater generality or facility, the rectangular coordinates x, y, z as functions of three other marks of position η_1, η_2, η_3, T will become a homogeneous function of the second dimension of their first differential coefficients $\eta_1', \eta_2', \eta_3'$ taken with respect to the time; and if we put, for abridgement,

$$\varpi_1 = \frac{\delta T}{\delta \eta_1'}, \quad \varpi_2 = \frac{\delta T}{\delta \eta_2'}, \quad \varpi_3 = \frac{\delta T}{\delta \eta_3'}, \qquad (80.)$$

T may be considered also as a function of the form

$$T = F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3), \qquad (81.)$$

which will be homogeneous of the second dimension with respect to $\varpi_1, \varpi_2, \varpi_3$. We may also put, for abridgement,

$$F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3) - U(\eta_1, \eta_2, \eta_3) = H; \qquad (82.)$$

and then, instead of the three differential equations of the second order (78.), we may employ the six following of the first order, analogous to the equations (A.), and obtained by a similar reasoning,

---

* [$C = \Sigma(\kappa\lambda - ep) - E$.]

$$\frac{d\eta_1}{dt} = +\frac{\delta H}{\delta \varpi_1}, \quad \frac{d\eta_2}{dt} = +\frac{\delta H}{\delta \varpi_2}, \quad \frac{d\eta_3}{dt} = +\frac{\delta H}{\delta \varpi_3},$$
$$\frac{d\varpi_1}{dt} = -\frac{\delta H}{\delta \eta_1}, \quad \frac{d\varpi_2}{dt} = -\frac{\delta H}{\delta \eta_2}, \quad \frac{d\varpi_3}{dt} = -\frac{\delta H}{\delta \eta_3}. \tag{83.}$$

20. The rigorous integrals of these six differential equations may be expressed under the following forms, analogous to (B.),

$$\varpi_1 = \frac{\delta S}{\delta \eta_1}, \quad \varpi_2 = \frac{\delta S}{\delta \eta_2}, \quad \varpi_3 = \frac{\delta S}{\delta \eta_3},$$
$$p_1 = -\frac{\delta S}{\delta e_1}, \quad p_2 = -\frac{\delta S}{\delta e_2}, \quad p_3 = -\frac{\delta S}{\delta e_3}, \tag{84.}$$

in which $e_1, e_2, e_3, p_1, p_2, p_3$ are the initial values, or values at the time 0, of $\eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3$; and S is the definite integral

$$S = \int_0^t \left(\varpi_1 \frac{\delta H}{\delta \varpi_1} + \varpi_2 \frac{\delta H}{\delta \varpi_2} + \varpi_3 \frac{\delta H}{\delta \varpi_3} - H \right) dt, \tag{85.}$$

considered as a function of $\eta_1, \eta_2, \eta_3, e_1, e_2, e_3$ and t. The quantity H does not change in the course of the motion, and the function S must satisfy the following pair of partial differential equations of the first order, analogous to the pair (C.),

$$\left.\begin{aligned} &\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \frac{\delta S}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right) = U(\eta_1, \eta_2, \eta_3); \\ &\frac{\delta S}{\delta t} + F\left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \frac{\delta S}{\delta e_3}, e_1, e_2, e_3\right) = U(e_1, e_2, e_3). \end{aligned}\right\} \tag{86.}$$

This important function S, which may be called the *principal function* of the motion, may hence be rigorously expressed under the following form, obtained by reasonings analogous to those of the seventh number of this Essay:

$$\left.\begin{aligned} S = S_1 &+ \int_0^t \left\{ -\frac{\delta S_1}{\delta t} + U(\eta_1, \eta_2, \eta_3) - F\left(\frac{\delta S_1}{\delta \eta_1}, \frac{\delta S_1}{\delta \eta_2}, \frac{\delta S_1}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right)\right\} dt \\ &+ \int_0^t F\left(\frac{\delta S}{\delta \eta_1} - \frac{\delta S_1}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2} - \frac{\delta S_1}{\delta \eta_2}, \frac{\delta S}{\delta \eta_3} - \frac{\delta S_1}{\delta \eta_3}, \eta_1, \eta_2, \eta_3\right) dt; \end{aligned}\right\} \tag{87.}$$

S_1 being any arbitrary function of the same quantities $\eta_1, \eta_2, \eta_3, e_1, e_2, e_3, t$, so chosen as to vanish with the time. And if this arbitrary function S_1 be chosen so as to be a first approximate value of the principal function S, we may neglect, in a second approximation, the second definite integral in (87.).

21. A first approximation of this kind can be obtained, whenever, by separating the expression H, (82.), into a predominant and a smaller part, H_1 and H_2, and by neglecting the part H_2, we have changed the differential equations (83.) to others, namely,

$$\left.\begin{aligned} &\frac{d\eta_1}{dt} = \frac{\delta H_1}{\delta \varpi_1}, \quad \frac{d\eta_2}{dt} = \frac{\delta H_1}{\delta \varpi_2}, \quad \frac{d\eta_3}{dt} = \frac{\delta H_1}{\delta \varpi_3}, \\ &\frac{d\varpi_1}{dt} = -\frac{\delta H_1}{\delta \eta_1}, \quad \frac{d\varpi_2}{dt} = -\frac{\delta H_1}{\delta \eta_2}, \quad \frac{d\varpi_3}{dt} = -\frac{\delta H_1}{\delta \eta_3}, \end{aligned}\right\} \tag{88.}$$

and have succeeded in integrating rigorously these simplified equations, belonging to a simpler motion, which may be called the *undisturbed motion* of the point. For the principal function of such undisturbed motion, namely, the definite integral

$$S_1 = \int_0^t \left(\varpi_1 \frac{\delta H_1}{\delta \varpi_1} + \varpi_2 \frac{\delta H_1}{\delta \varpi_2} + \varpi_3 \frac{\delta H_1}{\delta \varpi_3} - H_1 \right) dt, \tag{89.}$$

considered as a function of $\eta_1, \eta_2, \eta_3, e_1, e_2, e_3, t$, will then be an approximate value for the original function of disturbed motion S, which original function corresponds to the more complex differential equations,

$$\left.\begin{aligned} &\frac{d\eta_1}{dt} = \frac{\delta H_1}{\delta \varpi_1} + \frac{\delta H_2}{\delta \varpi_1}, \quad &&\frac{d\eta_2}{dt} = \frac{\delta H_1}{\delta \varpi_2} + \frac{\delta H_2}{\delta \varpi_2}, \quad &&\frac{d\eta_3}{dt} = \frac{\delta H_1}{\delta \varpi_3} + \frac{\delta H_2}{\delta \varpi_3}, \\ &\frac{d\varpi_1}{dt} = -\frac{\delta H_1}{\delta \eta_1} - \frac{\delta H_2}{\delta \eta_1}, \quad &&\frac{d\varpi_2}{dt} = -\frac{\delta H_1}{\delta \eta_2} - \frac{\delta H_2}{\delta \eta_2}, \quad &&\frac{d\varpi_3}{dt} = -\frac{\delta H_1}{\delta \eta_3} - \frac{\delta H_2}{\delta \eta_3}. \end{aligned}\right\} \tag{90.}$$

The function S_1 of undisturbed motion must satisfy a pair of partial differential equations of the first order, analogous to the pair (86.); and the integrals of undisturbed motion may be represented thus,

$$\begin{aligned} &\varpi_1 = \frac{\delta S_1}{\delta \eta_1}, \quad &&\varpi_2 = \frac{\delta S_1}{\delta \eta_2}, \quad &&\varpi_3 = \frac{\delta S_1}{\delta \eta_3}, \\ &p_1 = -\frac{\delta S_1}{\delta e_1}, \quad &&p_2 = -\frac{\delta S_1}{\delta e_2}, \quad &&p_3 = -\frac{\delta S_1}{\delta e_3}: \end{aligned} \tag{91.}$$

while the integrals of disturbed motion may be expressed with equal rigour under the following analogous forms,

$$\left.\begin{aligned} &\varpi_1 = \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1}, \quad &&\varpi_2 = \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2}, \quad &&\varpi_3 = \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3}, \\ &p_1 = -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1}, \quad &&p_2 = -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2}, \quad &&p_3 = -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3}, \end{aligned}\right\} \tag{92.}$$

if S_2 denote the rigorous correction of S_1, or the disturbing part of the whole principal function S. And by the foregoing general theory of approximation, this disturbing part or function S_2 may be approximately represented by the definite integral (T.),

$$S_2 = -\int_0^t H_2 \, dt; \tag{93.}$$

in calculating which definite integral the equations (91.) may be employed.

22. If the integrals of undisturbed motion (91.) have given

$$\left.\begin{aligned} \eta_1 &= \phi_1(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_2 &= \phi_2(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_3 &= \phi_3(t, e_1, e_2, e_3, p_1, p_2, p_3), \end{aligned}\right\} \tag{94.}$$

and

$$\left.\begin{aligned} \varpi_1 &= \psi_1(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \varpi_2 &= \psi_2(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \varpi_3 &= \psi_3(t, e_1, e_2, e_3, p_1, p_2, p_3), \end{aligned}\right\} \tag{95.}$$

then the integrals of disturbed motion (92.) may be rigorously transformed as follows,

$$\left.\begin{aligned}\eta_1&=\phi_1\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2},\ \ p_3+\frac{\delta S_2}{\delta e_3}\right),\\ \eta_2&=\phi_2\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2}\ \ \ p_3+\frac{\delta S_2}{\delta e_3}\right),\\ \eta_3&=\phi_3\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2},\ \ p_3+\frac{\delta S_2}{\delta e_3}\right),\end{aligned}\right\}\qquad(96.)$$

and

$$\begin{aligned}\varpi_1&=\frac{\delta S_2}{\delta\eta_1}+\psi_1\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2},\ \ p_3+\frac{\delta S_2}{\delta e_3}\right.\\ \varpi\ &=\frac{\delta S_2}{\delta\eta_2}+\psi_2\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2},\ \ p_3+\frac{\delta S_2}{\delta e_3}\right),\\ \varpi_3&=\frac{\delta S_2}{\delta\eta_3}+\psi_3\left(t,\ e_1,\ e_2,\ e_3,\ \ p_1+\frac{\delta S_2}{\delta e_1},\ \ p_2+\frac{\delta S_2}{\delta e_2},\ \ p_3+\frac{\delta S_2}{\delta e_3}\right),\end{aligned}\qquad(97.)$$

S_2 being here the rigorous disturbing function. And the perturbations of position, at any time t, may be approximately expressed by the following formula,

$$\left.\begin{aligned}\Delta\eta_1=\frac{\delta\eta_1}{\delta e_1}\int_0^t\frac{\delta H_2}{\delta p_1}dt+\frac{\delta\eta_1}{\delta e_2}\int_0^t\frac{\delta H_2}{\delta p_2}dt+\frac{\delta\eta_1}{\delta e_3}\int_0^t\frac{\delta H_2}{\delta p_3}dt\\ -\frac{\delta\eta_1}{\delta p_1}\int_0^t\frac{\delta H_2}{\delta e_1}dt-\frac{\delta\eta_1}{\delta p_2}\int_0^t\frac{\delta H_2}{\delta e_2}dt-\frac{\delta\eta_1}{\delta p_3}\int_0^t\frac{\delta H_2}{\delta e_3}dt,\end{aligned}\right\}\qquad(98.)$$

together with two similar formulæ for the perturbations of the two other coordinates, or marks of position η_2, η_3. In these formulæ, the coordinates and H_2 are supposed to be expressed, by the theory of undisturbed motion, as functions of the time t, and of the constants $e_1, e_2, e_3, p_1, p_2, p_3$.

23. Again, if the integrals of undisturbed motion have given, by elimination, expressions for these constants, of the forms

$$\left.\begin{aligned}e_1&=\eta_1+\Phi_1(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ e_2&=\eta_2+\Phi_2(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ e_3&=\eta_3+\Phi_3(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\end{aligned}\right\}\qquad(99.)$$

and

$$\left.\begin{aligned}p_1&=\varpi_1+\Psi_1(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ p_2&=\varpi_2+\Psi_2(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ p_3&=\varpi_3+\Psi_3(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3);\end{aligned}\right\}\qquad(100.)$$

and if, for disturbed motion, we establish the definitions

$$\left.\begin{aligned}\kappa_1&=\eta_1+\Phi_1(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ \kappa_2&=\eta_2+\Phi_2(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ \kappa_3&=\eta_3+\Phi_3(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\end{aligned}\right\}\qquad(101.)$$

and

$$\left.\begin{aligned}\lambda_1&=\varpi_1+\Psi_1(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ \lambda_2&=\varpi_2+\Psi_2(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3),\\ \lambda_3&=\varpi_3+\Psi_3(t,\ \eta_1,\ \eta_2,\ \eta_3,\ \varpi_1,\ \varpi_2,\ \varpi_3);\end{aligned}\right\}\qquad(102.)$$

we shall have, for such disturbed motion, the following rigorous equations, of the forms (94.) and (95.),

$$\left.\begin{aligned}\eta_1&=\phi_1(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),\\ \eta_2&=\phi_2(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),\\ \eta_3&=\phi_3(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),\end{aligned}\right\}\qquad(103.)$$

and

$$\left.\begin{aligned}\varpi_1&=\psi_1(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),\\ \varpi_2&=\psi_2(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3),\\ \varpi_3&=\psi_3(t,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ \lambda_1,\ \lambda_2,\ \lambda_3);\end{aligned}\right\}\qquad(104.)$$

and may call the quantities $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$ the 6 *varying elements* of the motion. To determine these six varying elements, we may employ the six following rigorous equations in ordinary differentials of the first order, in which H_2 is supposed to have been expressed by (103.) and (104.) as a function of the elements and of the time:

$$\begin{aligned}&\frac{d\kappa_1}{dt}=\frac{\delta H_2}{\delta\lambda_1},\quad \frac{d\kappa_2}{dt}=\frac{\delta H_2}{\delta\lambda_2}\quad \frac{d\kappa_3}{dt}=\frac{\delta H_2}{\delta\lambda_3}\\ &\frac{d\lambda_1}{dt}=-\frac{\delta H_2}{\delta\kappa_1},\ \frac{d\lambda_2}{dt}=-\frac{\delta H_2}{\delta\kappa_2},\ \frac{d\lambda_3}{dt}=-\frac{\delta H_2}{\delta\kappa_3}:\end{aligned}\qquad(105.)$$

and the rigorous integrals of these 6 equations may be expressed in the following manner,

$$\begin{aligned}&\lambda_1=\frac{\delta E}{\delta\kappa_1},\qquad \lambda_2=\frac{\delta E}{\delta\kappa_2},\qquad \lambda_3=\frac{\delta E}{\delta\kappa_3},\\ &p_1=\quad\frac{\delta E}{\delta e_1}\quad p_2=-\frac{\delta E}{\delta e_2},\ \ p_3=-\frac{\delta E}{\delta e_3},\end{aligned}\qquad(106.)$$

the constants $e_1, e_2, e_3, p_1, p_2, p_3$ retaining their recent meanings, and being therefore the initial values of the elements $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$; while the function E, which may be called the *function of elements*, because its form determines the laws of their variations, is the definite integral

$$E=\int_0^t\left(\lambda_1\frac{\delta H_2}{\delta\lambda_1}+\lambda_2\frac{\delta H_2}{\delta\lambda_2}+\lambda_3\frac{\delta H_2}{\delta\lambda_3}-H_2\right)dt,\qquad(107.)$$

considered as depending on $\kappa_1, \kappa_2, \kappa_3, e_1, e_2, e_3$ and t. The integrals of the equations (105.) may also be expressed in this other way,

$$\begin{aligned}&\kappa_1=+\frac{\delta C}{\delta\lambda_1},\quad \kappa_2=+\frac{\delta C}{\delta\lambda_2},\quad \kappa_3=+\frac{\delta C}{\delta\lambda_3},\\ &e_1=-\frac{\delta C}{\delta p_1},\quad e_2=-\frac{\delta C}{\delta p_2},\quad e_3=-\frac{\delta C}{\delta p_3},\end{aligned}\qquad(108.)$$

C being the definite integral

$$C=-\int_0^t\left(\kappa_1\frac{\delta H_2}{\delta\kappa_1}+\kappa_2\frac{\delta H_2}{\delta\kappa_2}+\kappa_3\frac{\delta H_2}{\delta\kappa_3}-H_2\right)dt,\qquad(109.)$$

regarded as a function of $\lambda_1, \lambda_2, \lambda_3, p_1, p_2, p_3$ and t: and it is easy to prove that each of these two *functions of elements*, C and E, must satisfy a partial differential equation of the first order,*

---

* [The partial differential equations are

$$-\frac{\partial C}{\partial t}+H_2\left(t,\ \frac{\partial C}{\partial\lambda},\ \lambda\right)=0,\qquad \frac{\partial E}{\partial t}+H_2\left(t,\kappa,\ \frac{\partial E}{\partial\kappa}\right)=0.]$$

which can be previously assigned, and which may assist in discovering the forms of these two functions, and especially in improving an approximate expression for either. All these results for the motion of a single point are analogous to the results already deduced in this Essay, for an attracting or repelling system.

Mathematical Example, suggested by the motion of Projectiles.

24. If the three marks of position η_1, η_2, η_3 of the moving point are the rectangular coordinates themselves, and if the function U has the form

$$U = -g\eta_3 - \tfrac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \tag{110.}$$

g, μ, ν being constants; then the expression

$$H = \tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g\eta_3 + \tfrac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\} \tag{111.}$$

is that which must be substituted in the general forms (83.), in order to form the 6 differential equations of motion of the first order, namely,

$$\begin{aligned} \frac{d\eta_1}{dt} &= \varpi_1, & \frac{d\eta_2}{dt} &= \varpi_2, & \frac{d\eta_3}{dt} &= \varpi_3, \\ \frac{d\varpi_1}{dt} &= -\mu^2\eta_1, & \frac{d\varpi_2}{dt} &= -\mu^2\eta_2, & \frac{d\varpi_3}{dt} &= -g - \nu^2\eta_3. \end{aligned} \tag{112.}$$

These differential equations have for their rigorous integrals the six following,

$$\begin{aligned} \eta_1 &= e_1 \cos \mu t + \frac{p_1}{\mu} \sin \mu t, \\ \eta_2 &= e_2 \cos \mu t + \frac{p_2}{\mu} \sin \mu t, \\ \eta_3 &= e_3 \cos \nu t + \frac{p_3}{\nu} \sin \nu t - \frac{g}{\nu^2} \operatorname{vers} \nu t, \end{aligned} \tag{113.}$$

and

$$\begin{aligned} \varpi_1 &= p_1 \cos \mu t - \mu e_1 \sin \mu t, \\ \varpi_2 &= p_2 \cos \mu t - \mu e_2 \sin \mu t, \\ \varpi_3 &= p_3 \cos \nu t - \left(\nu e_3 + \frac{g}{\nu}\right) \sin \nu t; \end{aligned} \tag{114.}$$

$e_1, e_2, e_3, p_1, p_2, p_3$ being still the initial values of $\eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3$.

Employing these rigorous integral equations to calculate the function S, that is, by (85.) and (110.), (111.), the definite integral

$$S = \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} + U\right) dt, \tag{115.}$$

we find

$$\begin{aligned} \tfrac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) &= \tfrac{1}{4}\left\{p_1^2 + p_2^2 + p_3^2 + \mu^2(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\} \\ &+ \tfrac{1}{4}\{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\} \cos 2\mu t - \tfrac{1}{2}\mu(e_1 p_1 + e_2 p_2) \sin 2\mu t \\ &+ \tfrac{1}{4}\left\{p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\} \cos 2\nu t - \tfrac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right) p_3 \sin 2\nu t, \end{aligned} \tag{116.}$$

and

$$\begin{aligned} U = \frac{g^2}{2\nu^2} - \tfrac{1}{4}\left\{p_1^2 + p_2^2 + p_3^2 + \mu^2\,(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\} & \\ + \tfrac{1}{4}\{p_1^2 + p_2^2 - \mu^2\,(e_1^2 + e_2^2)\}\cos 2\mu t - \tfrac{1}{2}\mu\,(e_1 p_1 + e_2 p_2)\sin 2\mu t & \qquad (117.) \\ + \tfrac{1}{4}\left\{p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\cos 2\nu t - \tfrac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right)p_3 \sin 2\nu t; & \end{aligned}$$

and therefore,

$$\left.\begin{aligned} S = \frac{g^2 t}{2\nu^2} + \{p_1^2 + p_2^2 - \mu^2\,(e_1^2 + e_2^2)\}\,\frac{\sin 2\mu t}{4\mu} - \tfrac{1}{2}\,(e_1 p_1 + e_2 p_2)\,\text{vers}\,2\mu t \\ + \left. p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2\right\}\frac{\sin 2\nu t}{4\nu} - \tfrac{1}{2}p_3\left(e_3 + \frac{g}{\nu^2}\right)\text{vers}\,2\nu t. \end{aligned}\right\} \qquad (118.)$$

In order, however, to express this function S, as supposed by our general method, in terms of the final and initial coordinates and of the time, we must employ the analogous expressions for the constants p_1, p_2, p_3, deduced from the integrals (113.), namely, the following:

$$\begin{aligned} p_1 &= \frac{\mu\eta_1 - \mu e_1 \cos\mu t}{\sin\mu t}, \\ p_2 &= \frac{\mu\eta_2 - \mu e_2 \cos\mu t}{\sin\mu t}, \\ p_3 &= \frac{\nu\eta_3 + \dfrac{g}{\nu} - \left(\nu e_3 + \dfrac{g}{\nu}\right)\cos\nu t}{\sin\nu t} \end{aligned} \qquad (119.)$$

and then we find

$$\begin{aligned} S = \frac{g^2 t}{2\nu^2} + \frac{\mu}{2}\cdot\frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2}{\tan\mu t} + \frac{\nu}{2}\cdot\frac{(\eta_3 - e_3)^2}{\tan\nu t} & \\ - \mu\,(\eta_1 e_1 + \eta_2 e_2)\tan\frac{\mu t}{2} - \nu\left(\eta_3 + \frac{g}{\nu^2}\right)\left(e_3 + \frac{g}{\nu^2}\right)\tan\frac{\nu t}{2}. & \end{aligned} \qquad (120.)$$

This *principal function* S satisfies the following pair of partial differential equations of the first order, of the kind (86.),

$$\left.\begin{aligned} \frac{\delta S}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S}{\delta \eta_1}\right)^2 + \left(\frac{\delta S}{\delta \eta_2}\right)^2 + \left(\frac{\delta S}{\delta \eta_3}\right)^2\right\} &= -g\eta_3 - \frac{\mu^2}{2}\,(\eta_1^2 + \eta_2^2) - \frac{\nu^2}{2}\,\eta_3^2, \\ \frac{\delta S}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S}{\delta e_1}\right)^2 + \left(\frac{\delta S}{\delta e_2}\right)^2 + \left(\frac{\delta S}{\delta e_3}\right)^2\right\} &= -g e_3 - \frac{\mu^2}{2}\,(e_1^2 + e_2^2) - \frac{\nu^2}{2}\,e_3^2; \end{aligned}\right\} \qquad (121.)$$

and *if its form had been previously found*, by the help of this pair, or in any other way, *the integrals of the equations of motion might* (*by our general method*) *have been deduced from it*, under the forms,

$$\begin{aligned} \varpi_1 &= \frac{\delta S}{\delta \eta_1} = \mu\,(\eta_1 - e_1)\,\text{cotan}\,\mu t - \mu e_1 \tan\frac{\mu t}{2}, \\ \varpi_2 &= \frac{\delta S}{\delta \eta_2} = \mu\,(\eta_2 - e_2)\,\text{cotan}\,\mu t - \mu e_2 \tan\frac{\mu t}{2}, \\ \varpi_3 &= \frac{\delta S}{\delta \eta_3} = \nu\,(\eta_3 - e_3)\,\text{cotan}\,\nu t - \left(\nu e_3 + \frac{g}{\nu}\right)\tan\frac{\nu t}{2}, \end{aligned} \qquad (122.)$$

and

$$\left.\begin{aligned} p_1 &= -\frac{\delta S}{\delta e_1} = \mu\,(\eta_1 - e_1)\,\text{cotan}\,\mu t + \mu\eta_1 \tan\frac{\mu t}{2}, \\ p_2 &= -\frac{\delta S}{\delta e_2} = \mu\,(\eta_2 - e_2)\,\text{cotan}\,\mu t + \mu\eta_2 \tan\frac{\mu t}{2}, \\ p_3 &= -\frac{\delta S}{\delta e_3} = \nu\,(\eta_3 - e_3)\,\text{cotan}\,\nu t + \left(\nu\eta_3 + \frac{g}{\nu}\right)\tan\frac{\nu t}{2}: \end{aligned}\right| \tag{123.}$$

the last of these two sets of equations coinciding with the set (119.), or (113.), and conducting, when combined with the first set, (122.), to the other former set of integrals, (114.).

25. Suppose now, to illustrate the theory of perturbation, that the constants μ, ν are small, and that, after separating the expression (111.) for H into the two parts,

$$H_1 = \tfrac{1}{2}\,(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g\eta_3, \tag{124.}$$

and

$$H_2 = \tfrac{1}{2}\,\{\mu^2\,(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \tag{125.}$$

we suppress at first the small part H_2, and so form, by (88.), these other and simpler differential equations of a motion which we shall call *undisturbed*:

$$\begin{aligned} \frac{d\eta_1}{dt} &= \varpi_1, & \frac{d\eta_2}{dt} &= \varpi_2, & \frac{d\eta_3}{dt} &= \varpi_3, \\ \frac{d\varpi_1}{dt} &= 0, & \frac{d\varpi_2}{dt} &= 0, & \frac{d\varpi_3}{dt} &= -g. \end{aligned} \tag{126.}$$

These new equations have for their rigorous integrals, of the forms (94.) and (95.),

$$\eta_1 = e_1 + p_1 t, \quad \eta_2 = e_2 + p_2 t, \quad \eta_3 = e_3 + p_3 t - \tfrac{1}{2}gt^2, \tag{127.}$$

and

$$\varpi_1 = p_1, \quad \varpi_2 = p_2, \quad \varpi_3 = p_3 - gt; \tag{128.}$$

and the *principal function* S_1 of the same undisturbed motion is, by (89.),

$$\begin{aligned} S_1 &= \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} - g\eta_3\right) dt \\ &= \int_0^t \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - ge_3 - 2gp_3 t + g^2 t^2\right) dt \\ &= \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - ge_3\right) t - gp_3 t^2 + \tfrac{1}{3}g^2 t^3, \end{aligned} \tag{129.}$$

or finally, by (127.),

$$S_1 = \frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2 + (\eta_3 - e_3)^2}{2t} - \tfrac{1}{2}gt\,(\eta_3 + e_3) - \tfrac{1}{24}g^2 t^3. \tag{130.}$$

This function satisfies, as it ought, the following pair of partial differential equations,

$$\left.\begin{aligned} \frac{\delta S_1}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S_1}{\delta \eta_1}\right)^2 + \left(\frac{\delta S_1}{\delta \eta_2}\right)^2 + \left(\frac{\delta S_1}{\delta \eta_3}\right)^2\right\} &= -g\eta_3, \\ \frac{\delta S_1}{\delta t} + \frac{1}{2}\left\{\left(\frac{\delta S_1}{\delta e_1}\right)^2 + \left(\frac{\delta S_1}{\delta e_2}\right)^2 + \left(\frac{\delta S_1}{\delta e_3}\right)^2\right\} &= -ge_3. \end{aligned}\right| \tag{131.}$$

And if by the help of this pair, or in any other way, the form (130.) of this *principal function* S_1 had been found, the integral equations (127.) and (128.) might have been deduced from it, by our general method, as follows:

$$\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} = \frac{\eta_1 - e_1}{t}, \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} = \frac{\eta_2 - e_2}{t} \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} = \frac{\eta_3 - e_3}{t} - \tfrac{1}{2}gt, \end{aligned} \qquad (132.)$$

and

$$\begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} = \frac{\eta_1 - e_1}{t} \\ p_2 &= -\frac{\delta S_1}{\delta e_2} = \frac{\eta_2 - e_2}{t} \\ p_3 &= -\frac{\delta S_1}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \tfrac{1}{2}gt; \end{aligned} \qquad (133.)$$

the latter of these two sets coinciding with (127.), and the former set conducting to (128.).

26. Returning now from this simpler motion to the more complex motion first mentioned, and denoting by S_2 the *disturbing part* or function which must be added to S_1 in order to make up the whole principal function S of that more complex motion; we have, by applying our general method, the following rigorous expression for this disturbing function,

$$S_2 = -\int_0^t H_2 dt + \int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_2}{\delta \eta_1}\right)^2 + \left(\frac{\delta S_2}{\delta \eta_2}\right)^2 + \left(\frac{\delta S_2}{\delta \eta_3}\right)^2\right\} dt, \qquad (134.)$$

in which we may, approximately, neglect the second definite integral, and calculate the first by the help of the equations of undisturbed motion. In this manner we find, approximately, by (125.), (127.),

$$-H_2 = -\frac{\mu^2}{2}\{(e_1 + p_1 t)^2 + (e_2 + p_2 t)^2\} - \frac{\nu^2}{2}(e_3 + p_3 t - \tfrac{1}{2}gt^2)^2, \qquad (135.)$$

and therefore, by integration,*

$$\left.\begin{aligned} S_2 = &-\tfrac{1}{2}\{\mu^2(e_1^2 + e_2^2) + \nu^2 e_3^2\}t - \tfrac{1}{2}\{\mu^2(e_1 p_1 + e_2 p_2) + \nu^2 e_3 p_3\}t^2 \\ &-\tfrac{1}{6}\{\mu^2(p_1^2 + p_2^2) + \nu^2(p_3^2 - g e_3)\}t^3 + \tfrac{1}{8}\nu^2 g p_3 t^4 - \tfrac{1}{40}\nu^2 g^2 t^5, \end{aligned}\right\} \qquad (136.)$$

* [The approximate S_2 of (136.) is primarily a solution of the partial differential equation

$$\frac{\delta S_2}{\delta t} + \frac{\delta S_1}{\delta \eta_1}\frac{\delta S_2}{\delta \eta_1} + \frac{\delta S_1}{\delta \eta_2}\frac{\delta S_2}{\delta \eta_2} + \frac{\delta S_1}{\delta \eta_3}\frac{\delta S_2}{\delta \eta_3} + H_2 = 0.$$

It must be a complete integral, the arbitrary constants being e_1, e_2, e_3 and an additive constant. The method employed by Hamilton in such cases is as follows. The value of S_2 is found by integrating along the path

$$p_1 = -\frac{\delta S_1}{\delta e_1}, \quad p_2 = -\frac{\delta S_1}{\delta e_2}, \quad p_3 = -\frac{\delta S_1}{\delta e_3}.$$

S_2 is then a function of t, e_1, e_2, e_3, p_1, p_2, p_3. Replacing p_1 by $-\frac{\delta S_1}{\delta e_1}$, etc., S_2 now satisfies the partial differential equation because the part arising from the differentiation of the p's vanishes on account of relations such as

$$\frac{\delta p}{\delta t} + \frac{\delta S_1}{\delta \eta_1}\frac{\delta p}{\delta \eta_1} + \frac{\delta S_1}{\delta \eta_2}\frac{\delta p}{\delta \eta_2} + \frac{\delta S_1}{\delta \eta_3}\frac{\delta p}{\delta \eta_3} = 0.]$$

or, by (133.),

$$S_2 = -\frac{\mu^2 t}{6}(\eta_1^2 + e_1\eta_1 + e_1^2 + \eta_2^2 + e_2\eta_2 + e_2^2) \\ -\frac{\nu^2 t}{6}\{\eta_3^2 + e_3\eta_3 + e_3^2 + \tfrac{1}{4}g(\eta_3 + e_3)t^2 + \tfrac{1}{40}g^2t^4\}: \qquad (137.)$$

the error being of the fourth order, with respect to the small quantities μ, ν. And neglecting this small error, we can deduce, by our general method, approximate forms for the integrals of the equations of disturbed motion, from the corrected function $S_1 + S_2$, as follows:

$$\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1} = \frac{\eta_1 - e_1}{t} - \frac{\mu^2 t}{3}(\eta_1 + \tfrac{1}{2}e_1), \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2} = \frac{\eta_2 - e_2}{t} - \frac{\mu^2 t}{3}(\eta_2 + \tfrac{1}{2}e_2), \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3} = \frac{\eta_3 - e_3}{t} - \tfrac{1}{2}gt - \frac{\nu^2 t}{3}(\eta_3 + \tfrac{1}{2}e_3 + \tfrac{1}{8}gt^2) \end{aligned} \qquad (138.)$$

and

$$\begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1} = \frac{\eta_1 - e_1}{t} + \frac{\mu^2 t}{3}(e_1 + \tfrac{1}{2}\eta_1), \\ p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2} = \frac{\eta_2 - e_2}{t} + \frac{\mu^2 t}{3}(e_2 + \tfrac{1}{2}\eta_2), \\ p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \tfrac{1}{2}gt + \frac{\nu^2 t}{3}(e_3 + \tfrac{1}{2}\eta_3 + \tfrac{1}{8}gt^2); \end{aligned} \qquad (139.)$$

or, in the same order of approximation,

$$\begin{aligned} \eta_1 &= e_1 + p_1 t - \tfrac{1}{2}\mu^2 t^2 (e_1 + \tfrac{1}{3}p_1 t), \\ \eta_2 &= e_2 + p_2 t - \tfrac{1}{2}\mu^2 t^2 (e_2 + \tfrac{1}{3}p_2 t), \\ \eta_3 &= e_3 + p_3 t - \tfrac{1}{2}gt^2 - \tfrac{1}{2}\nu^2 t^2 (e_3 + \tfrac{1}{3}p_3 t - \tfrac{1}{12}gt^2), \end{aligned} \qquad (140.)$$

and

$$\begin{aligned} \varpi_1 &= p_1 - \mu^2 t (e_1 + \tfrac{1}{2}p_1 t), \\ \varpi_2 &= p_2 - \mu^2 t (e_2 + \tfrac{1}{2}p_2 t), \\ \varpi_3 &= p_3 - gt - \nu^2 t (e_3 + \tfrac{1}{2}p_3 t - \tfrac{1}{6}gt^2). \end{aligned} \qquad (141.)$$

Accordingly, if we develope the rigorous integrals of disturbed motion, (113.) and (114.), as far as the squares (inclusive) of the small quantities μ and ν, we are conducted to these approximate integrals; and if we develope the rigorous expression (120.) for the principal function of such motion, to the same degree of accuracy, we obtain the sum of the two expressions (130.) and (137.).

27. To illustrate still further, in the present example, our general method of successive approximation, let S_3 denote the small unknown correction of the approximate expression (137.), so that we shall now have, rigorously, for the present disturbed motion,

$$S = S_1 + S_2 + S_3, \qquad (142.)$$

S_1 and S_2 being here determined rigorously by (130.) and (137.). Then, substituting $S_1 + S_2$ for S_1 in the general transformation (87.), we find, rigorously, in the present question,*

$$S_3 = -\int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_2}{\delta\eta_1}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_2}\right)^2 + \left(\frac{\delta S_2}{\delta\eta_3}\right)^2\right\} dt \\ + \int_0^t \frac{1}{2}\left\{\left(\frac{\delta S_3}{\delta\eta_1}\right)^2 + \left(\frac{\delta S_3}{\delta\eta_2}\right)^2 + \left(\frac{\delta S_3}{\delta\eta_3}\right)^2\right\} dt: \qquad (143.)$$

and if we neglect only terms of the eighth and higher dimensions with respect to the small quantities μ, ν, we may confine ourselves to the first of these two definite integrals, and may employ, in calculating it, the approximate expressions (140.) for the coordinates of disturbed motion. In this manner we obtain the very approximate expression,

$$\begin{aligned} S_3 = &-\frac{\mu^4}{18}\int_0^t t^2\{(\eta_1 + \tfrac{1}{2}e_1)^2 + (\eta_2 + \tfrac{1}{2}e_2)^2\}\, dt \\ &-\frac{\nu^4}{18}\int_0^t t^2(\eta_3 + \tfrac{1}{2}e_3 + \tfrac{1}{8}gt^2)^2\, dt \\ = &-\frac{\mu^4 t^3}{360}(4\eta_1^2 + 7\eta_1 e_1 + 4e_1^2 + 4\eta_2^2 + 7\eta_2 e_2 + 4e_2^2) \\ &-\frac{\nu^4 t^3}{360}(4\eta_3^2 + 7\eta_3 e_3 + 4e_3^2) - \frac{\nu^4 g t^5}{240}(\eta_3 + e_3) - \frac{17\nu^4 g^2 t^7}{40320} \\ &-\frac{\mu^6 t^5}{945}(\eta_1^2 + \tfrac{31}{16}\eta_1 e_1 + e_1^2 + \eta_2^2 + \tfrac{31}{16}\eta_2 e_2 + e_2^2) \\ &-\frac{\nu^6 t^5}{945}(\eta_3^2 + \tfrac{31}{16}\eta_3 e_3 + e_3^2) - \frac{17\nu^6 g t^7}{40320}(\eta_3 + e_3) - \frac{31\nu^6 g^2 t^9}{725760}; \end{aligned} \qquad (144.)$$

which is accordingly the sum of the terms of the fourth and sixth dimensions in the development of the rigorous expression (120.), and gives, by our general method, correspondingly approximate expressions for the integrals of disturbed motion, under the forms

$$\begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta\eta_1} + \frac{\delta S_2}{\delta\eta_1} + \frac{\delta S_3}{\delta\eta_1}, \\ \varpi_2 &= \frac{\delta S_1}{\delta\eta_2} + \frac{\delta S_2}{\delta\eta_2} + \frac{\delta S_3}{\delta\eta_2}, \\ \varpi_3 &= \frac{\delta S_1}{\delta\eta_3} + \frac{\delta S_2}{\delta\eta_3} + \frac{\delta S_3}{\delta\eta_3}, \end{aligned} \qquad (145.)$$

[S_1 and S_2 are rigorously determined by

$$\frac{\partial S_1}{\partial t} + \tfrac{1}{2}\Sigma\left(\frac{\partial S_1}{\partial\eta_1}\right)^2 = -H_1 \qquad \text{(i)}$$

and

$$\frac{\partial S_2}{\partial t} + \Sigma\frac{\partial S_1}{\partial\eta_1}\frac{\partial S_2}{\partial\eta_1} = -H_2. \qquad \text{(ii)}$$

In (143) $\frac{d}{dt}$ means $\frac{\partial}{\partial t} + \Sigma\frac{\partial S}{\partial\eta_1}\frac{\partial}{\partial\eta_1}$, where $S = S_1 + S_2 + S_3$, and so becomes

$$\frac{\partial S_3}{\partial t} + \Sigma\left(\frac{\partial S_1}{\partial\eta_1} + \frac{\partial S_2}{\partial\eta_1} + \frac{\partial S_3}{\partial\eta_1}\right)\frac{\partial S_3}{\partial\eta_1} + \tfrac{1}{2}\Sigma\left(\frac{\partial S_2}{\partial\eta_1}\right)^2 - \tfrac{1}{2}\Sigma\left(\frac{\partial S_3}{\partial\eta_1}\right)^2 = 0. \qquad \text{(iii)}$$

We see at once that the sum of three equations (i), (ii) and (iii) gives $\frac{\partial S}{\partial t} + \tfrac{1}{2}\Sigma\left(\frac{\partial S}{\partial\eta_1}\right)^2 = -H_1 - H_2$.]

and
$$\left.\begin{aligned}p_1&=-\frac{\delta S_1}{\delta e_1}-\frac{\delta S_2}{\delta e_1}-\frac{\delta S_3}{\delta e_1},\\ p_2&=-\frac{\delta S_1}{\delta e_2}-\frac{\delta S_2}{\delta e_2}-\frac{\delta S_3}{\delta e_2},\\ p_3&=-\frac{\delta S_1}{\delta e_3}-\frac{\delta S_2}{\delta e_3}-\frac{\delta S_3}{\delta e_3}.\end{aligned}\right\} \tag{146.}$$

28. To illustrate by the same example the theory of gradually varying elements, let us establish the following definitions, for the present disturbed motion,

$$\left.\begin{aligned}&\kappa_1=\eta_1-\varpi_1 t, &&\kappa_2=\eta_2-\varpi_2 t, &&\kappa_3=\eta_3-\varpi_3 t-\tfrac{1}{2}gt^2,\\ &\lambda_1=\varpi_1, &&\lambda_2=\varpi_2, &&\lambda_3=\varpi_3+gt,\end{aligned}\right\} \tag{147.}$$

and let us call these six quantities $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$ the *varying elements* of that motion, by analogy to the six constant quantities $e_1, e_2, e_3, p_1, p_2, p_3$, which may, for the undisturbed motion, be represented in a similar way, namely, by (127.) and (128.),

$$\begin{aligned}&e_1=\eta_1-\varpi_1 t, &&e_2=\eta_2-\varpi_2 t, &&e_3=\eta_3-\varpi_3 t-\tfrac{1}{2}gt^2,\\ &p_1=\varpi_1, &&p_2=\varpi_2, &&p_3=\varpi_3+gt.\end{aligned} \tag{148.}$$

We shall then have rigorously, for the six disturbed variables $\eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3$, expressions of the same forms as in the integrals (127.) and (128.) of undisturbed motion, but with variable instead of constant elements, namely, the following:

$$\left.\begin{aligned}&\eta_1=\kappa_1+\lambda_1 t, &&\eta_2=\kappa_2+\lambda_2 t, &&\eta_3=\kappa_3+\lambda_3 t-\tfrac{1}{2}gt^2,\\ &\varpi_1=\lambda_1, &&\varpi_2=\lambda_2, &&\varpi_3=\lambda_3-gt;\end{aligned}\right\} \tag{149.}$$

and the rigorous determination of the six varying elements $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$, as functions of the time and of their own initial values $e_1, e_2, e_3, p_1, p_2, p_3$, depends on the integration of the 6 following equations, in ordinary differentials of the first order, of the forms (105.):

$$\begin{aligned}\frac{d\kappa_1}{dt}&=\frac{\delta H_2}{\delta\lambda_1}=\mu^2 t\,(\kappa_1+\lambda_1 t),\\ \frac{d\kappa_2}{dt}&=\frac{\delta H_2}{\delta\lambda_2}=\mu^2 t\,(\kappa_2+\lambda_2 t),\\ \frac{d\kappa_3}{dt}&=\frac{\delta H_2}{\delta\lambda_3}=\nu^2 t\,(\kappa_3+\lambda_3 t-\tfrac{1}{2}gt^2),\end{aligned} \tag{150.}$$

and

$$\begin{aligned}\frac{d\lambda_1}{dt}&=-\frac{\delta H_2}{\delta\kappa_1}=-\mu^2(\kappa_1+\lambda_1 t),\\ \frac{d\lambda_2}{dt}&=-\frac{\delta H_2}{\delta\kappa_2}=-\mu^2(\kappa_2+\lambda_2 t),\\ \frac{d\lambda_3}{dt}&=-\frac{\delta H_2}{\delta\kappa_3}=-\nu^2(\kappa_3+\lambda_3 t-\tfrac{1}{2}gt^2),\end{aligned} \tag{151.}$$

H_2 being here the expression

$$H_2=\frac{\mu^2}{2}\{(\kappa_1+\lambda_1 t)^2+(\kappa_2+\lambda_2 t)^2\}+\frac{\nu^2}{2}(\kappa_3+\lambda_3 t-\tfrac{1}{2}gt^2)^2, \tag{152.}$$

which is obtained from (125.) by substituting for the disturbed coordinates η_1, η_2, η_3 their values (149.), as functions of the varying elements and of the time. It is not difficult to integrate rigorously this system of equations (150.) and (151.); and we shall soon have occasion to state their complete and accurate integrals: but we shall continue for a while to treat these rigorous integrals as unknown, that we may take this opportunity to exemplify our general method of indefinite approximation, for all such dynamical questions, founded on the properties of the *functions of elements* C and E. Of these two functions either may be employed, and we shall use here the function C.

29. This function, by (109.) and (152.), may rigorously be expressed as follows:

$$\left.\begin{aligned} C=\frac{\mu^2}{2}\int_0^t (\lambda_1^2 t^2-\kappa_1^2+\lambda_2^2 t^2-\kappa_2^2)\,dt \\ +\frac{\nu^2}{2}\int_0^t \{(\lambda_3 t-\tfrac{1}{2}gt^2)^2-\kappa_3^2\}\,dt; \end{aligned}\right\} \tag{153.}$$

and has therefore the following for a first approximate value, obtained by treating the elements $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$ as constant and equal to their initial values $e_1, e_2, e_3, p_1, p_2, p_3$,

$$\begin{aligned} C=-\frac{t}{2}\{\mu^2(e_1^2+e_2^2)+\nu^2 e_3^2\}+\frac{t^3}{6}\{\mu^2(p_1^2+p_2^2)+\nu^2 p_3^2\} \\ -\frac{t^4}{8}\nu^2 g p_3+\frac{t^5}{40}\nu^2 g^2. \end{aligned} \tag{154.}$$

In like manner we have, as first approximations, of the kind expressed by the general formula (Z^1.), the following results deduced from the equations (151.),

$$\left.\begin{aligned} \lambda_1&=p_1-\mu^2(e_1 t+\tfrac{1}{2}p_1 t^2), \\ \lambda_2&=p_2-\mu^2(e_2 t+\tfrac{1}{2}p_2 t^2), \\ \lambda_3&=p_3-\nu^2(e_3 t+\tfrac{1}{2}p_3 t^2-\tfrac{1}{6}gt^3), \end{aligned}\right\} \tag{155.}$$

and therefore, as approximations of the same kind,

$$\begin{aligned} e_1&=-\tfrac{1}{2}p_1 t-\frac{\lambda_1-p_1}{\mu^2 t}, \\ e_2&=-\tfrac{1}{2}p_2 t-\frac{\lambda_2-p_2}{\mu^2 t}, \\ e_3&=-\tfrac{1}{2}p_3 t+\tfrac{1}{6}gt^2-\frac{\lambda_3-p_3}{\nu^2 t}. \end{aligned} \tag{156.}$$

Substituting these values for the initial constants e_1, e_2, e_3 in the approximate value (154.) for the function of elements C, we obtain the following approximate expression C_1 for that function, of the form supposed by our theory:

$$\begin{aligned} C_1=-\frac{1}{2t}\left\{\frac{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2}{\mu^2}+\frac{(\lambda_3-p_3)^2}{\nu^2}\right\} \\ -\frac{t}{2}\{(\lambda_1-p_1)p_1+(\lambda_2-p_2)p_2+(\lambda_3-p_3)(p_3-\tfrac{1}{3}gt)\} \\ +\frac{t^3}{24}\{\mu^2(p_1^2+p_2^2)+\nu^2 p_3^2\}-\frac{t^4}{24}\nu^2 g p_3+\frac{t^5}{90} \end{aligned} \tag{157.}$$

The rigorous function C must satisfy, in the present question, by the principles of the eighteenth number, the partial differential equation,

$$\frac{\delta C}{\delta t}=\frac{\mu^2}{2}\left\{\left(\frac{\delta C}{\delta \lambda_1}+\lambda_1 t\right)^2+\left(\frac{\delta C}{\delta \lambda_2}+\lambda_2 t\right)^2\right\}+\frac{\nu^2}{2}\left(\frac{\delta C}{\delta \lambda_3}+\lambda_3 t-\tfrac{1}{2}gt^2\right)^2; \tag{158.}$$

and if it be put under the form (U^1.),

$$C=C_1+C_2,$$

C_1 being a first approximation, supposed to vanish with the time, then the correction C_2 must satisfy rigorously the condition

$$\begin{aligned}C_2=\int_0^t\left\{-\frac{\delta C_1}{\delta t}+\frac{\mu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_1}+\lambda_1 t\right)^2+\frac{\mu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_2}+\lambda_2 t\right)^2+\frac{\nu^2}{2}\left(\frac{\delta C_1}{\delta \lambda_3}+\lambda_3 t-\tfrac{1}{2}gt^2\right)^2\right\}dt\\ -\frac{1}{2}\int_0^t\left\{\mu^2\left(\frac{\delta C_2}{\delta \lambda_1}\right)^2+\mu^2\left(\frac{\delta C_2}{\delta \lambda_2}\right)^2+\nu^2\left(\frac{\delta C_2}{\delta \lambda_3}\right)^2\right\}dt.\end{aligned} \tag{159.}$$

In passing to a second approximation we may neglect the second definite integral, and may calculate the first by the help of the approximate equations (155.); which give, in this manner,

$$\begin{aligned}C_2&=-\int_0^t\{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2+(\lambda_3-p_3)^2\}\,dt\\ &\quad+\frac{\mu^2}{2}\int_0^t\{\lambda_1(\lambda_1-p_1)+\lambda_2(\lambda_2-p_2)\}\,dt\\ &\quad+\frac{\nu^2}{2}\int_0^t(\lambda_3-\tfrac{2}{3}gt)(\lambda_3-p_3)\,t^2\,dt\\ &=-\frac{t}{3}\{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2+(\lambda_3-p_3)^2\}\\ &\quad+\frac{t^3}{24}\{\mu^2p_1(\lambda_1-p_1)+\mu^2p_2(\lambda_2-p_2)+\nu^2p_3(\lambda_3-p_3)\}\\ &\quad-\frac{t^4}{45}\nu^2g(\lambda_3-p_3)+\frac{t^5}{240}(\mu^4p_1^2+\mu^4p_2^2+\nu^4p_3^2)\\ &\quad-\frac{t^6}{240}\nu^4gp_3+\frac{t^7}{945}\nu^4g^2.\end{aligned} \tag{160.}$$

We might improve this second approximation in like manner, by calculating a new definite integral C_3, with the help of the following more approximate forms for the relations between the varying elements λ_1, λ_2, λ_3 and the initial constants, deduced by our general method:

$$\begin{aligned}e_1&=-\frac{\delta C_1}{\delta p_1}-\frac{\delta C_2}{\delta p_1}=-\frac{\lambda_1-p_1}{\mu^2t}\left(1+\frac{\mu^2t^2}{6}+\frac{\mu^4t^4}{24}\right)-\frac{tp_1}{2}\left(1+\frac{\mu^2t^2}{12}+\frac{\mu^4t^4}{60}\right),\\ e_2&=-\frac{\delta C_1}{\delta p_2}-\frac{\delta C_2}{\delta p_2}=-\frac{\lambda_2-p_2}{\mu^2t}\left(1+\frac{\mu^2t^2}{6}+\frac{\mu^4t^4}{24}\right)-\frac{tp_2}{2}\left(1+\frac{\mu^2t^2}{12}+\frac{\mu^4t^4}{60}\right),\\ e_3&=-\frac{\delta C_1}{\delta p_3}-\frac{\delta C_2}{\delta p}=-\frac{\lambda_3-p_3}{\nu^2t}\left(1+\frac{\nu^2t^2}{6}+\frac{\nu^4t^4}{24}\right)-\frac{tp_3}{2}\left(1+\frac{\nu^2t^2}{12}+\frac{\nu^4t^4}{60}\right)\\ &\qquad+\frac{gt^2}{6}\left(1+\frac{7\nu^2t^2}{60}+\frac{\nu^4t^4}{40}\right);\end{aligned} \tag{161.}$$

in which we can only depend on the terms as far as the second order, but which acquire a correctness of the fourth order when cleared of the small divisors, and give then*

$$\left.\begin{aligned}\lambda_1 &= p_1 - \mu^2 t\,(e_1 + \tfrac{1}{2}p_1 t) + \tfrac{1}{6}\mu^4 t^3\,(e_1 + \tfrac{1}{4}p_1 t),\\ \lambda_2 &= p_2 - \mu^2 t\,(e_2 + \tfrac{1}{2}p_2 t) + \tfrac{1}{6}\mu^4 t^3\,(e_2 + \tfrac{1}{4}p_2 t),\\ \lambda_3 &= p_3 - \nu^2 t\,(e_3 + \tfrac{1}{2}p_3 t - \tfrac{1}{6}g t^2) + \tfrac{1}{6}\nu^4 t^3\,(e_3 + \tfrac{1}{4}p_3 t - \tfrac{1}{20}g t^2).\end{aligned}\right\} \tag{162.}$$

But a little attention to the nature of this process shows that all the successive corrections to which it conducts can be only rational and integer and homogeneous functions, of the second dimension, of the quantities $\lambda_1, \lambda_2, \lambda_3, p_1, p_2, p_3, g$, and that they may all be put under the following form, which is therefore the form of their sum, or of the whole sought function C;

$$\left.\begin{aligned}C = {} & \mu^{-2}a_\mu\,(\lambda_1 - p_1)^2 + b_\mu p_1\,(\lambda_1 - p_1) + \mu^2 c_\mu p_1^2\\ &+ \mu^{-2}a_\mu\,(\lambda_2 - p_2)^2 + b_\mu p_2\,(\lambda_2 - p_2) + \mu^2 c_\mu p_2^2\\ &+ \nu^{-2}a_\nu\,(\lambda_3 - p_3)^2 + b_\nu p_3\,(\lambda_3 - p_3) + \nu^2 c_\nu p_3^2\\ &+ f_\nu g\,(\lambda_3 - p_3) + \nu^2 h_\nu g p_3 + \nu^2 i_\nu g^2,\end{aligned}\right\} \tag{163.}$$

the coefficients a_μ, a_ν, &c. being functions of the small quantities μ, ν, and also of the time, of which it remains to discover the forms. Denoting therefore their differentials, taken with respect to the time, as follows,

$$da_\mu = a'_\mu\,dt, \quad da_\nu = a'_\nu\,dt, \text{ \&c.,} \tag{164.}$$

and substituting the expression (162.) in the rigorous partial differential equation (158.), we are conducted to the six following equations in ordinary differentials of the first order:

$$\left.\begin{aligned}2a'_\nu &= (2a_\nu + \nu^2 t)^2; & b'_\nu &= (2a_\nu + \nu^2 t)\,(b_\nu + t); & c'_\nu &= \tfrac{1}{2}\,(b_\nu + t)^2;\\ f'_\nu &= (2a_\nu + \nu^2 t)\,(f_\nu - \tfrac{1}{2}t^2); & h'_\nu &= (b_\nu + t)\,(f_\nu - \tfrac{1}{2}t^2); & i'_\nu &= \tfrac{1}{2}\,(f_\nu - \tfrac{1}{2}t^2)^2;\end{aligned}\right\} \tag{165.}$$

along with the 6 following conditions, to determine the 6 arbitrary constants introduced by integration,†

$$a_0 = -\frac{1}{2t}; \quad b_0 = -\frac{t}{2}; \quad f_0 = \frac{t^2}{6}; \quad c_0 = \frac{t^3}{24}; \quad h_0 = -\frac{t^4}{24}; \quad i_0 = \frac{t^5}{90}. \tag{166.}$$

In this manner we find, without difficulty, observing that a_μ, b_μ, c_μ may be formed from a_ν, b_ν, c_ν by changing ν to μ,

$$\begin{aligned}a_\nu &= -\tfrac{1}{2}\nu^2 t - \tfrac{1}{2}\nu \operatorname{cotan} \nu t, & a_\mu &= -\tfrac{1}{2}\mu^2 t - \tfrac{1}{2}\mu \operatorname{cotan} \mu t,\\ b_\nu &= -t + \frac{1}{\nu}\tan\frac{\nu t}{2}, & b_\mu &= -t + \frac{1}{\mu}\tan\frac{\mu t}{2},\\ c_\nu &= -\frac{t}{2\nu^2} + \frac{1}{\nu^3}\tan\frac{\nu t}{2}, & c_\mu &= -\frac{t}{2\mu^2} + \frac{1}{\mu^3}\tan\frac{\mu t}{2},\\ f_\nu &= \tfrac{1}{2}t^2 \quad \frac{1}{\nu^2} + \frac{t}{\nu}\operatorname{cotan} \nu t,\\ h_\nu &= \frac{t^2}{2\nu^2} - \frac{t}{\nu^3}\tan\frac{\nu t}{2},\\ i_\nu &= \frac{t}{2\nu^4} - \frac{t^3}{6\nu^2} - \frac{t^2}{2\nu^3}\operatorname{cotan} \nu t.\end{aligned} \tag{167.}$$

---

* [If we substitute on the right-hand side of (150.) and (151.) $e_1, e_2, e_3, p_1, p_2, p_3$ for $\kappa_1, \kappa_2, \kappa_3, \lambda_1, \lambda_2, \lambda_3$ as a first approximation, integration gives

$$\kappa_1 = e_1 + \tfrac{1}{2}\mu^2 t^2\,(e_1 + \tfrac{2}{3}p_1 t), \quad \lambda_1 = p_1 - \mu^2 t\,(e_1 + \tfrac{1}{2}p_1 t), \text{ etc.}$$

The insertion of these values in the same equations followed by an integration as before gives (162.).]

† [These coefficients come from (157.).]

The form of the function C is therefore entirely known, and we have for this *function of elements* the following rigorous expression,

$$\begin{aligned}C = &-\frac{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2}{2\mu\tan\mu t}-\frac{(\lambda_3-p_3)^2}{2\nu\tan\nu t}\\ &-\frac{t}{2}\{(\lambda_1-p_1)^2+(\lambda_2-p_2)^2+(\lambda_3-p_3)^2\}\\ &-t\{p_1(\lambda_1-p_1)+p_2(\lambda_2-p_2)+p_3(\lambda_3-p_3)\}\\ &+\frac{1}{\mu}\{p_1(\lambda_1-p_1)+p_2(\lambda_2-p_2)\}\tan\frac{\mu t}{2}+\frac{1}{\nu}p_3(\lambda_3-p_3)\tan\frac{\nu t}{2}\\ &-\frac{t}{2}(p_1^2+p_2^2+p_3^2)+\frac{1}{\mu}(p_1^2+p_2^2)\tan\frac{\mu t}{2}+\frac{1}{\nu}p_3^2\tan\frac{\nu t}{2}\\ &+\left(\frac{t^2}{2}-\frac{1}{\nu^2}+\frac{t}{\nu}\operatorname{cotan}\nu t\right)g(\lambda_3-p_3)+\left(\frac{t^2}{2}-\frac{t}{\nu}\tan\frac{\nu t}{2}\right)gp_3\\ &+\left(\frac{t}{2\nu^2}-\frac{t^3}{6}-\frac{t^2}{2\nu}\operatorname{cotan}\nu t\right)g^2,\end{aligned} \tag{168.}$$

which may be variously transformed, and gives by our general method the following systems of rigorous integrals of the differential equations of varying elements, (150.), (151.):

$$\begin{aligned}e_1 &= -\frac{\delta C}{\delta p_1} = -\frac{\lambda_1-p_1}{\mu\sin\mu t}-\frac{p_1}{\mu}\tan\frac{\mu t}{2},\\ e_2 &= -\frac{\delta C}{\delta p_2} = -\frac{\lambda_2-p_2}{\mu\sin\mu t}-\frac{p_2}{\mu}\tan\frac{\mu t}{2},\\ &\quad\frac{\delta C}{\delta p_3} = \quad\frac{\lambda_3-p_3}{\nu\sin\nu t}-\frac{p_3}{\nu}\tan\frac{\nu t}{2}+\frac{g}{\nu}\left(\frac{}{\sin\nu t}\quad\cdot\frac{1}{\nu}\right),\end{aligned} \tag{169.}$$

and
$$\begin{aligned}\kappa_1 &= \frac{\delta C}{\delta\lambda_1} = -(\lambda_1-p_1)\left(t+\frac{1}{\mu}\operatorname{cotan}\mu t\right)+p_1\left(-t+\frac{1}{\mu}\tan\frac{\mu t}{2}\right),\\ \kappa_2 &= \frac{\delta C}{\delta\lambda_2} = -(\lambda_2-p_2)\left(t+\frac{1}{\mu}\operatorname{cotan}\mu t\right)+p_2\left(-t+\frac{1}{\mu}\tan\frac{\mu t}{2}\right),\\ \kappa_3 &= \frac{\delta C}{\delta\lambda_3} = -(\lambda_3-p_3)\left(t+\frac{1}{\nu}\operatorname{cotan}\nu t\right)+p_3\left(-t+\frac{1}{\nu}\tan\frac{\nu t}{2}\right)+g\left(\frac{t^2}{2}-\frac{1}{\nu^2}+\frac{t}{\nu}\operatorname{cotan}\nu t\right);\end{aligned} \tag{170.}$$

that is,
$$\begin{aligned}\lambda_1 &= p_1\cos\mu t-e_1\mu\sin\mu t,\\ \lambda_2 &= p_2\cos\mu t-e_2\mu\sin\mu t,\\ \lambda_3 &= p_3\cos\nu t-e_3\nu\sin\nu t+g\left(t-\frac{1}{\nu}\sin\nu t\right),\end{aligned} \tag{171.}$$

and
$$\begin{aligned}\kappa_1 &= e_1(\cos\mu t+\mu t\sin\mu t)+p_1\left(\frac{1}{\mu}\sin\mu t-t\cos\mu t\right),\\ \kappa_2 &= e_2(\cos\mu t+\mu t\sin\mu t)+p_2\left(\frac{1}{\mu}\sin\mu t-t\cos\mu t\right),\\ \kappa_3 &= e_3(\cos\nu t+\nu t\sin\nu t)+p_3\left(\frac{1}{\nu}\sin\nu t-t\cos\nu t\right)-g\left(\text{vers}\,\frac{\nu t}{}-\frac{t}{}\sin\nu t+\frac{t^2}{2}\right).\end{aligned} \tag{172.}$$

Accordingly, these rigorous expressions for the 6 varying elements, in the present dynamical question, agree with the results obtained by the ordinary methods of integration from the 6 ordinary differential equations (150.) and (151.) and with those obtained by elimination from the equations (113.), (114.), (147.).

Remarks on the foregoing Example.

30. The example which has occupied us in the last six numbers is not altogether ideal, but is realised to some extent by the motion of a projectile in a void. For if we consider the earth as a sphere, of radius R, and suppose the accelerating force of gravity to vary inversely as the square of the distance r from its centre, and to be $=g$ at the surface, this force will be represented generally by $\frac{gR^2}{r^2}$; and to adapt the differential equations (78.) to the motion of a projectile in a void, it will be sufficient to make

$$U = gR^2\left(\frac{1}{r} - \frac{1}{R}\right). \tag{173.}$$

If we place the origin of rectangular coordinates at the earth's surface, and suppose the semiaxis of $+z$ to be directed vertically upwards, we shall have

$$r = \sqrt{(R+z)^2 + x^2 + y^2}, \tag{174.}$$

and

$$U = -gz + \frac{gz^2}{R} - \frac{g(x^2+y^2)}{2R}, \tag{175.}$$

neglecting only those very small terms which have the square of the earth's radius for a divisor: neglecting therefore such terms, the force-function U in this question is of that form (110.) on which all the reasonings of the example have been founded; the small constants μ, ν being the real and imaginary quantities $\sqrt{\frac{g}{R}}, \sqrt{\frac{-2g}{R}}$, respectively. We may therefore apply the results of the recent numbers to the motions of projectiles in a void, by substituting these values for the constants, and altering, where necessary, trigonometrical to exponential functions. But besides the theoretical facility and the little practical importance of researches respecting such projectiles, the results would only be accurate as far as the first negative power (inclusive) of the earth's radius, because the expression (110.) for the force-function U is only accurate so far; and therefore the rigorous and approximate investigations of the six preceding numbers, founded on that expression, are offered only as mathematical illustrations of a general *method*, extending to all problems of dynamics, at least to all those to which the law of living forces applies.

Attracting Systems resumed: Differential Equations of internal or Relative Motion; Integration by the Principal Function.

31. Returning now from this digression on the motion of a single point, to the more important study of an attracting or repelling system, let us resume the differential equations (A.), which may be thus summed up:

$$dt\,\delta H = \Sigma\,(d\eta\,\delta\varpi - d\varpi\,\delta\eta); \tag{A^2.}$$

and in order to separate the absolute motion of the whole system in space from the motions of its points among themselves, let us choose the following marks of position:

$$x_{,,}=\frac{\Sigma\,.\,mx}{\Sigma m},\quad y_{,,}=\frac{\Sigma\,.\,my}{\Sigma m},\quad z_{,,}=\frac{\Sigma\,.\,mz}{\Sigma m} \tag{176.}$$

and

$$\xi_i=x_i-x_n,\quad \eta_i=y_i-y_n,\quad \zeta_i=z_i-z_n; \tag{177.}$$

that is, the 3 rectangular coordinates of the centre of gravity of the system, referred to an origin fixed in space, and the $3n-3$ rectangular coordinates of the $n-1$ masses $m_1, m_2, \dots m_{n-1}$, referred to the nth mass m_n, as an internal and moveable origin, but to axes parallel to the former.* We then find, as in the former Essay,†

$$\left.\begin{aligned}T&=\tfrac{1}{2}(x'^2_{,,}+y'^2_{,,}+z'^2_{,,})\,\Sigma m\\&\quad+\tfrac{1}{2}\Sigma_{,}\,.\,m\,(\xi'^2+\eta'^2+\zeta'^2)-\frac{1}{2\Sigma m}\{(\Sigma_{,}\,.\,m\xi')^2+(\Sigma_{,}\,.\,m\eta')^2+(\Sigma_{,}\,.\,m\zeta')^2\},\end{aligned}\right\} \tag{178.}$$

the sign of summation $\Sigma_{,}$ referring to the first $n-1$ masses only; and therefore,

$$\begin{aligned}T=\frac{1}{2\Sigma m}\left\{\left(\frac{\delta T}{\delta x'_{,,}}\right)^2+\left(\frac{\delta T}{\delta y'_{,,}}\right)^2+\left(\frac{\delta T}{\delta z'_{,,}}\right)^2\right\}+\tfrac{1}{2}\Sigma_{,}\,.\,\frac{1}{m}\left\{\left(\frac{\delta T}{\delta \xi'}\right)^2+\left(\frac{\delta T}{\delta \eta'}\right)^2+\left(\frac{\delta T}{\delta \zeta'}\right)^2\right.\\ \frac{1}{2m_n}\left\{\left(\Sigma_{,}\frac{\delta T}{\delta\xi'}\right)^2+\left(\Sigma_{,}\frac{\delta T}{\delta\eta'}\right)^2+\left(\Sigma_{,}\frac{\delta T}{\delta\zeta'}\right.\right.\end{aligned} \tag{179.}$$

If then we put for abridgement,

$$\begin{aligned}x'_{,}&=\frac{1}{m}\frac{\delta T}{\delta\xi'}=\xi'-\frac{\Sigma_{,}\,.\,m\xi'}{\Sigma m}\\ y'_{,}&=\frac{1}{m}\frac{\delta T}{\delta\eta'}=\eta'-\frac{\Sigma_{,}\,.\,m\eta'}{\Sigma m}\\ z'_{,}&=\frac{1}{m}\frac{\delta T}{\delta\zeta'}=\zeta'-\frac{\Sigma_{,}\,.\,m\zeta'}{\Sigma m},\end{aligned} \tag{180.}$$

we shall have the expression

$$\left.\begin{aligned}H&=\tfrac{1}{2}(x'^2_{,,}+y'^2_{,,}+z'^2_{,,})\,\Sigma m+\tfrac{1}{2}\Sigma_{,}\,.\,m\,(x'^2_{,}+y'^2_{,}+z'^2_{,})\\&\quad-\frac{1}{2m_n}\{(\Sigma_{,}\,.\,mx'_{,})^2+(\Sigma_{,}\,.\,my'_{,})^2+(\Sigma_{,}\,.\,mz'_{,})^2\}-U,\end{aligned}\right\} \tag{B^2.}$$

of which the variation is to be compared with the following form of (A².),

$$\left.\begin{aligned}dt\,\delta H&=(dx_{,,}\,\delta x'_{,,}-dx'_{,,}\,\delta x_{,,}+dy_{,,}\,\delta y'_{,,}-dy'_{,,}\,\delta y_{,,}+dz_{,,}\,\delta z'_{,,}-dz'_{,,}\,\delta z_{,,})\,\Sigma m\\&\quad+\Sigma_{,}\,.\,m\,(d\xi\,\delta x'_{,}-dx'_{,}\,\delta\xi+d\eta\,\delta y'_{,}-dy'_{,}\,\delta\eta+d\zeta\,\delta z'_{,}-dz'_{,}\,\delta\zeta),\end{aligned}\right\} \tag{C^2.}$$

in order to form, by our general process, $6n$ differential equations of motion of the first order, between the $6n$ quantities $x_{,,}, y_{,,}, z_{,,}, x'_{,,}, y'_{,,}, z'_{,,}, \xi, \eta, \zeta, x'_{,}, y'_{,}, z'_{,}$ and the time t. In thus taking the variation of H, we are to remember that the force-function U depends only on the $3n-3$ internal coordinates ξ, η, ζ, being of the form

$$\left.\begin{aligned}U&=m_n(m_1f_1+m_2f_2+\dots+m_{n-1}f_{n-1})\\&\quad+m_1m_2f_{1,2}+m_1m_3f_{1,3}+\dots+m_{n-2}m_{n-1}f_{n-2,n-1},\end{aligned}\right\} \tag{D^2.}$$

---

* $[x_n=x_{,,}-\frac{\Sigma_{,}m\xi}{\Sigma m}$, etc.]

† [P. 127, equation (69.).]

in which f_i is a function of the distance of m_i from m_n, and $f_{i,k}$ is a function of the distance of m_i from m_k, such that their derived functions or first differential coefficients, taken with respect to the distances, express the laws of mutual repulsion, being negative in the case of attraction; and then we obtain, as we desired, two separate groups of equations, for the motion of the whole system of points in space, and for the motions of those points among themselves; namely, first, the group

$$\left.\begin{aligned} dx_{,,} &= x'_{,,}\,dt, & dx'_{,,} &= 0,\\ dy_{,,} &= y'_{,,}\,dt, & dy'_{,,} &= 0,\\ dz_{,,} &= z'_{,,}\,dt, & dz'_{,,} &= 0,\end{aligned}\right\} \tag{181.}$$

and secondly the group

$$\begin{aligned} d\xi &= \left(x'_{,} + \frac{1}{m_n}\Sigma_{,}\,.\,mx'_{,}\right)dt, & dx'_{,} &= \frac{1}{m}\frac{\delta U}{\delta\xi}dt,\\ d\eta &= \left(y'_{,} + \frac{1}{m_n}\Sigma\quad my'_{,}\right)dt, & dy'_{,} &= \frac{1}{m}\frac{\delta U}{\delta\eta}dt,\\ d\zeta &= \left(z'_{,} + \frac{1}{m_n}\Sigma\quad mz'_{,}\right)dt, & dz'_{,} &= \frac{1}{m}\frac{\delta U}{\delta\zeta}dt.\end{aligned} \tag{182.}$$

The six differential equations of the first order, (181.), between $x_{,,}, y_{,,}, z_{,,}, x'_{,,}, y'_{,,}, z'_{,,}$ and t, contain the law of rectilinear and uniform motion of the centre of gravity of the system; and the $6n-6$ equations of the same order, (182.), between the $6n-6$ variables $\xi, \eta, \zeta, x'_{,}, y'_{,}, z'_{,}$ and the time, are forms for the differential equations of internal or relative motion. We might eliminate the $3n-3$ auxiliary variables $x'_{,}, y'_{,}, z'_{,}$ between these last equations, and so obtain the following other group of $3n-3$ equations of the second order, involving only the relative coordinates and the time,

$$\begin{aligned} \xi'' &= \frac{1}{m}\frac{\delta U}{\delta\xi} + \frac{1}{m_n}\Sigma_{,}\frac{\delta U}{\delta\xi},\\ \eta'' &= \frac{1}{m}\frac{\delta U}{\delta\eta} + \frac{1}{m_n}\Sigma_{,}\frac{\delta U}{\delta\eta},\\ \zeta'' &= \frac{1}{m}\frac{\delta U}{\delta\zeta} + \frac{1}{m_n}\Sigma_{,}\frac{\delta U}{\delta\zeta};\end{aligned} \tag{183.}$$

but it is better for many purposes to retain them under the forms (182.), omitting, however, for simplicity, the lower accents of the auxiliary variables $x'_{,}, y'_{,}, z'_{,}$, because it is easy to prove that these auxiliary variables (180.) are the components of centrobaric velocity,* and because, in investigating the properties of internal or relative motion, we are at liberty to suppose that the centre of gravity of the system is fixed in space, at the origin of x, y, z. We may also, for simplicity, omit the lower accent of $\Sigma_{,}$, understanding that the summations are to fall only on the first $n-1$ masses, and denoting for greater distinctness the nth mass by a separate symbol M; and then we may comprise the differential equations of relative motion in the following simplified formula,

$$dt\,\delta H = \Sigma\,.\,m\,(d\xi\,\delta x' - dx'\,\delta\xi + d\eta\,\delta y' - dy'\,\delta\eta + d\zeta\,\delta z' - dz'\,\delta\zeta), \tag{E^2.}$$

in which

$$H = \tfrac{1}{2}\Sigma\,.\,m\,(x'^2 + y'^2 + z'^2) + \frac{1}{2M}\{(\Sigma\,.\,mx')^2 + (\Sigma\,.\,my')^2 + (\Sigma\,.\,mz')^2\} - U. \tag{F^2.}$$

---

* [By centrobaric velocity is meant the velocity relative to the mean centre: $x'_i - x'_{,,} = (x'_i - x'_n) + (x'_n - x'_{,,}) = \xi'_i - \Sigma_{,} m\xi'/\Sigma m = x'_{,i}$.]

And the integrals of these equations of relative motion are contained (by our general method) in the formula

$$\delta S = \Sigma \,.\, m\,(x'\delta\xi - a'\delta\alpha + y'\delta\eta - b'\delta\beta + z'\delta\zeta - c'\delta\gamma), \qquad (\mathrm{G}^2.)$$

in which α, β, γ, a', b', c' denote the initial values of ξ, η, ζ, x', y', z', and S is the *principal function of relative motion* of the system; that is, the former function S, simplified by the omission of the part which vanishes when the centre of gravity is fixed, and which gives in general the laws of motion of that centre, or the integrals of the equations (181.).

Second Example: Case of a Ternary or Multiple System with one Predominant Mass; Equations of the undisturbed motions of the other masses about this, in their several Binary Systems; Differentials of all their Elements, expressed by the coefficients of one Disturbing Function.

32. Let us now suppose that the $n-1$ masses m are small in comparison with the nth mass M; and let us separate the expression (F^2.) for H into the two following parts,

$$\left.\begin{aligned} H_1 &= \Sigma \,.\, \frac{m}{2}\left(1+\frac{m}{M}\right)(x'^2+y'^2+z'^2) - M\Sigma \,.\, mf, \\ H_2 &= \frac{m_1 m_2}{M}(x'_1x'_2 + y'_1y'_2 + z'_1z'_2 - Mf_{1,2}) + \dots \\ &\quad + \frac{m_i m_k}{M}(x'_ix'_k + y'_iy'_k + z'_iz'_k - Mf_{i,k}) + \dots, \end{aligned}\right\} \qquad (\mathrm{H}^2.)$$

of which the latter is small in comparison with the former, and may be neglected in a first approximation. Suppressing it accordingly, we are conducted to the following $6n-6$ differential equations of the 1st order, belonging to a simpler motion, which may be called the *undisturbed*:

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta x'} = \left(1+\frac{m}{M}\right)x'; & \frac{dx'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \xi} = M\frac{\delta f}{\delta \xi}; \\ \frac{d\eta}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta y'} = \left(1+\frac{m}{M}\right)y'; & \frac{dy'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \eta} = M\frac{\delta f}{\delta \eta}; \\ \frac{d\zeta}{dt} &= \frac{1}{m}\frac{\delta H_1}{\delta z'} = \left(1+\frac{m}{M}\right)z'; & \frac{dz'}{dt} &= -\frac{1}{m}\frac{\delta H_1}{\delta \zeta} = M\frac{\delta f}{\delta \zeta}. \end{aligned}\right\} \qquad (\mathrm{I}^2.)$$

These equations arrange themselves in $n-1$ groups, corresponding to the $n-1$ binary systems (m, M); and it is easy to integrate the equations of each group separately. We may suppose, then, these integrals found, under the forms,

$$\left.\begin{aligned} \kappa &= \chi^{(1)}(t, \xi, \eta, \zeta, x', y', z'), & \nu &= \chi^{(4)}(t, \xi, \eta, \zeta, x', y', z'), \\ \lambda &= \chi^{(2)}(t, \xi, \eta, \zeta, x', y', z'), & \tau &= \chi^{(5)}(t, \xi, \eta, \zeta, x', y', z'), \\ \mu &= \chi^{(3)}(t, \xi, \eta, \zeta, x', y', z'), & \omega &= \chi^{(6)}(t, \xi, \eta, \zeta, x', y', z'), \end{aligned}\right\} \qquad (\mathrm{K}^2.)$$

the six quantities κ, λ, μ, ν, τ, ω being constant for the undisturbed motion of any one binary system; and therefore the six functions $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(4)}$, $\chi^{(5)}$, $\chi^{(6)}$, or κ, λ, μ, ν, τ, ω, being such as to satisfy *identically* the following equation,

$$0 = m\frac{\delta\kappa}{\delta t} + \frac{\delta\kappa}{\delta\xi}\frac{\delta H_1}{\delta x'} - \frac{\delta\kappa}{\delta x'}\frac{\delta H_1}{\delta\xi} + \frac{\delta\kappa}{\delta\eta}\frac{\delta H_1}{\delta y'} - \frac{\delta\kappa}{\delta y'}\frac{\delta H_1}{\delta\eta} + \frac{\delta\kappa}{\delta\zeta}\frac{\delta H_1}{\delta z'} - \frac{\delta\kappa}{\delta z'}\frac{\delta H_1}{\delta\zeta}, \qquad (\mathrm{L}^2.)$$

with five other equations analogous, for the five other elements λ, μ, ν, τ, ω, in any one binary system (m, M).

33. Returning now to the original multiple system, we may retain as definitions the equations (K².), but then we can no longer consider the elements $\kappa_i, \lambda_i, \mu_i, \nu_i, \tau_i, \omega_i$ of the binary system (m_i, M) as constant, because this system is now disturbed by the other masses m_k; however, the $6n-6$ equations of disturbed relative motion, when put under the forms

$$\begin{aligned} m\frac{d\xi}{dt}&=\frac{\delta H_1}{\delta x'}+\frac{\delta H_2}{\delta x'}, \quad & m\frac{dx'}{dt}&=-\frac{\delta H_1}{\delta \xi}-\frac{\delta H_2}{\delta \xi},\\ m\frac{d\eta}{dt}&=\frac{\delta H_1}{\delta y'}+\frac{\delta H_2}{\delta y'}, \quad & m\frac{dy'}{dt}&=-\frac{\delta H_1}{\delta \eta}-\frac{\delta H_2}{\delta \eta},\\ m\frac{d\zeta}{dt}&=\frac{\delta H_1}{\delta z'}+\frac{\delta H_2}{\delta z'}, \quad & m\frac{dz'}{dt}&=-\frac{\delta H_1}{\delta \zeta}-\frac{\delta H_2}{\delta \zeta},\end{aligned} \tag{M^2.}$$

and combined with the identical equations of the kind (L².), give the following simple expression for the differential of the element κ, in its disturbed and variable state,

$$m\frac{d\kappa}{dt}=\frac{\delta\kappa}{\delta\xi}\frac{\delta H_2}{\delta x'}-\frac{\delta\kappa}{\delta x'}\frac{\delta H_2}{\delta\xi}+\frac{\delta\kappa}{\delta\eta}\frac{\delta H_2}{\delta y'}-\frac{\delta\kappa}{\delta y'}\frac{\delta H_2}{\delta\eta}+\frac{\delta\kappa}{\delta\zeta}\frac{\delta H_2}{\delta z'}-\frac{\delta\kappa}{\delta z'}\frac{\delta H_2}{\delta\zeta}, \tag{N^2.}$$

together with analogous expressions for the differentials of the other elements. And if we express $\xi, \eta, \zeta, x', y', z'$, and therefore H_2 itself, as depending on the time and on these varying elements, we may transform the $6n-6$ differential equations of the 1st order, (M².), between $\xi, \eta, \zeta, x', y', z', t$, into the same number of equations of the same order between the varying elements and the time; which will be of the forms

$$\begin{aligned} m\frac{d\kappa}{dt}&=\{\kappa,\lambda\}\frac{\delta H_2}{\delta\lambda}+\{\kappa,\mu\}\frac{\delta H_2}{\delta\mu}+\{\kappa,\nu\}\frac{\delta H_2}{\delta\nu}+\{\kappa,\tau\}\frac{\delta H_2}{\delta\tau}+\{\kappa,\omega\}\frac{\delta H_2}{\delta\omega},\\ m\frac{d\lambda}{dt}&=\{\lambda,\kappa\}\frac{\delta H_2}{\delta\kappa}+\{\lambda,\mu\}\frac{\delta H_2}{\delta\mu}+\{\lambda,\nu\}\frac{\delta H_2}{\delta\nu}+\{\lambda,\tau\}\frac{\delta H_2}{\delta\tau}+\{\lambda,\omega\}\frac{\delta H_2}{\delta\omega}\\ m\frac{d\mu}{dt}&=\{\mu,\kappa\}\frac{\delta H_2}{\delta\kappa}+\{\mu,\lambda\}\frac{\delta H_2}{\delta\lambda}+\{\mu,\nu\}\frac{\delta H_2}{\delta\nu}+\{\mu,\tau\}\frac{\delta H_2}{\delta\tau}+\{\mu,\omega\}\frac{\delta H_2}{\delta\omega},\\ m\frac{d\nu}{dt}&=\{\nu,\kappa\}\frac{\delta H_2}{\delta\kappa}+\{\nu,\lambda\}\frac{\delta H_2}{\delta\lambda}+\{\nu,\mu\}\frac{\delta H_2}{\delta\mu}+\{\nu,\tau\}\frac{\delta H_2}{\delta\tau}+\{\nu,\omega\}\frac{\delta H_2}{\delta\omega}\\ m\frac{d\tau}{dt}&=\{\tau,\kappa\}\frac{\delta H_2}{\delta\kappa}+\{\tau,\lambda\}\frac{\delta H_2}{\delta\lambda}+\{\tau,\mu\}\frac{\delta H_2}{\delta\mu}+\{\tau,\nu\}\frac{\delta H_2}{\delta\nu}+\{\tau,\omega\}\frac{\delta H_2}{\delta\omega}\\ m\frac{d\omega}{dt}&=\{\omega,\kappa\}\frac{\delta H_2}{\delta\kappa}+\{\omega,\lambda\}\frac{\delta H_2}{\delta\lambda}+\{\omega,\mu\}\frac{\delta H_2}{\delta\mu}+\{\omega,\nu\}\frac{\delta H_2}{\delta\nu}+\{\omega,\tau\}\frac{\delta H_2}{\delta\tau},\end{aligned} \tag{O^2.}$$

if we put, for abridgement,

$$\{\kappa,\lambda\}=\frac{\delta\kappa}{\delta\xi}\frac{\delta\lambda}{\delta x'}-\frac{\delta\kappa}{\delta x'}\frac{\delta\lambda}{\delta\xi}+\frac{\delta\kappa}{\delta\eta}\frac{\delta\lambda}{\delta y'}-\frac{\delta\kappa}{\delta y'}\frac{\delta\lambda}{\delta\eta}+\frac{\delta\kappa}{\delta\zeta}\frac{\delta\lambda}{\delta z'}-\frac{\delta\kappa}{\delta z'}\frac{\delta\lambda}{\delta\zeta}, \tag{P^2.}$$

and form the other symbols $\{\kappa,\mu\}$, $\{\lambda,\kappa\}$, &c., from this, by interchanging the letters. It is evident that these symbols have the properties,

$$\{\lambda,\kappa\}=-\{\kappa,\lambda\}, \quad \{\kappa,\kappa\}=0; \tag{184.}$$

and it results from the principles of the 15th number, that these combinations $\{\kappa,\lambda\}$, &c., when expressed as functions of the elements, do not contain the time explicitly. There are in general, by (184.), only 15 such distinct combinations for each of the $n-1$ binary systems; but there would thus be, in all, $15n-15$, if they admitted of no further reductions: however, it results from the principles of the 16th number, that $12n-12$ of these combinations may be made to vanish by a suitable choice of the elements. The following is another way of effecting as great a simplification, at least for that extensive class of cases in which the undisturbed distance between the two points of each binary system (m, M) admits of a minimum value.

Simplification of the Differential Expressions by a suitable choice of the Elements.

34. When the undisturbed distance r of m from M admits of such a minimum q, corresponding to a time τ, and satisfying at that time the conditions

$$r' = 0, \quad r'' > 0, \tag{185.}$$

then the integrals of the group (I².), or the known rules of the undisturbed motion of m about M, may be presented in the following manner:*

$$\begin{aligned}
&\kappa = \sqrt{\{(\xi y' - \eta x')^2 + (\eta z' - \zeta y')^2 + (\zeta x' - \xi z')^2\}};\\
&\lambda = \kappa - \xi y' + \eta x';\\
&\mu = \frac{M+m}{2M}(x'^2 + y'^2 + z'^2) - Mf(r);\\
&\nu = \tan^{-1}.\frac{\eta z' - \zeta y'}{\xi z' - \zeta x'};\\
&\tau = t - \int_q \frac{\sqrt{\dfrac{M}{M+m}}\cdot\dfrac{dr}{\sqrt{dr^2}}}{\sqrt{\left\{2\mu + 2Mf(r) - \left(1+\dfrac{m}{M}\right)\dfrac{\kappa^2}{r^2}\right\}}},\\
&\omega = \nu + \sin^{-1}.\frac{\kappa\zeta r^{-1}}{\sqrt{2\lambda\kappa - \lambda^2}} - \int_q \frac{\sqrt{\dfrac{M+m}{M}}\cdot\dfrac{dr}{\sqrt{dr^2}}\cdot\dfrac{\kappa}{r^2}.dr}{\sqrt{\left\{2\mu + 2Mf(r) - \left(1+\dfrac{m}{M}\right)\dfrac{\kappa^2}{r^2}\right\}}}
\end{aligned} \tag{Q².}$$

the minimum distance q being a function of the two elements κ, μ, which must satisfy the conditions

$$2\mu + 2Mf(q) - \left(1+\frac{m}{M}\right)\frac{\kappa^2}{q^2} = 0, \quad Mf'(q) + \left(1+\frac{m}{M}\right)\frac{\kappa^2}{q^3} > 0; \tag{186.}$$

and $\sin^{-1}s$, $\tan^{-1}t$, being used (according to Sir John Herschel's notation)† to express, *not* the cosecant and cotangent, but the *inverse functions* corresponding to sine and cosine, or the arcs which are more commonly called arc $(\sin = s)$, arc $(\tan = t)$. It must also be observed that the factor $\dfrac{dr}{\sqrt{dr^2}}$, which we have introduced under the signs of integration, is not superfluous, but is designed to be taken as equal to positive or negative unity, according as dr is positive or negative; that is, according as r is increasing or diminishing, so as to make the element under each integral sign constantly positive. In general, it appears to be a useful rule, though not always followed by analysts, to employ the real radical symbol $\sqrt{R}$ only for positive quantities, unless the negative sign be expressly prefixed; and then $\dfrac{r}{\sqrt{r^2}}$ will denote positive or negative unity, according as r is positive or negative. The arc given by its sine, in the expression of the element ω, is supposed to be so chosen as to increase continually with the time.

---

* [See Appendix, Note 4, p. 623, and also footnote on p. 206.]

† [For the history of the notation for the inverse trigonometrical functions see Cajori, *History of Mathematical Notations*, Vol. II, pp. 175–178.]

35. After these remarks on the notation, let us apply the formula (P^2.) to calculate the values of the 15 combinations such as $\{\kappa, \lambda\}$, of the 6 constants or elements (Q^2.).

Since

$$r = \sqrt{(\xi^2 + \eta^2 + \zeta^2)}, \tag{187.}$$

it is easy to perceive that the six combinations of the 4 first elements are as follows:

$$\{\kappa, \lambda\} = 0, \quad \{\kappa, \mu\} = 0, \quad \{\kappa, \nu\} = 0, \quad \{\lambda, \mu\} = 0, \quad \{\lambda, \nu\} = 1, \quad \{\mu, \nu\} = 0. \tag{188.}$$

To form the 4 combinations of these 4 first elements with τ, we may observe, that this 5th element τ, as expressed in (Q^2.), involves explicitly (besides the time) the distance r, and the two elements κ, μ; but the combinations already determined show that these two elements may be treated as constant in forming the four combinations now sought; we need only attend, therefore, to the variation of r, and if we interpret by the rule (P^2.) the symbols $\{\kappa, r\}$, $\{\lambda, r\}$, $\{\mu, r\}$, $\{\nu, r\}$, and attend to the equations (I^2.), we see that

$$\{\kappa, r\} = 0, \quad \{\lambda, r\} = 0, \quad \{\mu, r\} = -\frac{dr}{dt}, \quad \{\nu, r\} = 0, \tag{189.}$$

$\dfrac{dr}{dt}$ being the total differential coefficient of r in the undisturbed motion, as determined by the equations (I^2.); and, therefore, that

$$\{\kappa, \tau\} = 0, \quad \{\lambda, \tau\} = 0, \quad \{\nu, \tau\} = 0, \tag{190.}$$

and

$$\{\mu, \tau\} = -\frac{\delta\tau}{\delta r}\frac{dr}{dt} = +\frac{dt}{dr}\frac{dr}{dt} = 1: \tag{191.}$$

observing that in differentiating the expressions of the elements (Q^2.), we may treat those elements as constant, if we change the differentials of ξ, η, ζ, x', y', z' to their undisturbed values. It remains to calculate the 5 combinations of these 5 elements with the last element ω; which is given by (Q^2.) as a function of the distance r, the coordinate ζ, and the 4 elements κ, λ, μ, ν; so that we may employ this formula,

$$\{e, \omega\} = \frac{\delta\omega}{\delta r}\{e, r\} + \frac{\delta\omega}{\delta\zeta}\{e, \zeta\} + \frac{\delta\omega}{\delta\kappa}\{e, \kappa\} + \frac{\delta\omega}{\delta\lambda}\{e, \lambda\} + \frac{\delta\omega}{\delta\mu}\{e, \mu\} + \frac{\delta\omega}{\delta\nu}\{e, \nu\}, \tag{192.}$$

in which, if e be any of the first five elements, or the distance r,

$$\{e, r\} = -\frac{1}{r}\left(\xi\frac{\delta e}{\delta x'} + \eta\frac{\delta e}{\delta y'} + \zeta\frac{\delta e}{\delta z'}\right), \quad \{e, \zeta\} = -\frac{\delta e}{\delta z'}, \quad \{e, \kappa\} = 0, \tag{193.}$$

and

$$\frac{\delta\omega}{\delta\zeta} = \left(\frac{\delta\kappa}{\delta z'}\right)^{-1}, \quad \frac{\delta\omega}{\delta r} = -\frac{d\zeta}{dr}\frac{\delta\omega}{\delta\zeta}, \quad \frac{\delta\omega}{\delta\nu} = 1; \tag{194.}$$

the formula (192.) may therefore be thus written:

$$\{e, \omega\} = \left\{\frac{z'\left(\xi\dfrac{\delta e}{\delta x'} + \eta\dfrac{\delta e}{\delta y'} + \zeta\dfrac{\delta e}{\delta z'}\right)}{\xi x' + \eta y' + \zeta z'} - \frac{\delta e}{\delta z'}\right\}\left(\frac{\delta\kappa}{\delta z'}\right)^{-1} \tag{195.}$$
$$+ \{e, \nu\} + \frac{\delta\omega}{\delta\lambda}\{e, \lambda\} + \frac{\delta\omega}{\delta\mu}\{e, \mu\}.$$

We easily find, by this formula, that

$$\{\kappa, \omega\} = -1; \quad \{\lambda, \omega\} = 0; \quad \{\mu, \omega\} = 0; \quad \{r, \omega\} = \frac{dr}{dt}\frac{\delta\omega}{\delta\mu}; \tag{196.}$$

and

$$\{\nu, \omega\} = -\frac{\delta\nu}{\delta z'}\frac{\delta\omega}{\delta\zeta} - \frac{\delta\omega}{\delta\lambda} = 0. \tag{197.}$$

The formula (195.) extends to the combination $\{\tau, \omega\}$ also; but in calculating this last combination we are to remember that τ is given by (Q^2.) as a function of κ, μ, r, such that

$$\frac{\delta\tau}{\delta r} = -\frac{dt}{dr}; \tag{198.}$$

and thus we see, with the help of the combinations (196.) already determined, that

$$\{\tau, \omega\} = -\frac{\delta\tau}{\delta\kappa} - \frac{\delta\omega}{\delta\mu} = \frac{\delta}{\delta\kappa}\int_q^r \Theta_r dr + \frac{\delta}{\delta\mu}\int_q^r \Omega_r dr, \tag{199.}$$

if we represent for abridgement by Θ_r and Ω_r the coefficients of dr under the integral signs in (Q^2.), namely,

$$\begin{aligned} \Theta_r &= \sqrt{\frac{M}{M+m}}\frac{dr}{\sqrt{dr^2}}\Big\{2\mu + 2Mf(r) - \frac{M+m}{M}\kappa^2\Big\}^{-\frac{1}{2}} \\ \Omega_r &= \frac{\kappa}{r^2}\sqrt{\frac{M+m}{M}}\frac{dr}{\sqrt{dr^2}}\Big\{2\mu + 2Mf(r) - \frac{M+m}{M}\kappa^2 . \end{aligned} \tag{200.}$$

These coefficients are evidently connected by the relation

$$\frac{\delta\Theta_r}{\delta\kappa} + \frac{\delta\Omega_r}{\delta\mu} = 0, \tag{201.}$$

which gives

$$\frac{\delta}{\delta\kappa}\int_{r_{,}}^r \Theta_r dr + \frac{\delta}{\delta\mu}\int_{r_{,}}^r \Omega_r dr = 0, \tag{202.}$$

$r_{,}$ being any quantity which does not vary with the elements κ and μ; we might therefore at once conclude by (199.) that the combination $\{\tau, \omega\}$ vanishes, if a difficulty were not occasioned by the necessity of varying the lower limit q, which depends on those two elements, and by the circumstance that at this lower limit the coefficients Θ_r, Ω_r become infinite. However, the relation (202.) shows that we may express this combination $\{\tau, \omega\}$ as follows:

$$\{\tau, \omega\} = \frac{\delta}{\delta\kappa}\int_q^{r_{,}} \Theta_r dr + \frac{\delta}{\delta\mu}\int_q^{r_{,}} \Omega_r dr, \tag{203.}$$

$r_{,}$ being an auxiliary and arbitrary quantity, which cannot really affect the result, but may be made to facilitate the calculation; or in other words, we may assign to the distance r any arbitrary value, not varying for infinitesimal variations of κ, μ, which may assist in calculating the value of the expression (199.). We may therefore suppose that the increase of distance $r-q$ is small, and corresponds to a small positive interval of time $t-\tau$, during which the distance r and its differential coefficient r' are constantly increasing; and then after the first moment τ, the quantity

$$\Theta_r = \frac{1}{r'} \tag{204.}$$

will be constantly finite, positive, and decreasing, during the same interval, so that its integral must be greater than if it had constantly its final value; that is,

$$t-\tau=\int_q^r \Theta_r dr > (r-q)\,\Theta_r. \tag{205.}$$

Hence, although Θ_r tends to infinity, yet $(r-q)\,\Theta_r$ tends to zero, when by diminishing the interval we make r tend to q; and therefore the following difference

$$\int_q^r \Omega_r dr - \frac{M+m}{M}\frac{\kappa}{q^2}\int_q^r \Theta_r dr = \frac{M+m}{M}\int_q^r \left(\frac{\kappa}{r^2}-\frac{\kappa}{q^2}\right)\Theta_r dr \tag{206.}$$

will also tend to 0, and so will also its partial differential coefficient of the first order, taken with respect to μ.* We find therefore the following formula for $\{\tau, \omega\}$, (remembering that this combination has been shown to be independent of r,)

$$\{\tau, \omega\} = \underset{r=q}{\Lambda}\left\{\frac{\delta}{\delta\kappa}\int_q^r \Theta_r dr + \frac{M+m}{M}\frac{\kappa}{q^2}\frac{\delta}{\delta\mu}\int_q^r \Theta_r dr\right\}; \tag{207.}$$

the sign $\underset{r=q}{\Lambda}$ implying that the limit is to be taken to which the expression tends when r tends to q. In this last formula, as in (199.), the integral $\int_q^r \Theta_r dr$ may be considered as a known function of r, q, κ, μ, or simply of r, q, κ, if μ be eliminated by the first condition (186.); and since it vanishes independently of κ when $r=q$, it may be thus denoted:

$$\int_q^r \Theta_r dr = \phi(r, q, \kappa) - \phi(q, q, \kappa), \tag{208.}$$

the form of the function ϕ depending on the law of attraction or repulsion. This integral therefore, when considered as depending on κ and μ, by depending on κ and q, need not be varied with respect to κ, in calculating $\{\tau, \omega\}$ by (207.), because its partial differential coefficient $\left(\frac{\delta}{\delta\kappa}\int_q^r \Theta_r dr\right)$, obtained by treating q as constant, vanishes at the limit $r=q$; nor need it be varied with respect to q, because, by (186.),

$$\frac{\delta q}{\delta\kappa} + \frac{M+m}{M}\frac{\kappa}{q^2}\frac{\delta q}{\delta\mu} = 0: \tag{209.}$$

it may therefore be treated as constant, and we find at last

$$\{\tau, \omega\} = 0, \tag{210.}$$

the two terms (199.) or (203.) both tending to infinity when r tends to q, but always destroying each other.

---

* $\left[\frac{\partial\Theta_r}{\partial\mu}\right.$ is of the form $\frac{\psi}{(r-q)^{\frac{3}{2}}}$, where ψ remains finite, $\therefore \left(\frac{\kappa}{r^2}-\frac{\kappa}{q^2}\right)\frac{\partial\Theta_r}{\partial\mu}$ is of the form $\frac{f}{\sqrt{r-q}}$, where f is finite in the interval of integration. It follows that $\frac{\partial}{\partial\mu}\int_q^r \left(\frac{\kappa}{r^2}-\frac{\kappa}{q^2}\right)\Theta_r dr \to 0$ as $r \to q.\left.\right]$

36. Collecting now our results, and presenting for greater clearness each combination under the two forms in which it occurs when the order of the elements is changed, we have, for each binary system, the following thirty expressions:*

$$\left.\begin{array}{lllll}
\{\kappa, \lambda\}=0, & \{\kappa, \mu\}=0, & \{\kappa, \nu\}=0, & \{\kappa, \tau\}=0, & \{\kappa, \omega\}=-1, \\
\{\lambda, \kappa\}=0, & \{\lambda, \mu\}=0, & \{\lambda, \nu\}=1, & \{\lambda, \tau\}=0, & \{\lambda, \omega\}=0, \\
\{\mu, \kappa\}=0, & \{\mu, \lambda\}=0, & \{\mu, \nu\}=0, & \{\mu, \tau\}=1, & \{\mu, \omega\}=0, \\
\{\nu, \kappa\}=0, & \{\nu, \lambda\}=-1, & \{\nu, \mu\}=0, & \{\nu, \tau\}=0, & \{\nu, \omega\}=0, \\
\{\tau, \kappa\}=0, & \{\tau, \lambda\}=0, & \{\tau, \mu\}=-1, & \{\tau, \nu\}=0, & \{\tau, \omega\}=0, \\
\{\omega, \kappa\}=1, & \{\omega, \lambda\}=0, & \{\omega, \mu\}=0, & \{\omega, \nu\}=0, & \{\omega, \tau\}=0;
\end{array}\right\} \qquad (\mathrm{R}^2.)$$

so that the three combinations

$$\{\mu,\tau\}, \quad \{\omega,\kappa\}, \quad \{\lambda,\nu\}$$

are each equal to positive unity; the three inverse combinations

$$\{\tau,\mu\}, \quad \{\kappa,\omega\}, \quad \{\nu,\lambda\}$$

are each equal to negative unity; and all the others vanish. The six differential equations of the first order, for the 6 varying elements of any one binary system (m, M), are therefore, by $(\mathrm{O}^2.)$,

$$\left.\begin{array}{ll}
m\dfrac{d\mu}{dt}=\dfrac{\delta H_2}{\delta\tau} & m\dfrac{d\tau}{dt}=-\dfrac{\delta H_2}{\delta\mu}, \\
m\dfrac{d\omega}{dt}=\dfrac{\delta H_2}{\delta\kappa} & m\dfrac{d\kappa}{dt}=-\dfrac{\delta H_2}{\delta\omega}, \\
m\dfrac{d\lambda}{dt}=\dfrac{\delta H_2}{\delta\nu} & m\dfrac{d\nu}{dt}=-\dfrac{\delta H_2}{\delta\lambda}.
\end{array}\right\} \qquad (\mathrm{S}^2.)$$

and, if we still omit the variation of t, they may all be summed up in this form for the variation of H_2,

$$\delta H_2=\Sigma\,.\,m\,(\mu'\delta\tau-\tau'\delta\mu+\omega'\delta\kappa-\kappa'\delta\omega+\lambda'\delta\nu-\nu'\delta\lambda), \qquad (\mathrm{T}^2.)$$

which single formula enables us to derive all the $6n-6$ differential equations of the first order, for all the varying elements of all the binary systems, from the variation or from the partial differential coefficients of a single quantity H_2, expressed as a function of those elements.

If we choose to introduce into the expression $(\mathrm{T}^2.)$, for δH_2, the variation of the time t, we have only to change $\delta\tau$ to $\delta\tau-\delta t$, because, by $(\mathrm{Q}^2.)$, δt enters only so accompanied; that is, t enters only under the form $t-\tau_i$, in the expressions of $\xi_i, \eta_i, \zeta_i, x_i', y_i', z_i'$ as functions of the time and of the elements; we have, therefore,

$$\frac{\delta H_2}{\delta t}=-\Sigma\,\frac{\delta H_2}{\delta\tau}=-\Sigma\,.\,m\mu'; \qquad (211.)$$

* [If we put $\xi=r\cos l\cos\phi$, $\eta=r\cos l\sin\phi$, $\zeta=r\sin l$, the principal function S satisfies

$$\frac{\partial S}{\partial t}+\frac{M+m}{2M}\left\{\left(\frac{\partial S}{\partial r}\right)^2+\left(\frac{\partial S}{r\,\partial l}\right)^2+\left(\frac{\partial S}{r\cos l\,\partial\phi}\right)^2\right\}-Mf(r)=0.$$

The Jacobi Complete Integral can be put in the form

$$S=-\mu t+(\kappa-\lambda)\,\phi+\int\sqrt{\kappa^2-\frac{(\kappa-\lambda)^2}{\cos^2 l}}\,.\,dl+\sqrt{\frac{M}{M+m}}\int\left\{2\mu+2Mf(r)-\left(1+\frac{m}{M}\right)\frac{\kappa^2}{r^2}\right\}^{\frac{1}{2}}dr+\text{const.},$$

where κ, λ, μ are arbitrary constants. We, then, easily obtain $\dfrac{\partial S}{\partial\kappa}=\omega$, $\dfrac{\partial S}{\partial\lambda}=-\nu$, $\dfrac{\partial S}{\partial\mu}=-\tau$. Hence by Donkin's Theorem (Routh's *Advanced Rigid Dynamics*, Art. 496), $(\kappa, -\omega)$, (λ, ν), (μ, τ) form a canonical set of constants and hence relations $(\mathrm{R}^2.)$ hold.]

and since, by (H².), (Q².),

$$H_1 = \Sigma \,.\, m\mu, \tag{212.}$$

we find finally,

$$\frac{dH_1}{dt} = \frac{\delta H_2}{\delta t}. \tag{U².}$$

This remarkable form for the differential of H_1, considered as a varying element, is general for all problems of dynamics. It may be deduced by the general method from the formulæ of the 13th and 14th numbers, which give*

$$\left.\begin{aligned} \frac{dH_1}{dt} &= \frac{\delta H_2}{\delta \kappa_1} \Sigma \left(\frac{\delta H_1}{\delta \eta} \frac{\delta \kappa_1}{\delta \varpi} - \frac{\delta H_1}{\delta \varpi} \frac{\delta \kappa_1}{\delta \eta} \right) + \ldots + \frac{\delta H_2}{\delta \kappa_{6n}} \Sigma \left(\frac{\delta H_1}{\delta \eta} \frac{\delta \kappa_{6n}}{\delta \varpi} - \frac{\delta H_1}{\delta \varpi} \frac{\delta \kappa_{6n}}{\delta \eta} \right) \\ &= \frac{\delta H_2}{\delta \kappa_1} \frac{\delta \kappa_1}{\delta t} + \frac{\delta H_2}{\delta \kappa_2} \frac{\delta \kappa_2}{\delta t} + \ldots + \frac{\delta H_2}{\delta \kappa_{6n}} \frac{\delta \kappa_{6n}}{\delta t} = \frac{\delta H_2}{\delta t} \end{aligned}\right\} \tag{213.}$$

$\kappa_1, \kappa_2, \ldots \kappa_{6n}$ being any $6n$ elements of a system expressed as functions of the time and of the quantities η, ω; or more concisely by this special consideration, that $H_1 + H_2$ is constant in the disturbed motion, and that in taking the first total differential coefficient of H_2 with respect to the time, the elements may by (F¹.) be treated as constant. It is also a remarkable corollary of the general principles just referred to, but one not difficult to verify, that the first partial differential coefficient $\frac{\delta \kappa_s}{\delta t}$ of any element κ_s, taken with respect to the time, may be expressed as a function of the elements alone, not involving the time explicitly.

On the essential distinction between the Systems of Varying Elements considered in this Essay and those hitherto employed by mathematicians.

37. When we shall have integrated the differential equations of varying elements (S².), we can then calculate the varying relative coordinates ξ, η, ζ, for any binary system (m, M), by the rules of undisturbed motion, as expressed by the equations (I².), (Q².), or by the following connected formulæ:

$$\left.\begin{aligned} \xi &= r \left(\cos\theta + \frac{\lambda}{\kappa} \sin(\theta - \nu) \sin\nu \right), \\ \eta &= r \left(\sin\theta - \frac{\lambda}{\kappa} \sin(\theta - \nu) \cos\nu \right), \\ \zeta &= \frac{r}{\kappa} \sqrt{2\lambda\kappa - \lambda^2} \sin(\theta - \nu): \end{aligned}\right\} \tag{V².}$$

---

* [To deduce (213.) from §§ 13, 14. We have (since H_1 remains constant for undisturbed motion)

$$\begin{aligned} \frac{dH_1}{dt} &= \Sigma \frac{\partial H_1}{\partial \eta_s} \frac{\partial \eta_s}{\partial \kappa_i} \frac{d\kappa_i}{dt} + \Sigma \frac{\partial H_1}{\partial \varpi_s} \frac{\partial \varpi_s}{\partial \kappa_i} \frac{d\kappa_i}{dt} = \Sigma \left(\frac{\partial H_1}{\partial \eta_s} \frac{\partial \eta_s}{\partial \kappa_i} + \frac{\partial H_1}{\partial \varpi_s} \frac{\partial \varpi_s}{\partial \kappa_i} \right. \; a_{ij} \frac{\partial H_2}{\partial \kappa_j} \\ &= \Sigma \left(\frac{\partial H_1}{\partial \eta_s} \frac{\partial \eta_s}{\partial \kappa_i} + \frac{\partial H_1}{\partial \varpi_s} \frac{\partial \varpi_s}{\partial \kappa_i} \right) \left(\frac{\partial \kappa_i}{\partial \eta_t} \frac{\partial \kappa_j}{\partial \varpi_t} - \frac{\partial \kappa_i}{\partial \varpi_t} \frac{\partial \kappa_j}{\partial \eta_t} \right) \frac{\partial H_2}{\partial \kappa_j} \\ &= \Sigma \frac{\partial H_2}{\partial \kappa_j} \left[\frac{\partial H_1}{\partial \eta_s} \left\{ \frac{\partial \eta_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \eta_t} \frac{\partial \kappa_j}{\partial \varpi_t} - \frac{\partial \eta_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \varpi_t} \frac{\partial \kappa_j}{\partial \eta_t} \right\} + \Sigma \frac{\partial H_1}{\partial \varpi_s} \left\{ \frac{\partial \varpi_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \eta_t} \frac{\partial \kappa_j}{\partial \varpi_t} - \frac{\partial \varpi_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \varpi_t} \frac{\partial \kappa_j}{\partial \eta_t} \right\} \right]. \end{aligned}$$

But $\Sigma \frac{\partial \eta_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \eta_t} = \delta_{st}$, $\Sigma \frac{\partial \eta_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \varpi_t} = 0$, $\Sigma \frac{\partial \varpi_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \eta_t} = 0$, $\Sigma \frac{\partial \varpi_s}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial \varpi_t} = \delta_{st}$, where δ_{st} equals 1 or 0 according as s is or is not equal to t.

$$\therefore \frac{dH_1}{dt} = \Sigma \frac{\partial H_2}{\partial \kappa_i} \left\{ \frac{\partial H_1}{\partial \eta_s} \frac{\partial \kappa_i}{\partial \varpi_s} - \frac{\partial H_1}{\partial \varpi_s} \frac{\partial \kappa_i}{\partial \eta_s} \right\} = \Sigma \frac{\partial H_2}{\partial \kappa_i} \frac{\partial \kappa_i}{\partial t}.]$$

in which the distance r is determined as a function of the time t and of the elements τ, κ, μ, by the 5th equation (Q^2.), and in which*

$$\theta = \omega + \int_q \frac{\sqrt{\frac{M+m}{M}} \cdot \frac{dr}{\sqrt{dr^2}} \cdot \frac{\kappa}{r^2}\, dr}{\sqrt{\left\{2\mu + 2Mf(r) - \frac{M+m}{M} \cdot \frac{\kappa^2}{r^2}\right\}}}, \qquad (\mathrm{W}^2.)$$

q being still the minimum of r, when the orbit is treated as constant, and being still connected with the elements κ, μ, by the first equation of condition (186.). In astronomical language, M is the sun, m a planet, ξ, η, ζ are the heliocentric rectangular coordinates, r is the radius vector, θ the longitude in the orbit, ω the longitude of the perihelion, ν of the node, $\theta - \omega$ is the true anomaly, $\theta - \nu$ the argument of latitude, μ the constant part of the half square of undisturbed heliocentric velocity, diminished in the ratio of the sun's mass (M) to the sum ($M + m$) of masses of sun and planet, κ is the double of the areal velocity diminished in the same ratio, $\frac{\lambda}{\kappa}$ is the versed sine of the inclination of the orbit, q the perihelion distance, and τ the time of perihelion passage. The law of attraction or repulsion is here left undetermined; for Newton's law, μ is the sun's mass divided by the axis major of the orbit taken negatively, and κ is the square root of the semiparameter, multiplied by the sun's mass, and divided by the square root of the sum of the masses of sun and planet. But the varying ellipse or other orbit, which the foregoing formulæ require, differs essentially (though little) from that hitherto employed by astronomers: because it gives correctly the heliocentric coordinates, but *not* the heliocentric components of velocity, without differentiating the elements in the calculation; and therefore does *not touch* but *cuts*, (though under a very small angle,) the actual heliocentric orbit, described under the influence of all the disturbing forces.

38. For it results from the foregoing theory, that if we differentiate the expressions (V^2.) for the heliocentric coordinates, without differentiating the elements, and then assign to those new varying elements their values as functions of the time, obtained from the equations (S^2.), and deduce the centrobaric components of velocity by the formulæ (I^2.), or by the following:

$$x' = \frac{M\xi'}{M+m}, \quad y' = \frac{M\eta'}{M+m}, \quad z' = \frac{M\zeta'}{M+m}; \qquad (214.)$$

---

* [To deduce (V^2.) from (Q^2.) we have

$$(\eta z' - \zeta y')^2 + (\zeta x' - \xi z') = \kappa^2 - (\kappa - \lambda)^2 = (2\kappa\lambda - \lambda^2).$$

Therefore
$$\eta z' - \zeta y' = \sqrt{2\kappa\lambda - \lambda^2}\,\sin\nu, \quad \xi z' - \zeta x' = \sqrt{2\kappa\lambda - \lambda^2}\,\cos\nu.$$

Consequently
$$\sqrt{2\kappa\lambda - \lambda^2}\,(\xi\sin\nu - \eta\cos\nu) + (\kappa - \lambda)\,\zeta = \Sigma\xi\,(\eta z' - \zeta y') = 0,$$

that is,
$$\xi\sin\nu - \eta\cos\nu = -\frac{\kappa - \lambda}{\sqrt{2\kappa\lambda - \lambda^2}}\,\zeta.$$

Also
$$\xi^2 + \eta^2 = r^2 - \zeta^2.$$

Hence,
$$\xi\cos\nu + \eta\sin\nu = \left[r^2 - \zeta^2 - \frac{(\kappa-\lambda)^2}{2\kappa\lambda - \lambda^2}\,\zeta^2\right]^{\frac{1}{2}} = \left(r^2 - \frac{\kappa^2\zeta^2}{2\kappa\lambda - \lambda^2}\right)^{\frac{1}{2}}.$$

From (W^2.) and (Q^2.) we see that $\frac{\kappa\zeta}{\sqrt{2\kappa\lambda - \lambda^2}} = r\sin(\theta - \nu)$, and therefore

$$\xi\sin\nu - \eta\cos\nu = \left(-1 + \frac{\lambda}{\kappa}\right) r\sin(\theta - \nu),$$

$$\xi\cos\nu + \eta\sin\nu = r\cos(\theta - \nu),$$

from which (V^2.) follows immediately.]

then these centrobaric components will be the same functions of the time and of the new varying elements which might be otherwise deduced by elimination from the integrals (Q^2.), and will represent rigorously (by the extension given in the theory to those last-mentioned integrals) the components of velocity of the disturbed planet m, relatively to the centre of gravity of the whole solar system. We chose, as more suitable to the general course of our method, that these centrobaric components of velocity should be the auxiliary variables to be combined with the heliocentric coordinates, and to have their disturbed values rigorously expressed by the formulæ of undisturbed motion; but in making this choice it became necessary to modify these latter formulæ, and to determine a varying orbit essentially distinct in theory (though little differing in practice) from that conceived so beautifully by Lagrange.* The orbit which he imagined was more simply connected with the heliocentric motion of a *single planet*, since it gave, for such heliocentric motion, the velocity as well as the position; the orbit which we have chosen is perhaps more closely combined with the conception of a *multiple system*, moving about its common centre of gravity, and influenced in every part by the actions of all the rest. Whichever orbit shall be hereafter adopted by astronomers, they will remember that both are equally fit to represent the celestial appearances, if the numeric elements of either set be suitably determined by observation, and the elements of the other set of orbits be deduced from these by calculation. Meantime mathematicians will judge, whether in sacrificing a part of the simplicity of that geometrical conception on which the theories of Lagrange and Poisson are founded, a simplicity of another kind has not been introduced, which was wanting in those admirable theories; by our having succeeded in expressing rigorously the differentials of *all* our own new varying elements through the coefficients of a *single* function: whereas it has seemed necessary hitherto to employ one function for the Earth disturbed by Venus, and another function for Venus disturbed by the Earth.

Integration of the Simplified Equations, which determine the new varying Elements.

39. The simplified differential equations of varying elements, (S^2.), are of the same form as the equations (A.), and may be integrated in a similar manner. If we put, for abridgement,

$$(\tau, \kappa, \nu) = \int_0^t \left\{ \Sigma \left(\tau \frac{\delta H_2}{\delta \tau} + \kappa \frac{\delta H_2}{\delta \kappa} + \nu \frac{\delta H_2}{\delta \nu} \right) - H_2 \right\} dt, \qquad (X^2.)$$

and interpret similarly the symbols (μ, ω, λ), &c., we can easily assign the variations of the following 8 combinations, (τ, κ, ν), (μ, ω, λ), (μ, κ, ν), (τ, ω, λ), (τ, ω, ν), (μ, κ, λ), (τ, κ, λ), (μ, ω, ν); namely,

$$\left.\begin{aligned}
\delta(\tau, \kappa, \nu) &= \Sigma . m(\tau \delta\mu - \tau_0 \delta\mu_0 + \kappa \delta\omega - \kappa_0 \delta\omega_0 + \nu \delta\lambda - \nu_0 \delta\lambda_0) - H_2 \delta t, \\
\delta(\mu, \omega, \lambda) &= \Sigma . m(\mu_0 \delta\tau_0 - \mu \delta\tau + \omega_0 \delta\kappa_0 - \omega \delta\kappa + \lambda_0 \delta\nu_0 - \lambda \delta\nu) - H_2 \delta t, \\
\delta(\mu, \kappa, \nu) &= \Sigma . m(\mu_0 \delta\tau_0 - \mu \delta\tau + \kappa \delta\omega - \kappa_0 \delta\omega_0 + \nu \delta\lambda - \nu_0 \delta\lambda_0) - H_2 \delta t, \\
\delta(\tau, \omega, \lambda) &= \Sigma . m(\tau \delta\mu - \tau_0 \delta\mu_0 + \omega_0 \delta\kappa_0 - \omega \delta\kappa + \lambda_0 \delta\nu_0 - \lambda \delta\nu) - H_2 \delta t, \\
\delta(\tau, \omega, \nu) &= \Sigma . m(\tau \delta\mu - \tau_0 \delta\mu_0 + \omega_0 \delta\kappa_0 - \omega \delta\kappa + \nu \delta\lambda - \nu_0 \delta\lambda_0) - H_2 \delta t, \\
\delta(\mu, \kappa, \lambda) &= \Sigma . m(\mu_0 \delta\tau_0 - \mu \delta\tau + \kappa \delta\omega - \kappa_0 \delta\omega_0 + \lambda_0 \delta\nu_0 - \lambda \delta\nu) - H_2 \delta t, \\
\delta(\tau, \kappa, \lambda) &= \Sigma . m(\tau \delta\mu - \tau_0 \delta\mu_0 + \kappa \delta\omega - \kappa_0 \delta\omega_0 + \lambda_0 \delta\nu_0 - \lambda \delta\nu) - H_2 \delta t, \\
\delta(\mu, \omega, \nu) &= \Sigma . m(\mu_0 \delta\tau_0 - \mu \delta\tau + \omega_0 \delta\kappa_0 - \omega \delta\kappa + \nu \delta\lambda - \nu_0 \delta\lambda_0) - H_2 \delta t,
\end{aligned}\right\} \qquad (Y^2.)$$

---

* [See Appendix, Note 7, p. 628.]

$\kappa_0, \lambda_0, \mu_0, \nu_0, \tau_0, \omega_0$ being the initial values of the varying elements κ, λ, μ, ν, τ, ω. If, then, we consider, for example, the first of these 8 combinations (τ, κ, ν), as a function of all the $3n-3$ elements $\mu_i, \omega_i, \lambda_i$, and of their initial values $\mu_{0,i}, \omega_{0,i}, \lambda_{0,i}$, involving also in general the time explicitly, we shall have the following forms for the $6n-6$ rigorous integrals of the $6n-6$ equations (S².):

$$\left.\begin{aligned} m_i\tau_i &= \frac{\delta}{\delta\mu_i}(\tau, \kappa, \nu); & m_i\tau_{0,i} &= -\frac{\delta}{\delta\mu_{0,i}}(\tau, \kappa, \nu); \\ m_i\kappa_i &= \frac{\delta}{\delta\omega_i}(\tau, \kappa, \nu); & m_i\kappa_{0,i} &= -\frac{\delta}{\delta\omega_{0,i}}(\tau, \kappa, \nu); \\ m_i\nu_i &= \frac{\delta}{\delta\lambda_i}(\tau, \kappa, \nu); & m_i\nu_{0,i} &= -\frac{\delta}{\delta\lambda_{0,i}}(\tau, \kappa, \nu); \end{aligned}\right\} \qquad (\text{Z}^2.)$$

and in like manner we can deduce forms for the same rigorous integrals, from any one of the eight combinations (Y².). The determination of all the varying elements would therefore be fully accomplished, if we could find the complete expression for any one of these 8 combinations.

40. A first approximate expression for any one of them can be found from the form under which we have supposed H_2 to be put, namely, as a function of the elements and of the time, which may be thus denoted:

$$H_2 = H_2(t, \kappa_1, \lambda_1, \mu_1, \nu_1, \tau_1, \omega_1, \dots \kappa_{n-1}, \lambda_{n-1}, \mu_{n-1}, \nu_{n-1}, \tau_{n-1}, \omega_{n-1}); \qquad (\text{A}^3.)$$

by changing in this function the varying elements to their initial values, and employing the following approximate integrals of the equations (S².),

$$\left.\begin{aligned} \mu &= \mu_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\tau_0}dt, & \tau &= \tau_0 - \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\mu_0}dt, \\ \omega &= \omega_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\kappa_0}dt, & \kappa &= \kappa_0 - \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\omega_0}dt, \\ \lambda &= \lambda_0 + \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\nu_0}dt, & & \frac{1}{m}\int_0^t \frac{\delta H_2}{\delta\lambda_0}dt. \end{aligned}\right\} \qquad (\text{B}^3.)$$

For if we denote, for example, the first of the 8 combinations (Y².) by G, so that

$$G = (\tau, \kappa, \nu), \qquad (\text{C}^3.)$$

we shall have, as a first approximate value,

$$G_1 = \int_0^t \left\{\Sigma\left(\tau_0\frac{\delta H_2}{\delta\tau_0} + \kappa_0\frac{\delta H_2}{\delta\kappa_0} + \nu_0\frac{\delta H_2}{\delta\nu_0}\right) - H_2\right\}dt; \qquad (\text{D}^3.)$$

and after thus expressing G_1 as a function of the time, and of the initial elements, we can eliminate the initial quantities of the forms τ_0, κ_0, ν_0, and introduce in their stead the final quantities μ, ω, λ, so as to obtain an expression for G_1 of the kind supposed in (Z².), namely, a function of the time t, the varying elements μ, ω, λ, and their initial values $\mu_0, \omega_0, \lambda_0$. An approximate expression thus found may be corrected by a process of that kind, which has often been employed in this Essay for other similar purposes. For the function G, or the combination (τ, κ, ν), must satisfy rigorously, by (Y².), (A³.), the following partial differential equation:

$$0 = \frac{\delta G}{\delta t} + H_2\left(t, \frac{1}{m_1}\frac{\delta G}{\delta\omega_1}, \lambda_1, \mu_1, \frac{1}{m_1}\frac{\delta G}{\delta\lambda_1}, \frac{1}{m_1}\frac{\delta G}{\delta\mu_1}, \omega_1, \frac{1}{m_2}\frac{\delta G}{\delta\omega_2}, \dots \omega_{n-1}\right); \qquad (\text{E}^3.)$$

and each of the other analogous functions or combinations (Y².) must satisfy an analogous equation: if then we change G to $G_1 + G_2$, and neglect the squares and products of the coefficients of the small correction G_2, G_1 being a first approximation such as that already found, we are conducted, as a second approximation on principles already explained, to the following expression for this correction G_2:

$$G_2 = -\int_0^t \left\{ \frac{\delta G_1}{\delta t} + H_2 \left(t, \frac{1}{m_1 \, \delta\omega_1}, \ldots_1, \mu_1, m_1 \, \delta\lambda_1, \frac{1}{m_1} \frac{\delta G_1}{\delta\mu_1}, \ldots \right) \right\} dt: \qquad (\mathrm{F}^3.)$$

which may be continually and indefinitely improved by a repetition of the same process of correction. We may therefore, theoretically, consider the problem as solved; but it must remain for future consideration, and perhaps for actual trial, to determine which of all these various processes of successive and indefinite approximation, deduced in the present Essay and in the former, as corollaries of one general Method, and as consequences of one central Idea, is best adapted for numeric application, and for the mathematical study of phenomena.*

---

* [Two memoirs by M. Houël presented to the Faculté des Sciences de Paris (1855) are of interest. In the first he employs the Principal Function to prove various known theorems about planetary perturbations and in the second applies this method to detailed calculations of the perturbations of Jupiter. A complete bibliography of works relating to Hamilton's dynamical methods prior to that of Houël was given by Cayley, *Brit. Ass. Report* (1857), p. 40.]

IV.

ON THE APPLICATION TO DYNAMICS OF A GENERAL MATHEMATICAL METHOD PREVIOUSLY APPLIED TO OPTICS

[*British Association Report*, 1834, pp. 513–518.]

The method is founded on a combination of the principles of variations with those of partial differentials, and may suggest to analysts a separate branch of algebra, which may be called, perhaps, the *Calculus of Principal Functions*; because, in all the chief applications of algebra to physics and in a very extensive class of purely mathematical questions, it reduces the determination of many mutually connected functions to the search and study of one principal or central relation. In applying this method to Dynamics, (having previously applied it to Optics,) Professor Hamilton has discovered the existence of a principal function which, if its form were fully known, would give by its partial differential coefficients all the intermediate and all the final integrals of the known equations of motion.

Professor Hamilton is of opinion that the mathematical explanation of all the phenomena of matter distinct from the phenomena of life will ultimately be found to depend on the properties of systems of attracting and repelling points. And he thinks that those who do not adopt this opinion in all its extent must yet admit the properties of such systems to be more highly important in the present state of science than any other part of the application of mathematics to physics. He therefore accounts it the capital problem of Dynamics "to determine the $3n$ rectangular coordinates, or other marks of position, of a free system of n attracting or repelling points as functions of the time," involving also $6n$ initial constants which depend on the initial circumstances of the motion and involving besides n other constants called the masses, which measure, for a standard distance, the attractive or repulsive energies.

Denoting these n masses by $m_1, m_2, \dots m_n$ and their $3n$ rectangular coordinates by $x_1, y_1, z_1, \dots x_n, y_n, z_n$ and also the $3n$ component accelerations, or second differential coefficients of these coordinates taken with respect to the time, by $x_1'', y_1'', z_1'', \dots x_n'', y_n'', z_n''$, he adopts Lagrange's statement of this problem; namely, a formula of the following kind,

$$\Sigma m\,(x''\delta x + y''\delta y + z''\delta z) = \delta U, \tag{1.}$$

in which U is the sum of the products of the masses, taken two by two, and then multiplied by each other and by certain functions of their mutual distances such that their first derived functions express the laws of their mutual repulsion, being negative in the case of attraction. Thus, for the solar system, each product of two masses is to be multiplied by the reciprocal of their distance and the results are to be added in order to compose the function U.

Mr. Hamilton next multiplies this formula of Lagrange by the element of the time dt, and integrates from the time 0 to the time t, considering the time and its element as not subject at present to the variation δ. He denotes the initial values, or values at the time 0, of the coordinates x, y, z and of their first differential coefficients x', y', z' by a, b, c and a', b', c'; and thus he obtains from Lagrange's formula (1.) this other important formula

$$\Sigma m\,(x'\delta x + y'\delta y + z'\delta z - a'\delta a - b'\delta b - c'\delta c) = \delta S, \tag{2.}$$

S being the definite integral

$$S = \int_0^t \left\{ U + \Sigma \frac{m}{2}\,(x'^2 + y'^2 + z'^2) \right\} dt. \tag{3.}$$

If the known equations of motion, of the forms

$$m_i x_i'' = \frac{\delta U}{\delta x_i}, \quad m_i y_i'' = \frac{\delta U}{\delta y_i}, \quad m_i z_i'' = \frac{\delta U}{\delta z_i}, \tag{4.}$$

had been completely integrated, they would give the $3n$ coordinates x, y, z and therefore also S as a function of the time t, the masses $m_1, \ldots m_n$ and the $6n$ initial constants a, b, c, a', b', c'; so that, by eliminating the $3n$ initial components of velocities a', b', c', we should in general obtain a relation between the $7n+2$ quantities S, t, m, x, y, z, a, b, c which would give S as a function of the time, the masses and the final and initial coordinates. We do not yet know the form of this last function, but we know its variation (2.) taken with respect to the $6n$ coordinates; and on account of the independence of their $6n$ variations we can resolve this expression (2.) into two groups containing each $3n$ equations: namely,

$$\frac{\delta S}{\delta x_i} = m_i x_i', \quad \frac{\delta S}{\delta y_i} = \qquad\qquad \frac{\delta S}{\delta z_i} \tag{5.}$$

and

$$\frac{\delta S}{\delta a_i} = -m_i a_i', \quad \frac{\delta S}{\delta b_i} = -m_i b_i', \quad \frac{\delta S}{\delta c_i} = -m_i c_i'; \tag{6.}$$

the first members being partial differential coefficients of the function S, which Mr. Hamilton calls the *Principal Function* of motion of the attracting or repelling system. He thinks that, if analysts had perceived this principal function S and these groups of equations (5.) and (6.), they must have perceived their importance. For the group (5.) expresses the $3n$ intermediate integrals of the known equations of motion (4.) under the form of $3n$ relations between the time t, the masses m, the varying coordinates x, y, z, the varying components of velocities x', y', z' and the $3n$ initial constants a, b, c; while the group (6.) expresses the $3n$ final integrals of the same known differential equations as $3n$ relations, with $6n$ initial and arbitrary constants a, b, c, a', b', c', between the time, the masses and the $3n$ varying coordinates. These $3n$ intermediate and $3n$ final integrals it was the problem of dynamics to discover. Mathematicians had found seven intermediate and none of the final integrals.

Professor Hamilton's solution of this long celebrated problem contains, indeed, one unknown function, namely, the *principal function* S, to the search and study of which he has reduced mathematical dynamics. This function must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations. Lagrange's function *states*, Mr. Hamilton's function would *solve* the problem. The one serves to form the *differential* equations of motion, the other would give their *integrals*. To assist in pursuing this new track and in discovering the form of this new function, Mr.

Hamilton remarks that it must satisfy the following partial differential equation of the first order and second degree, (the time being now made to vary,)

$$\frac{\delta S}{\delta t}+\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S}{\delta x}\right)^2+\left(\frac{\delta S}{\delta y}\right)^2+\left(\frac{\delta S}{\delta z}\right)^2\right\}=U; \qquad (7.)$$

which may rigorously be thus transformed, by the help of the equations (5.)

$$S=S_1+\int_0^t\left(U-\frac{\delta S_1}{\delta t}-\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S_1}{\delta x}\right)^2+\left(\frac{\delta S_1}{\delta y}\right)^2+\left(\frac{\delta S_1}{\delta z}\right)^2\right\}\right)dt$$
$$+\int_0^t\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S}{\delta x}-\frac{\delta S_1}{\delta x}\right)^2+\left(\frac{\delta S}{\delta y}-\frac{\delta S_1}{\delta y}\right)^2+\left(\frac{\delta S}{\delta z}-\frac{\delta S_1}{\delta z}\right)^2\right\}dt, \qquad (8.)$$

S_1 being any arbitrary function of the same quantities t, m, x, y, z, a, b, c, supposed only to vanish (like S) at the origin of time. If this arbitrary function S_1 be so chosen as to be an approximate value of the sought function S, (and it is always easy so to choose it,) then the two definite integrals in the formula (8.) are small, but the second is in general much smaller than the first; it may, therefore, be neglected in passing to a second approximation, and in calculating the first definite integral the following approximate forms of the equations (6.) may be used,

$$\frac{\delta S_1}{\delta a}=-ma', \quad \frac{\delta S_1}{\delta b}=-mb', \quad \frac{\delta S_1}{\delta c}=-mc' \qquad (9.)$$

In this manner, a first approximation may be successively and indefinitely corrected. And for the practical perfection of the method nothing further seems to be required, except to make this process of correction more easy and rapid in its applications.

Professor Hamilton has written two Essays on this new method in Dynamics, and one of them is already printed in the second part of the *Philosophical Transactions* (of London) for 1834.* The method did not at first present itself to him under quite so simple a form. He used at first a *Characteristic Function V*, more closely analogous to that optical function which he had discovered, and had denoted by the same letter, in his *Theory of Systems of Rays*. In both optics and dynamics this function was the quantity called *Action*, considered as depending (chiefly) on the final and initial coordinates. But when this *Action-Function* was employed in dynamics it involved an auxiliary quantity H, namely, the known constant part in the expression of half the living force of a system; and many troublesome eliminations were required in consequence, which are avoided by the new form of the method.

Mr. Hamilton thinks it worth while, however, to point out briefly a new property of this constant H, which suggests a new manner of expressing the differential and integral equations of motion of an attracting or repelling system. It is often useful to express the $3n$ rectangular coordinates $x_1, y_1, z_1, \ldots x_n, y_n, z_n$ as functions of $3n$ other marks of position, which may be thus denoted, $\eta_1, \eta_2, \ldots \eta_{3n}$; and if $3n$ other new variables, $\varpi_1, \varpi_2, \ldots \varpi_{3n}$, be introduced and defined as follows,

$$\varpi_i=\Sigma m\left(x'\frac{\delta x}{\delta\eta_i}+y'\frac{\delta y}{\delta\eta_i}+z'\frac{\delta z}{\delta\eta_i}\right), \qquad (10.)$$

[Pp. 103–161 of this volume.]

it is in general possible to express, reciprocally, the $6n$ variables x, y, z, x', y', z' as functions of these $6n$ new variables η, ϖ; it is, therefore, possible to express as such a function the quantity

$$H = \Sigma \frac{m}{2}(x'^2 + y'^2 + z'^2) - U \tag{11.}$$

under the form

$$H = F(\varpi_1, \dots \varpi_{3n}, \eta_1, \dots \eta_{3n}) - U(\eta_1, \dots \eta_{3n}), \tag{12.}$$

in which the part F is rational, integer and homogeneous of the second dimension with respect to the variables ϖ. Now Mr. Hamilton has found that when the quantity H is expressed in this last way as a function of these $6n$ new variables, η, ϖ, its variation may be put under this form,

$$\delta H = \Sigma(\eta' \delta\varpi - \varpi' \delta\eta), \tag{13.}$$

η', ϖ' denoting the first differential coefficients of these new variables η, ϖ, considered as functions of the time. The $3n$ differential equations of motion of the second order, (4.), between the rectangular coordinates and the time, for any attracting or repelling system, may therefore be generally transformed into twice that number of equations of the first order between these $6n$ variables and the time of the forms

$$\eta'_i = \frac{\delta H}{\delta \varpi_i}, \quad \varpi'_i = -\frac{\delta H}{\delta \eta_i}. \tag{14.}$$

To integrate this system of equations is to assign from them $6n$ relations between the time t, the $6n$ variables η_i, ϖ_i and their $6n$ initial values which may be called e_i, p_i. Mr. Hamilton resolves the problem, under this more general form, by the same *principal function* S as before, regarding it however as depending now on the new marks η, e of final and initial positions of the various points of the system. For, putting in this new notation

$$S = \int_0^t \left(\Sigma \varpi \frac{\delta H}{\delta \varpi} - H \right) dt, \tag{15.}$$

and considering the time as given, he finds now the formula of variation

$$\delta S = \Sigma(\varpi \delta\eta - p \delta e), \tag{16.}$$

and therefore the $6n$ separate equations

$$\varpi_i = \frac{\delta S}{\delta \eta_i}, \quad p_i = -\frac{\delta S}{\delta e_i}, \tag{17.}$$

which are forms for the sought relations.

Professor Hamilton thinks that these two formulae of variation, (13.) and (16.), namely,

$$\delta H = \Sigma(\eta' \delta\varpi - \varpi' \delta\eta) \tag{A.}$$

and

$$\delta S = \Sigma(\varpi \delta\eta - p \delta e), \tag{B.}$$

are worthy of attention as expressing, under concise and simple forms, the one the differential and the other the integral equations of motion of an attracting or repelling system. They may be extended to other problems of dynamics besides this capital problem. The expression H can always easily be found and the function S can be determined with indefinite accuracy by a method of successive approximation of the kind already explained.

The properties of his *Principal Function* are treated of more fully in his "Second Essay on a General Method in Dynamics*"; in which he has introduced several forms of a certain *Function of Elements*, connected with the Principal Function and with each other, and adapted to questions of perturbation; and has shown that for the perturbations of a ternary or multiple system with any laws of attraction or repulsion and with one predominant mass the differential equations of the varying elements of *all* the smaller masses may be expressed together, and as simply as in the usual way, by the coefficients of *one* disturbing function, (namely, the disturbing part of the whole expression H,) and may be integrated rigorously by a corollary of his general method.

V.

[ON NEARLY CIRCULAR ORBITS]

[1836.]

[Note Book 42.]

[CONTENTS

PAGE
1. The principal function for an elliptic orbit. 217
2. Approximations to the principal function in the case of nearly circular orbits. 220
3. The general expansion of κ in terms of ι. 225
4. The results expressed in polar coordinates. 228
5. The principal function in the three-body problem. 230
6. Approximations in the case of nearly circular orbits.] 235

[Jan. 6, 1836.]

[1. *The principal function for an elliptic orbit.*]

For the undisturbed motion of a planet, with Newton's law of attraction, let*

$$s=\int_0^t\left\{\frac{\xi_t'^2+\eta_t'^2}{2}+\frac{\mu}{r_t}\right\}dt,$$

ξ_t, η_t, r_t being the two rectangular heliocentric coordinates and the heliocentric distance of a planet, and μ being the sum of the two masses; so that the differential equations of motion of the second order are

$$\xi_t''=-\mu\xi_t r_t^{-3},\quad \eta_t''=-\mu\eta_t r_t^{-3},$$

and, by my general theorem of dynamics,†

$$\delta s_t=\xi_t'\delta\xi_t-\xi_0'\delta\xi_0+\eta_t'\delta\eta_t-\eta_0'\delta\eta_0+\left(\frac{\mu}{r_t}-\frac{\xi_t'^2+\eta_t'^2}{2}\right)\delta t,$$

in which the coefficient of δt is constant,

$$\frac{\mu}{r_t}-\tfrac{1}{2}(\xi_t'^2+\eta_t'^2)=\frac{\mu}{r_0}-\tfrac{1}{2}(\xi_0'^2+\eta_0'^2).$$

The function s will therefore satisfy the two partial differential equations following:†

$$\boxed{\frac{\delta s}{\delta t}+\frac{1}{2}\left(\frac{\delta s}{\delta\xi_t}\right)^2+\frac{1}{2}\left(\frac{\delta s}{\delta\eta_t}\right)^2=\frac{\mu}{r_t},\quad \frac{\delta s}{\delta t}+\frac{1}{2}\left(\frac{\delta s}{\delta\xi_0}\right)^2+\frac{1}{2}\left(\frac{\delta s}{\delta\eta_0}\right)^2=\frac{\mu}{r_0};}$$

s being considered as dependent on the 5 independent variables ξ_0, η_0, ξ_t, η_t and t, besides the constant μ.

---

* [See Appendix, Note 5, p. 624.]

† [Cf. pp. 160, 166.]

Let $r_t=\alpha+\beta$, $r_0=\alpha-\beta$, $(\xi_t-\xi_0)^2+(\eta_t-\eta_0)^2=(r+r_0)^2\sin\lambda^2$. I have found,* first, that s depends only on the three variables α, λ, t, being independent of β; and, secondly, that when λ is given, $s/\sqrt{\mu\alpha}$ depends only on $t\sqrt{\mu/\alpha^3}$; so that if we put

$$s=\kappa\sqrt{\mu\alpha}, \quad \frac{t}{2}\sqrt{\frac{\mu}{\alpha^3}}=\nu,$$

κ will be a certain function, of which the form is to be discovered, of the *two* variables λ and ν. And if the motion is *nearly circular*, these two variables λ, ν will be *nearly equal* to each other. (In some loose sheets of investigations begun in May, 1834, I have used the symbol ν to denote $t\sqrt{\frac{\mu}{\alpha^3}}$.)

Taking the variation of the expression $s=\kappa\sqrt{\mu\alpha}$, we find

$$=\sqrt{\mu\alpha}\left(\frac{\delta\kappa}{\delta\lambda}\delta\lambda+\frac{\delta\kappa}{\delta\nu}\delta\nu\right)+\frac{\kappa}{2}\sqrt{\frac{\mu}{\alpha}}\delta\alpha;$$

in which

$$\delta\alpha=\tfrac{1}{2}(\delta r_t+\delta r_0)=\frac{\xi_t\delta\xi_t+\eta_t\delta\eta_t}{2r_t}+\frac{\xi_0\delta\xi_0+\eta_0\delta\eta_0}{2r_0},$$

$$\delta\nu=\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\delta t-\frac{3t}{4}\sqrt{\frac{\mu}{\alpha^5}}\left(\frac{\xi_t\delta\xi_t+\eta_t\delta\eta_t}{2r_t}+\frac{\xi_0\delta\xi_0+\eta_0\delta\eta_0}{2r_0}\right),$$

and

$$\delta\lambda=\frac{(\xi_t-\xi_0)(\delta\xi_t-\delta\xi_0)+(\eta_t-\eta_0)(\delta\eta_t-\delta\eta_0)-(r+r_0)\sin\lambda^2(\delta r+\delta r_0)}{(r+r_0)^2\sin\lambda\cos\lambda}.$$

Hence

$$\delta s=\left(\frac{\kappa}{2}\sqrt{\frac{\mu}{\alpha}}-\frac{3t}{4}\frac{\mu}{\alpha^2}\frac{\delta\kappa}{\delta\nu}-\tan\lambda\sqrt{\frac{\mu}{\alpha}}\frac{\delta\kappa}{\delta\lambda}\right)\left(\frac{\xi_t\delta\xi_t+\eta_t\delta\eta_t}{2r_t}+\frac{\xi_0\delta\xi_0+\eta_0\delta\eta_0}{2r_0}\right)$$

$$\frac{\operatorname{cosec}2\lambda}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta\kappa}{\delta\lambda}\{(\xi_t-\xi_0)(\delta\xi_t-\delta\xi_0)+(\eta_t-\eta_0)(\delta\eta_t-\delta\eta_0)\}+\frac{\mu}{2\alpha}\frac{\delta\kappa}{\delta\nu}\delta t$$

therefore

$$\frac{\delta s}{\delta\xi_t}=A\frac{\xi_t}{r_t}+B(\xi_t-\xi_0), \quad \frac{\delta s}{\delta\eta_t}=A\frac{\eta_t}{r_t}+B(\eta_t-\eta_0),$$

$$\frac{\delta s}{\delta\xi_0}=A\frac{\xi_0}{r_0}-B(\xi_t-\xi_0), \quad \frac{\delta s}{\delta\eta_0}=A\frac{\eta_0}{r_0}-B(\eta_t-\eta_0), \quad \frac{\delta s}{\delta t}=C,$$

if we put, for abridgment,

$$A=\left(\frac{\kappa}{4}-\frac{\tan\lambda}{2}\frac{\delta\kappa}{\delta\lambda}-\frac{3\nu}{4}\frac{\delta\kappa}{\delta\nu}\right)\sqrt{\frac{\mu}{\alpha}}, \quad B=\frac{\operatorname{cosec}2\lambda}{2\alpha}\frac{\delta\kappa}{\delta\lambda}\sqrt{\frac{\mu}{\alpha}}, \quad C=\frac{\mu}{2\alpha}\frac{\delta\kappa}{\delta\nu};$$

and since

$$\left(\frac{\xi_t}{r_t}\right)^2+\left(\frac{\eta_t}{r_t}\right)^2=1, \quad \left(\frac{\xi_0}{r_0}\right)^2+\left(\frac{\eta_0}{r_0}\right)^2=1, \quad (\xi_t-\xi_0)^2+(\eta_t-\eta_0)^2=4\alpha^2\sin\lambda^2,$$

$$\frac{2}{r_t}\{\xi_t(\xi_t-\xi_0)+\eta_t(\eta_t-\eta_0)\}=\frac{r_t^2-r_0^2+4\alpha^2\sin\lambda^2}{r_t},$$

$$-\frac{2}{r_0}\{\xi_0(\xi_t-\xi_0)+\eta_0(\eta_t-\eta_0)\}=\frac{r_0^2-r_t^2+4\alpha^2\sin\lambda^2}{r_0},$$

---

* [See Appendix, Note 5, p. 624.]

the two partial differential equations become, when multiplied by r_t, r_0, and then added and subtracted, and divided by 2α, 2β,

$$\frac{\mu}{\alpha} = C + \tfrac{1}{2}A^2 + 2B^2\alpha^2 \sin\lambda^2 + 2AB\alpha\sin\lambda^2,$$

$$0 = C + \tfrac{1}{2}A^2 + 2B^2\alpha^2\sin\lambda^2 + 2AB\alpha.$$

Resolving these two equations for the two quantities $C + \frac{1}{2}A^2 + 2B^2\alpha^2\sin\lambda^2$ and $2AB\alpha$, we find

$$\frac{\alpha}{\mu}(2C + A^2 + 4B^2\alpha^2\sin\lambda^2) = -\frac{4AB}{\mu}\alpha^2 = 2\sec\lambda^2;$$

therefore

$$(A + 2B\alpha\sin\lambda)^2 = 2\left(\frac{\mu}{\alpha}\sec\lambda^2 - C - \frac{\mu}{\alpha}\sin\lambda\sec\lambda^2\right),$$

$$(A - 2B\alpha\sin\lambda)^2 = 2\left(\frac{\mu}{\alpha}\sec\lambda^2 - C + \frac{\mu}{\alpha}\sin\lambda\sec\lambda^2\right);$$

and, substituting for A, B, C their values, the equations become

$$\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\Big)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2 + 16\frac{\delta\kappa}{\delta\nu} = 32\sec\lambda^2,$$

$$\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\Big)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda = 0.$$

[Hence

$$\left(\kappa - 3\nu\frac{\delta\kappa}{\delta\nu}\right)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 16\left(\frac{\delta\kappa}{\delta\nu} - 2\right) = 0.]$$

These two equations ought to be compatible; that is, they ought to give equal values of $\frac{\delta^2\kappa}{\delta\lambda\,\delta\nu}$, whether we calculate by elimination $\frac{\delta\kappa}{\delta\lambda}$ and differentiate it with respect to ν, or calculate by elimination $\frac{\delta\kappa}{\delta\nu}$ and differentiate it with respect to λ. Without this previous elimination, we may differentiate the two partial differential equations separately and successively with respect to λ and ν, and then eliminate, between the four resulting equations, the three partial differential coefficients of the second order $\frac{\delta^2\kappa}{\delta\lambda^2}$, $\frac{\delta^2\kappa}{\delta\lambda\,\delta\nu}$, $\frac{\delta^2\kappa}{\delta\nu^2}$; the resulting equation ought to be true either identically or at least in virtue of the two partial differential equations of the first order. After some reduction it is easy to see that this condition of compatibility is

$$\begin{aligned}0 = -4\frac{\delta\kappa}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 16\tan\lambda^2(4\sec\lambda^2 - 1) + 2\left\{\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 - 8\right\}\tan\lambda^2\sec\lambda^2\left\{\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 4\right\} \\ - \sec\lambda^2\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\left\{(2\sec\lambda^2 - 1)\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 - 8\sec\lambda^2\right\},\end{aligned}$$

in which, by the first partial differential equation,

$$-4\frac{\delta\kappa}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 = 16\tan\lambda^2 + \left(\frac{\delta\kappa}{\delta\lambda}\right)^4\sec\lambda^2 - 8\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2,$$

so that the condition is in fact satisfied, and the two partial differential equations are compatible.

[2. *Approximations to the principal function in the case of nearly circular orbits.*]

Of these two partial differential equations, the second, namely

$$\left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda = 0,$$

seems well adapted for determining the coefficients of a series such as the following:

$$\kappa = \kappa_0 + \kappa_1(\nu-\lambda) + \kappa_2\frac{(\nu-\lambda)^2}{2} + \kappa_3\frac{(\nu-\lambda)^3}{2\,.\,3} + \text{etc.},$$

if it is possible, as it seems likely that it is, to represent κ by such a series for the case of nearly circular orbits.* Assuming this series, we have

$$\frac{\delta\kappa}{\delta\lambda} = \kappa_0' - \kappa_1 + (\nu-\lambda)(\kappa_1' - \kappa_2) + \frac{(\nu-\lambda)^2}{2}(\kappa_2' - \kappa_3) + \frac{(\nu-\lambda)^3}{2\,.\,3}(\kappa_3' - \kappa_4) + \text{etc.};$$

$$\frac{\delta\kappa}{\delta\nu} = \kappa_1 + (\nu-\lambda)\kappa_2 + \frac{(\nu-\lambda)^2}{2}\kappa_3 + \frac{(\nu-\lambda)^3}{2.\,3}\kappa_4 + \text{etc.};$$

κ_i being a function of λ only, and κ_i' being its first derivative with respect to λ. Thus, comparing the terms independent of $\nu-\lambda$, we find the equation

$$0 = \{\kappa_0 - 2\tan\lambda(\kappa_0' - \kappa_1) - 3\lambda\kappa_1\}(\kappa_0' - \kappa_1) + 8\tan\lambda;$$

in which, from the theory of circular motion, $\kappa_0 = 3\lambda$, and therefore $\kappa_0' = 3$, so that the equation becomes

$$0 = (\kappa_1 - 3)\{\kappa_1(3\lambda - 2\tan\lambda) + 3(2\tan\lambda - \lambda)\} + 8\tan\lambda,$$

that is,

$$\begin{aligned} 0 &= \kappa_1^2(3\lambda - 2\tan\lambda) + 12\kappa_1(\tan\lambda - \lambda) + 9\lambda - 10\tan\lambda \\ &= (\kappa_1 - 1)\{\kappa_1(3\lambda - 2\tan\lambda) - (9\lambda - 10\tan\lambda)\}; \end{aligned}$$

therefore either $\kappa_1 = 1$, or else

$$\kappa_1 = \frac{9\lambda - 10\tan\lambda}{3\lambda - 2\tan\lambda};$$

and in order to decide which of these two roots we are to take, it seems necessary, in the present method, to employ the other partial differential equation, namely,

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2 + 16\left(\frac{\delta\kappa}{\delta\nu} - 2\sec\lambda^2\right).$$

This equation gives, by making $\nu = \lambda$,

$$0 = \{\kappa_0 - 3\lambda\kappa_1 + 2\tan\lambda(\kappa_1 - \kappa_0')\}^2 + 4(\kappa_1 - \kappa_0')^2\sec\lambda^2 + 16(\kappa_1 - 2\sec\lambda^2);$$

therefore, since $\kappa_0 = 3\lambda$, $\kappa_0' = 3$,

$$\begin{aligned} 0 &= \{(3\lambda - 2\tan\lambda)\kappa_1 + 3(2\tan\lambda - \lambda)\}^2 + 4(\kappa_1 - 3)^2\sec\lambda^2 + 16(\kappa_1 - 2\sec\lambda^2) \\ &= \kappa_1^2\{(3\lambda - 2\tan\lambda)^2 + 4\sec\lambda^2\} + 2\kappa_1\{3(3\lambda - 2\tan\lambda)(2\tan\lambda - \lambda) - 12\sec\lambda^2 + 8\} \\ &\qquad + 9(2\tan\lambda - \lambda)^2 + 4\sec\lambda^2 \\ &= (\kappa_1 - 1)\{\kappa_1(9\lambda^2 - 12\lambda\tan\lambda + 8\tan\lambda^2 + 4) - (9\lambda^2 - 36\lambda\tan\lambda + 40\tan\lambda^2 + 4)\}; \end{aligned}$$

---

* [$\iota = \nu - \lambda$ is small when the orbit is nearly circular. See Appendix, Note 5, p. 626.]

therefore by this equation we have either $\kappa_1 = 1$, or

$$\kappa_1 = \frac{9(2\tan\lambda - \lambda)^2 + 4\sec\lambda^2}{(3\lambda - 2\tan\lambda)^2 + 4\sec\lambda^2};$$

and, since, by the former equation, we had either $\kappa_1 = 1$ or $\kappa_1 = \frac{9\lambda - 10\tan\lambda}{3\lambda - 2\tan\lambda}$, we must conclude that $\kappa_1 = 1$. If we then neglect $(\nu - \lambda)^2$, we have simply

$$\kappa = \kappa_0 + \kappa_1(\nu - \lambda) = 2\lambda + \nu,$$

because $\kappa_0 = 3\lambda$, $\kappa_1 = 1$.

Since the two partial differential equations separately gave, as we just now said, ambiguous results, but such that the ambiguity was removed when the results of the one equation were compared with those of the other, it seems evident that some combination of the two equations must be capable of giving unambiguous results, at least for the coefficient κ_1, and probably for the other coefficients. Accordingly, if we put $\frac{\delta\kappa}{\delta\lambda}$ under the form $\frac{\delta\kappa}{\delta\lambda} + \frac{\delta\kappa}{\delta\nu} - \frac{\delta\kappa}{\delta\nu}$, and then eliminate the square of that $\frac{\delta\kappa}{\delta\nu}$ which is not added to $\frac{\delta\kappa}{\delta\lambda}$, we shall accomplish the purpose proposed; and the same object may be accomplished still more simply by introducing a new variable ι, which shall be equal to $\nu - \lambda$, and shall therefore be small for nearly circular orbits, and thus changing $\frac{\delta\kappa}{\delta\nu}$ to $\frac{\delta\kappa}{\delta\iota}$ and $\frac{\delta\kappa}{\delta\lambda}$ to $\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}$ in the two partial differential equations, and then eliminating $\left(\frac{\delta\kappa}{\delta\iota}\right)^2$ between them.

In this manner we have

$$\kappa = \kappa_0 + \kappa_1\iota + \kappa_2\frac{\iota^2}{2} + \kappa_3\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

$$\frac{\delta\kappa}{\delta\lambda} = \kappa_0' + \kappa_1'\iota + \kappa_2'\frac{\iota^2}{2} + \kappa_3'\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

$$\frac{\delta\kappa}{\delta\iota} = \kappa_1 + \kappa_2\iota + \kappa_3\frac{\iota^2}{2} + \kappa_4\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

κ_i being some function of λ, and $\kappa_i' = \frac{\delta\kappa_i}{\delta\lambda}$; also the two partial differential equations become

$$0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}^2 + 4\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right)^2\sec\lambda^2 + 16\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right),$$

$$0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right) + 8\tan\lambda.$$

The coefficient of $\left(\frac{\delta\kappa}{\delta\iota}\right)^2$ in the first is $(2\tan\lambda - 3\lambda - 3\iota)^2 + 4\sec\lambda^2$, and in the second it is $-(2\tan\lambda - 3\lambda - 3\iota)$; and since the terms of the form const. $\times\,\iota\left(\frac{\delta\kappa}{\delta\iota}\right)^2$ cannot cause ambiguity, it is sufficient for our present purpose to multiply the first equation by $2\tan\lambda - 3\lambda$ and the second by $(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2$, and to add the products together. In this manner we get

$$\begin{aligned}0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}(2\tan\lambda - 3\lambda)\left\{\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}&\\ + 4\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right)\sec\lambda^2\left\{\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\} + 16\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right)(2\tan\lambda - 3\lambda)&\\ + 8\tan\lambda\{(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2\};&\end{aligned}$$

that is,

$$\begin{aligned}0 = \left\{\left(\kappa - 3\iota\frac{\delta\kappa}{\delta\iota}\right)(2\tan\lambda - 3\lambda) + 2(2 + 3\lambda\tan\lambda)\frac{\delta\kappa}{\delta\lambda} + (9\lambda^2 - 12\lambda\tan\lambda - 4)\frac{\delta\kappa}{\delta\iota}\right\}\left(3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)&\\ + 16(2\tan\lambda - 3\lambda)\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right) + 8\tan\lambda\{(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2\};&\end{aligned}$$

that is,

$$\begin{aligned}0 = \left(\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)^2(2\tan\lambda - 3\lambda) + (9\lambda^2 - 12\lambda\tan\lambda - 4)\left(\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa}{\delta\iota} - \frac{\delta\kappa}{\delta\lambda}\right)&\\ + 16(2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda.&\end{aligned}$$

Making, in this, $\kappa = \kappa_0$, $\frac{\delta\kappa}{\delta\lambda} = \kappa_0'$, $\frac{\delta\kappa}{\delta\iota} = \kappa_1$, $\iota = 0$, we get

$$\begin{aligned}0 = (\kappa_0 - 3\lambda\kappa_0')^2(2\tan\lambda - 3\lambda) + (9\lambda^2 - 12\lambda\tan\lambda - 4)(\kappa_0 - 3\lambda\kappa_0')(\kappa_1 - \kappa_0')&\\ + 16(2\tan\lambda - 3\lambda)\kappa_1 + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda;&\end{aligned}$$

and if we further employ the values $\kappa_0 = 3\lambda$, $\kappa_0' = 3$, we find

$$\begin{aligned}0 &= 36\lambda^2(2\tan\lambda - 3\lambda) - 6\lambda(9\lambda^2 - 12\lambda\tan\lambda - 4)(\kappa_1 - 3)\\ &\qquad + 16(2\tan\lambda - 3\lambda)\kappa_1 + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda\\ &= 2(\kappa_1 - 1)(9\lambda^2 + 4)\{4\tan\lambda - 3\lambda\};\end{aligned}$$

therefore $\kappa_1 = 1$, as before.

In general,

$$\begin{aligned}0 = \left\{-\kappa + 3\lambda\frac{\delta\kappa}{\delta\lambda} + 3\iota\frac{\delta\kappa}{\delta\iota}\right\}^2 + \left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)\left(\frac{\delta\kappa}{\delta\iota} - \frac{\delta\kappa}{\delta\lambda}\right)\left(-\kappa + 3\lambda\frac{\delta\kappa}{\delta\lambda} + 3\iota\frac{\delta\kappa}{\delta\iota}\right)&\\ + 16\frac{\delta\kappa}{\delta\iota} + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};&\end{aligned}$$

a formula which, when $\iota = 0$, $\kappa = 3\lambda$, $\frac{\delta\kappa}{\delta\lambda} = 3$, $\frac{\delta\kappa}{\delta\iota} = \kappa_1$, becomes

$$0 = 36\lambda^2 + 6\lambda\left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)(\kappa_1 - 3) + 16\kappa_1 + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda},$$

and gives $\kappa_1 = 1$. When we make $\kappa = 3\lambda + \iota$, $\frac{\delta\kappa}{\delta\lambda} = 3$, $\frac{\delta\kappa}{\delta\iota} = 1 + \kappa_2\iota$, and neglect ι^2, the formula becomes

$$0 = (6\lambda + 2\iota)^2 + \left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)(-2 + \kappa_2\iota)(6\lambda + 2\iota) + 16(1 + \kappa_2\iota) + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};$$

that is,

$$\begin{aligned}0 &= 24\lambda + (6\lambda\kappa_2 - 4)\left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right) + 16\kappa_2\\ &= \kappa_2\left\{36\lambda^2 + 6\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda} + 16\right\} - 4\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};\end{aligned}$$

that is, finally,

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda}.$$

Thus

$$\kappa = 3\lambda + \iota + \quad 4\tan\lambda - 3\lambda,$$

neglecting ι^3.

If we make the same substitutions, $\left(\kappa = 3\lambda + \iota,\ \frac{\delta\kappa}{\delta\lambda} = 3,\ \frac{\delta\kappa}{\delta\iota} = 1 + \kappa_2\iota\right)$, and still neglect ι^2, in the partial differential equation

$$0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda - 3\iota)\frac{\delta\kappa}{\delta\iota}\right\}\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right) + 8\tan\lambda,$$

we find

$$0 = \{3\lambda + \iota - 6\tan\lambda + (2\tan\lambda - 3\lambda - 3\iota)(1 + \kappa_2\iota)\}(2 - \kappa_2\iota) + 8\tan\lambda,$$

that is,

$$0 = 2\{1 - 3 + (2\tan\lambda - 3\lambda)\kappa_2\} + 4\kappa_2\tan\lambda,$$

or

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before. But it is remarkable that this value has been deduced *without ambiguity* from the tolerably simple partial differential equation which we set aside before as likely to give only ambiguous results.

If we return to the system of variables κ, λ, ν, instead of κ, λ, ι, and therefore resume the old partial differential equation

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda,$$

together with the old development

$$\kappa = \kappa_0 + \kappa_1(\nu - \lambda) + \kappa_2\frac{(\nu - \lambda)^2}{2} + \kappa_3\frac{(\nu - \lambda)^3}{2.3} + \text{etc.},$$

we may propose to determine the several coefficients κ_0, κ_1, κ_2, ... as the values of the partial differential coefficients

$$\frac{\delta^0\kappa}{\delta\nu^0},\quad \frac{\delta^1\kappa}{\delta\nu^1},\quad \frac{\delta^2\kappa}{\delta\nu^2},\quad \ldots$$

when ν becomes equal to λ. Thus, differentiating the equation with respect to ν once, we find

$$\begin{aligned} 0 &= -\frac{\delta}{\delta\nu}\left(\kappa\frac{\delta\kappa}{\delta\lambda}\right) + 2\tan\lambda\frac{\delta}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 3\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu} + 3\nu\frac{\delta}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu}\right) \\ &= 3\nu\left(\frac{\delta\kappa}{\delta\lambda}\frac{\delta^2\kappa}{\delta\nu^2} + \frac{\delta\kappa}{\delta\nu}\frac{\delta^2\kappa}{\delta\lambda\delta\nu}\right) + 2\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu} - \kappa\frac{\delta^2\kappa}{\delta\lambda\delta\nu} + 4\tan\lambda\frac{\delta\kappa}{\delta\lambda}\frac{\delta^2\kappa}{\delta\lambda\delta\nu}; \end{aligned}$$

which, if we make $\nu = \lambda$, $\kappa = \kappa_0$, $\frac{\delta\kappa}{\delta\lambda} = \kappa_0' - \kappa_1$, $\frac{\delta\kappa}{\delta\nu} = \kappa_1$, $\frac{\delta^2\kappa}{\delta\lambda\delta\nu} = \kappa_1' - \kappa_2$, $\frac{\delta^2\kappa}{\delta\nu^2} = \kappa_2$, becomes

$$0 = 3\lambda\{(\kappa_0' - \kappa_1)\kappa_2 + \kappa_1(\kappa_1' - \kappa_2)\} + 2\kappa_1(\kappa_0' - \kappa_1) - \kappa_0(\kappa_1' - \kappa_2) + 4\tan\lambda(\kappa_0' - \kappa_1)(\kappa_1' - \kappa_2),$$

that is, $0 = \{(3\lambda - 4\tan\lambda)(\kappa_0' - \kappa_1) - (3\lambda\kappa_1 - \kappa_0)\}(\kappa_2 - \kappa_1') + (3\lambda\kappa_1' + 2\kappa_1)(\kappa_0' - \kappa_1)$;
and this, when we make $\kappa_0 = 3\lambda$, $\kappa_1 = 1$, $\kappa_0' = 3$, $\kappa_1' = 0$, becomes

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before.

Accurately, for elliptic orbits, the relation between κ, λ, ι may be found by eliminating the auxiliary variables e and υ between the three following equations:*

$$\kappa = \frac{3\upsilon + e\sin\upsilon}{(1 - e\cos\upsilon)^{\frac{1}{2}}}; \quad \tan\frac{\lambda}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\upsilon}{2}; \quad \lambda + \iota = \frac{\upsilon - e\sin\upsilon}{(1 - e\cos\upsilon)^{\frac{3}{2}}},$$

and if the orbit be nearly circular, then ι and e will be small and of one common order.

Moreover, if a be the semiaxis major of the elliptic orbit and α (as before) the semi-sum of the two given distances r and r_0, then

$$\frac{\alpha}{\mathrm{a}} = \frac{\delta\kappa}{\delta\nu} = \frac{\delta\kappa}{\delta\iota},$$

λ being treated as a constant. Hence, in series,

$$\frac{\alpha}{\mathrm{a}} = \kappa_1 + \kappa_2\iota + \kappa_3\frac{\iota^2}{2} + \kappa_4\frac{\iota^3}{2\,.\,3} + \text{etc.};$$

and hence we might have easily foreseen that κ_1 must be equal to 1, because it is the value of α/a for circular motion. The finding of this value $\kappa_1 = 1$ was almost the only difficulty in the use of the partial differential equation

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda;$$

and if we assume, as we now see that we may,

$$\kappa = 2\lambda + \nu + \kappa_2\frac{(\nu-\lambda)^2}{2} + \kappa_3\frac{(\nu-\lambda)^3}{2\,.\,3} + \text{etc.},$$

the labour of calculating κ_2, κ_3 will be but trifling and the operation will become one of elementary facility. In fact, to resume, we have, neglecting ι^3,

$$\frac{\delta\kappa}{\delta\lambda} = 2 - \kappa_2\iota + (\kappa_2' - \kappa_3)\frac{\iota^2}{2},$$

$$\frac{\delta\kappa}{\delta\nu} = 1 + \kappa_2\iota + \kappa_3\frac{\iota^2}{2},$$

$$\kappa = 3\lambda + \iota + \kappa_2\frac{\iota^2}{2}, \quad \nu = \lambda + \iota;$$

and, if we substitute these values in the partial differential equation, we find

$$\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu} = -4\tan\lambda + \iota\{\kappa_2(2\tan\lambda - 3\lambda) - 2\}$$
$$+ \frac{\iota^2}{2}\{\kappa_3(2\tan\lambda - 3\lambda) - 5\kappa_2 - 2\kappa_2'\tan\lambda\}.$$

---

* [Cf. Appendix, Note 5, p. 625.]

Therefore, multiplying this by $\frac{\delta\kappa}{\delta\lambda} = 2 - \kappa_2\iota - \frac{\iota^2}{2}(\kappa_3 - \kappa_2')$, adding $8\tan\lambda$, and equating to zero the coefficients of ι and ι^2, we have

$$0 = \kappa_2(4\tan\lambda - 3\lambda) - 2, \quad 0 = \kappa_3(4\tan\lambda - 3\lambda) - \kappa_2^2(2\tan\lambda - 3\lambda) - 3\kappa_2 - 4\kappa_2'\tan\lambda;$$

the first equation gives

$$\kappa_2 = \frac{}{4\tan\lambda - 3\lambda},$$

as before, and therefore

$$\kappa_2' = \frac{-2(1 + 4\tan\lambda^2)}{(4\tan\lambda - 3\lambda)^2},$$

and then the second equation gives

$$\begin{aligned}\kappa_3(4\tan\lambda - 3\lambda)^3 &= 4(2\tan\lambda - 3\lambda) + 6(4\tan\lambda - 3\lambda) - 8(1 + 4\tan\lambda^2)\tan\lambda\\ &= -30\lambda + 24\tan\lambda - 32\tan\lambda^3,\end{aligned}$$

that is,

$$\kappa_3 = -\frac{32\tan\lambda^3 - 24\tan\lambda + 30\lambda}{(4\tan\lambda - 3\lambda)^3}$$

Hence, neglecting ι^4,

$$\kappa = 3\lambda + \iota + \frac{}{4\tan\lambda - 3\lambda} \; \frac{\iota^3}{3} \; \frac{16\tan\lambda^3 - 12\tan\lambda + 15\lambda}{(4\tan\lambda - 3\lambda)^3}$$

It is extremely remarkable how little laborious this process is, as compared with any other likely to occur to a mathematician, for the elimination of e and υ between the three equations

$$\kappa = \frac{3\upsilon + e\sin\upsilon}{(1 - e\cos\upsilon)^{\frac{1}{2}}}, \quad \tan\frac{\lambda}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\upsilon}{2}, \quad \nu = \lambda + \iota = \frac{\upsilon - e\sin\upsilon}{(1 - e\cos\upsilon)^{\frac{3}{2}}}.$$

[3. *The general expansion of κ in terms of ι.*]

To calculate now the general expression for κ_4 from the partial differential equation

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda.$$

We have*

$$\kappa = \Sigma_{(i)0}^{\infty}\kappa_i[0]^{-i}\iota^i;$$

$$\frac{\delta\kappa}{\delta\lambda} = \Sigma_{(i)0}^{\infty}\kappa_i'[0]^{-i}\iota^i - \Sigma_{(i)1}^{\infty}\kappa_i[0]^{-(i-1)}\iota^{i-1} = \Sigma_{(i)0}^{\infty}(\kappa_i' - \kappa_{i+1})[0]^{-i}\iota^i = -\Sigma_{(i)0}^{\infty}a_i[0]^{-i}\iota^i;$$

$$\begin{aligned}\nu\frac{\delta\kappa}{\delta\nu} = (\lambda + \iota)\frac{\delta\kappa}{\delta\iota} &= \Sigma_{(i)1}^{\infty}(\lambda + \iota)\kappa_i[0]^{-(i-1)}\iota^{(i-1)} = \Sigma_{(i)0}^{\infty}(\lambda + \iota)\kappa_{i+1}[0]^{-i}\iota^i\\ &= \Sigma_{(i)0}^{\infty}(\lambda\kappa_{i+1} + i\kappa_i)[0]^{-i}\iota^i;\end{aligned}$$

therefore,

$$-\kappa + 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + 3\nu\frac{\delta\kappa}{\delta\nu} = \Sigma_{(i)0}^{\infty}[0]^{-i}\iota^i\{-\kappa_i + 2\tan\lambda(\kappa_i' - \kappa_{i+1}) + 3\lambda\kappa_{i+1} + 3i\kappa_i\} = \Sigma_{(i)0}^{\infty}b_i[0]^{-i}\iota^i$$

---

* $[[0]^{-i} = 1/i!$ See *Mathematical Papers*, Vol. I, Appendix, Note 4, p. 468.]

The partial differential equation gives

$$\{\Sigma_{(i)0}^{\infty} a_i [0]^{-i} \iota^i\} \times \{\Sigma_{(i)0}^{\infty} b_i [0]^{-i} \iota^i\} = -8 \tan \lambda;$$

that is,

$$a_0 b_0 = -8 \tan \lambda;$$

and when $i > 0$,

$$\Sigma_{(i')0}^{i} a_{i-i'} b_{i'} [0]^{-(i-i')} [0]^{-i'} = 0,$$

a formula which may also be thus written*

$$0 = (a+b)_i .$$

Here

$$a_i = \kappa_{i+1} - \kappa_i', \quad b_i = (3\lambda - 2\tan\lambda)\,\kappa_{i+1} + 2\kappa_i' \tan\lambda + (3i-1)\,\kappa_i;$$
$$a_0 = \kappa_1 - \kappa_0', \quad b_0 = -\kappa_0 + 2\kappa_0' \tan\lambda + (3\lambda - 2\tan\lambda)\,\kappa_1;$$

and, since we know by the circular values of κ and a that $\kappa_0 = 3\lambda$, $\kappa_1 = 1$, $\kappa_0' = 3$, $\kappa_1' = 0$, we have

$$a_0 = -2, \quad b_0 = 4\tan\lambda,$$

values which satisfy, accordingly, the condition $a_0 b_0 = -8\tan\lambda$. Again

$$a_1 = \kappa_2 - \kappa_1' = \kappa_2, \quad b_1 = 2\kappa_1 + 2\kappa_1' \tan\lambda + (3\lambda - 2\tan\lambda)\,\kappa_2 = (3\lambda - 2\tan\lambda)\,\kappa_2 + 2;$$

therefore the condition

$$(a+b)_1 = a_0 b_1 + a_1 b_0 = 0$$

becomes

$$0 = 2\,(2\tan\lambda - 3\lambda)\,\kappa_2 - 4 + 4\kappa_2 \tan\lambda,$$

that is,

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before. Again,

$$b_2 = (3\lambda - 2\tan\lambda)\,\kappa_3 + 2\kappa_2' \tan\lambda + 5\kappa_2, \quad a_2 = \kappa_3 - \kappa_2',$$
$$b_3 = (3\lambda - 2\tan\lambda)\,\kappa_4 + 2\kappa_3' \tan\lambda + 8\kappa_3, \quad a_3 = \kappa_4 - \kappa_3';$$

therefore the condition

$$(a+b)_2 = a_2 b_0 + 2a_1 b_1 + a_0 b_2 = 0$$

becomes

$$0 = 4\tan\lambda\,(\kappa_3 - \kappa_2') + 2\kappa_2^2\,(3\lambda - 2\tan\lambda) + 4\kappa_2 + 2\,(2\tan\lambda - 3\lambda)\,\kappa_3 - 4\kappa_2' \tan\lambda - 10\kappa_2,$$

that is,

$$0 = (4\tan\lambda - 3\lambda)\,\kappa_3 - 4\kappa_2' \tan\lambda + \kappa_2^2\,(3\lambda - 2\tan\lambda) - 3\kappa_2,$$

in which

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda}, \quad \kappa_2' = \frac{-2\,(1 + 4\tan\lambda^2)}{(4\tan\lambda - 3\lambda)^2}.$$

Therefore

$$0 = \tfrac{1}{2}\kappa_3\,(4\tan\lambda - 3\lambda)^3 + 4\tan\lambda\,(1 + 4\tan\lambda^2) + 2\,(3\lambda - 2\tan\lambda) - 3\,(4\tan\lambda - 3\lambda),$$

and so

$$\kappa_3 = \frac{-2\,(16\tan\lambda^3 - 12\tan\lambda + 15\lambda)}{(4\tan\lambda - 3\lambda)^3},$$

as before.

---

* $\left[(a+b)_i = a_i b_0 + i a_{i-1} b_i + \frac{i\,(i-1)}{2} a_{i-2} b_2 + \ldots + a_0 b_i .\right]$

It might have been convenient to have calculated a_1, b_1 as functions of λ, and a_2, b_2 as functions of λ and κ_3, before we proceeded to calculate κ_3 from the equation $0=(a+b)_2$. In this way we should have had

$$\begin{aligned}
&a_0=-2, \quad b_0=4\tan\lambda;\\
&a_1=\kappa_2=\frac{2}{4\tan\lambda-3\lambda}, \quad b_1=(3\lambda-2\tan\lambda)\,\kappa_2+2=\frac{4\tan\lambda}{4\tan\lambda-3\lambda},\\
&a_2=\kappa_3-\kappa_2'=\kappa_3+\frac{2\,(1+4\tan\lambda^2)}{(4\tan\lambda-3\lambda)^2},\\
&b_2=(3\lambda-2\tan\lambda)\,\kappa_3+2\kappa_2'\tan\lambda+5\kappa_2\\
&\quad=(3\lambda-2\tan\lambda)\,\kappa_3 \quad \frac{16\tan\lambda^3-36\tan\lambda+30\lambda}{(4\tan\lambda-3\lambda)^2},
\end{aligned}$$

so that the condition $0=(a+b)_2=a_2b_0+2a_1b_1+a_0b_2$ would have become

$$0=2\kappa_3\,\{4\tan\lambda-3\lambda\}+\frac{4}{(4\tan\lambda-3\lambda)^2}\{2\tan\lambda\,(1+4\tan\lambda^2)+4\tan\lambda+8\tan\lambda^3-18\tan\lambda+15\lambda\},$$

that is,

$$\kappa_3=\frac{-2\,\{16\tan\lambda^3-12\tan\lambda+15\lambda\}}{(4\tan\lambda-3\lambda)^3},$$

as before.

In like manner, substituting this value of κ_3 in a_2 and b_2, we find

$$\begin{aligned}
\tfrac{1}{2}a_2\,(4\tan\lambda-3\lambda)^3&=-(16\tan\lambda^3-12\tan\lambda+15\lambda)+(4\tan\lambda^2+1)\,(4\tan\lambda-3\lambda)\\
&=16\tan\lambda-12\lambda\tan\lambda^2-18\lambda,
\end{aligned}$$

therefore

$$a_2=\frac{4\,(8\tan\lambda-6\lambda\tan\lambda^2-9\lambda)}{(4\tan\lambda-3\lambda)^3};$$

$$\begin{aligned}
\tfrac{1}{2}b_2\,(4\tan\lambda-3\lambda)^3&=(2\tan\lambda-3\lambda)\,(16\tan\lambda^3-12\tan\lambda+15\lambda)\\
&\qquad-(4\tan\lambda-3\lambda)\,(8\tan\lambda^3-18\tan\lambda+15\lambda)\\
&=48\tan\lambda^2-24\lambda\tan\lambda^3-48\lambda\tan\lambda,
\end{aligned}$$

therefore

$$b_2=\frac{48\tan\lambda\,(2\tan\lambda-\lambda\tan\lambda^2-2\lambda)}{(4\tan\lambda-3\lambda)^3}.$$

Hence

$$3a_2b_1+3a_1b_2=\frac{48\tan\lambda\,(20\tan\lambda-12\lambda\tan\lambda^2-21\lambda)}{(4\tan\lambda-3\lambda)^4}.$$

Also

$$\begin{aligned}
a_3b_0+a_0b_3&=4\tan\lambda\,(\kappa_4-\kappa_3')+2\,(2\tan\lambda-3\lambda)\,\kappa_4-4\kappa_3'\tan\lambda-16\kappa_3\\
&=2\,(4\tan\lambda-3\lambda)\,\kappa_4-8\,(\kappa_3'\tan\lambda+2\kappa_3);
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{6}\,(4\tan\lambda-3\lambda)^4\kappa_3'&=(4\tan\lambda^2+1)\,(16\tan\lambda^3-12\tan\lambda+15\lambda)\\
&\qquad-(4\tan\lambda-3\lambda)\,(16\tan\lambda^4+12\tan\lambda^2+1)\\
&=-80\tan\lambda^3-16\tan\lambda+48\lambda\tan\lambda^4+96\lambda\tan\lambda^2+18\lambda,
\end{aligned}$$

therefore

$$\kappa_3'=\frac{12\,(-40\tan\lambda^3-8\tan\lambda+24\lambda\tan\lambda^4+48\lambda\tan\lambda^2+9\lambda)}{(4\tan\lambda-3\lambda)^4}.$$

Substituting these values in the condition

$$(a+b)_3 = a_0 b_3 + 3a_1 b_2 + 3a_2 b_1 + a_3 b_0 = 0,$$

we get

$$\begin{aligned} -\tfrac{1}{8}(4\tan\lambda - 3\lambda)^5 \kappa_4 &= 3\tan\lambda\,(80\tan\lambda^3 + 36\tan\lambda - 48\lambda\tan\lambda^4 - 108\lambda\tan\lambda^2 - 39\lambda) \\ &\quad + 2\,(64\tan\lambda^4 - 48\tan\lambda^2 - 48\lambda\tan\lambda^3 + 96\lambda\tan\lambda - 45\lambda^2) \\ &= 368\tan\lambda^4 + 12\tan\lambda^2 - 144\lambda\tan\lambda^5 - 420\lambda\tan\lambda^3 + 75\lambda\tan\lambda - 90\lambda^2, \end{aligned}$$

that is, finally,

$$\boxed{\kappa_4 = \frac{8}{(4\tan\lambda - 3\lambda)^5}\{-4\tan\lambda^2(92\tan\lambda^2 + 3) + 3\lambda\tan\lambda\,(48\tan\lambda^4 + 140\tan\lambda^2 - 25) + 90\lambda^2\}.}$$

Thus if we only neglect ι^5, or $(\nu-\lambda)^5$, we have

$$\boxed{\begin{aligned} \kappa = 3\lambda + \iota + \frac{\iota^2}{4\tan\lambda - 3\lambda} - \frac{\iota^3}{3}\cdot\frac{16\tan\lambda^3 - 12\tan\lambda + 15\lambda}{(4\tan\lambda - 3\lambda)^3} \\ + \frac{\iota^4}{3}\cdot\frac{-4\tan\lambda^2(92\tan\lambda^2+3) + 3\lambda\tan\lambda\,(48\tan\lambda^4 + 140\tan\lambda^2 - 25) + 90\lambda^2}{(4\tan\lambda - 3\lambda)^5}. \end{aligned}}$$

[4. *The results expressed in polar coordinates.*]

Here, to recapitulate, λ is the half of the angle $\theta_t - \theta_0$ described in the time t, *reduced* to the case of equal final and initial radii vectores r_t, r_0; so that

$$(\xi_t - \xi_0)^2 + (\eta_t - \eta_0)^2 + (\zeta_t - \zeta_0)^2 = (r_t + r_0)^2 \sin\lambda^2,$$

(ξ_t, η_t, ζ_t being the three rectangular heliocentric coordinates of a planet at the time t, and ξ_0, η_0, ζ_0 being their values at the time 0), and

$$\sin\lambda^2 = \left(\sin\frac{\theta_t - \theta_0}{2}\right)^2 + \left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \left(\cos\frac{\theta_t - \theta_0}{2}\right)^2;$$

ι is the following quantity, which is small for nearly circular orbits (such as here considered),

$$\iota = -\lambda + t\sqrt{\frac{2\mu}{(r_t + r_0)^3}},$$

μ being a constant such that the differential equations of relative motion are

$$\xi_t'' = -\mu\xi_t r_t^{-3}, \quad \eta_t'' = -\mu\eta_t r_t^{-3}, \quad \zeta_t'' = -\mu\zeta_t r_t^{-3};$$

and

$$\kappa = \frac{\sqrt{2}s}{\sqrt{\mu(r_t + r_0)}},$$

s being the principal function of the motion, or the definite integral

$$s = \int_0^t \left\{\frac{\xi_t'^2 + \eta_t'^2}{2} + \frac{\mu}{r_t}\right\} dt,$$

which, when considered as depending on ξ_t, η_t, ζ_t, ξ_0, η_0, ζ_0 and t, has (by my general theorem) its variation equal to

$$\xi_t'\delta\xi_t - \xi_0'\delta\xi_0 + \eta_t'\delta\eta_t - \eta_0'\delta\eta_0 + \zeta_t'\delta\zeta_t - \zeta_0'\delta\zeta_0 + \frac{\mu}{2a}\delta t,$$

2a being the axis major of the ellipse.

Accurately $$\cos 2\lambda = \cos(\theta_t-\theta_0) - 2\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2$$

therefore

$$\sin\{\lambda-\tfrac{1}{2}(\theta_t-\theta_0)\}\sin\{\lambda+\tfrac{1}{2}(\theta_t-\theta_0)\} = \left(\frac{r_t-r_0}{r_t+r_0}\right)^2\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2$$

Therefore, nearly, $\lambda = \frac{\theta_t-\theta_0}{2}$; more nearly

$$\lambda = \frac{\theta_t-\theta_0}{2} + \frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2 \operatorname{cotan}\frac{\theta_t-\theta_0}{2};$$

more nearly still

$$\begin{aligned}\lambda &= \frac{\theta_t-\theta_0}{2} + \left(\frac{r_t-r_0}{r_t+r_0}\right)^2\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2 \operatorname{cosec}\left\{\theta_t-\theta_0+\frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\operatorname{cotan}\frac{\theta_t-\theta}{2}\right. \\ &= \frac{\theta_t-\theta_0}{2} + \frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\operatorname{cotan}\frac{\theta_t-\theta_0}{2}\left\{1-\frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\operatorname{cotan}\frac{\theta_t-\theta_0}{2}\operatorname{cotan}(\theta_t-\theta_0)\right\} \\ &\quad \frac{\theta_t-\theta_0}{2} + \frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\operatorname{cotan}\frac{\theta_t-\theta_0}{2} - \frac{1}{4}\left(\frac{r_t-r_0}{r_t+r_0}\right)^4\left(\operatorname{cotan}\frac{\theta_t-\theta_0}{2}\right)^2\operatorname{cotan}(\theta_t-\theta_0);\end{aligned}$$

and since, in general,

$$(\operatorname{cotan} x)^2\operatorname{cotan} 2x = \frac{(\cos x)^2\cos 2x}{(\sin x)^2\sin 2x} = \tfrac{1}{2}(\operatorname{cotan} x^3 - \operatorname{cotan} x)$$

we have, neglecting $\left(\frac{r_t-r_0}{r_t+r_0}\right)^6$,

$$\boxed{\lambda = \frac{\theta_t-\theta_0}{2} + \frac{1}{2}\left(\frac{r_t-r_0}{r_t+r_0}\right)^2\operatorname{cotan}\frac{\theta_t-\theta_0}{2} + \frac{1}{8}\left(\frac{r_t-r_0}{r_t+r_0}\right)^4\left\{\operatorname{cotan}\frac{\theta_t-\theta_0}{2} - \left(\operatorname{cotan}\frac{\theta_t-\theta_0}{2}\right)^3\right\}.}$$

If we neglect the square of the eccentricity, then

$$\lambda = \theta_t-\theta_0, \quad \kappa = \theta_t-\theta_0+t\ \overline{(r_t+r_0)^3},$$

$$s = (\theta_t-\theta_0)\sqrt{\frac{\mu(r_t+r_0)}{2}} + \frac{\mu t}{r_t+r_0}.$$

And if, in this expression, we change r_t into $r_0+r_t-r_0$, and still neglect the square of the eccentricity, we find more simply (but less symmetrically)

$$s = (\theta_t-\theta_0)\sqrt{\mu r_0} + \frac{\mu t}{2r_0};$$

because the coefficient of r_t-r_0 is equal to

$$\frac{\theta_t-\theta_0}{4}\sqrt{\frac{\mu}{r_0}} - \frac{\mu t}{4r_0^2} = \left(\theta_t-\theta_0-t\sqrt{\frac{\mu}{r_0^3}}\right)\sqrt{\frac{\mu}{16r_0}},$$

and $\theta_t-\theta_0-t\sqrt{\frac{\mu}{r_0^3}}$ is small of the order of the eccentricity.

We can carry s to a further approximation as follows. We have

$$\frac{s}{\sqrt{\mu r_0}} = \kappa\sqrt{\frac{}{r_0}} = \kappa \qquad \frac{}{2r_0} = \kappa\sqrt{1 + \frac{}{2r_0}},$$

accurately; therefore, nearly,

$$\frac{}{\mu r_0} = \kappa\left\{1 + \frac{r_t - r_0}{4r_0} - \frac{1}{2}\left(\frac{r_t - r_0}{4r_0}\right)^2\right.$$

neglecting the cube of the eccentricity. Neglecting the same cube,

$$\kappa = 3\lambda + \iota + \frac{}{4\tan\lambda - 3\lambda}; \quad \lambda = \frac{\vartheta_t - \theta_0}{2} + \frac{1}{2}\left(\frac{r_t - r_0}{2r_0}\right)^2 \operatorname{cotan}\frac{nt}{2};$$

where we have put $n = \sqrt{\mu/r_0^3}$;

$$\iota = -\lambda + t\sqrt{\frac{2\mu}{(r_t + r_0)^3}} = -\lambda + \frac{nt}{2}\left\{1 + \frac{r_t - r_0}{2r_0}\right\}^{-\frac{3}{2}} = -\lambda + \frac{nt}{2}\left\{1 - \frac{3}{2}\left(\frac{r_t - r_0}{2r_0}\right) + \frac{15}{8}\left(\frac{r_t - r_0}{2r_0}\right)^2\right\};$$

therefore

$$\kappa = \frac{3nt}{2} + (\theta_t - \theta_0 - nt) - \frac{3nt}{4}\left(\frac{r_t - r_0}{2r_0}\right) + \left(\frac{r_t - r_0}{2r_0}\right)^2\left\{\frac{15nt}{16} + \operatorname{cotan}\frac{nt}{2}\right\} + \frac{\frac{1}{2}\left\{\theta_t - \theta_0 - nt + \frac{3nt}{2}\left(\frac{r_t - r_0}{2r_0}\right)\right\}^2}{8\tan\frac{1}{2}nt - 3nt}$$

and hence

$$\frac{s}{\sqrt{\mu r_0}} = \frac{3nt}{2} + (\theta_t - \theta_0 - nt) + \left(\frac{r_t - r_0}{2r_0}\right)^2\left\{\frac{3nt}{8} + \operatorname{cotan}\frac{nt}{2} + \frac{\frac{9}{8}n^2t^2}{8\tan\frac{1}{2}nt - 3nt}\right.$$

$$+ \frac{4\tan\frac{nt}{2}\left(\frac{r_t - r_0}{2r_0}\right)(\theta_t - \theta_0 - nt) + \frac{1}{2}(\theta_t - \theta_0 - nt)^2}{8\tan\frac{1}{2}nt - 3nt}.$$

[5. *The principal function in the three-body problem.*]

Also, if we consider the system of the sun, M, and 2 planets, m, $m_,$, the Principal Function S of the relative motion of this system must satisfy the two partial differential equations following:*

$$\frac{\delta S}{\delta t} + \frac{1}{2}\left(\frac{1}{m} + \frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta \xi_t}\right)^2 + \left(\frac{\delta S}{\delta \eta_t}\right)^2 + \left(\frac{\delta S}{\delta \zeta_t}\right)^2\right\} + \frac{1}{2}\left(\frac{1}{m_,} + \frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta \xi_{,t}}\right)^2 + \left(\frac{\delta S}{\delta \eta_{,t}}\right)^2 + \left(\frac{\delta S}{\delta \zeta_{,t}}\right)^2\right.$$

$$+ \frac{1}{M}\left(\frac{\delta S}{\delta \xi_t}\frac{\delta S}{\delta \xi_{,t}} + \frac{\delta S}{\delta \eta_t}\frac{\delta S}{\delta \eta_{,t}} + \frac{\delta S}{\delta \zeta_t}\frac{\delta S}{\delta \zeta_{,t}}\right) = \frac{mM}{\sqrt{r_t^2 + \zeta_t^2}} + \frac{m_, M}{\sqrt{r_{,t}^2 + \zeta_{,t}^2}} + \frac{mm_,}{\sqrt{\{(\xi_{,t} - \xi_t)^2 + (\eta_{,t} - \eta_t)^2 + (\zeta_{,t} - \zeta_t)^2\}}};$$

$$\frac{\delta S}{\delta t} + \frac{1}{2}\left(\frac{1}{m} + \frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta \xi_0}\right)^2 + \left(\frac{\delta S}{\delta \eta_0}\right)^2 + \left(\frac{\delta S}{\delta \zeta_0}\right)^2\right\} + \frac{1}{2}\left(\frac{1}{m_,} + \frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta \xi_{,0}}\right)^2 + \left(\frac{\delta S}{\delta \eta_{,0}}\right)^2 + \left(\frac{\delta S}{\delta \zeta_{,0}}\right)^2\right.$$

$$+ \frac{1}{M}\left(\frac{\delta S}{\delta \xi_0}\frac{\delta S}{\delta \xi_{,0}} + \frac{\delta S}{\delta \eta_0}\frac{\delta S}{\delta \eta_{,0}} + \frac{\delta S}{\delta \zeta_0}\frac{\delta S}{\delta \zeta_{,0}}\right) = \frac{mM}{\sqrt{r_0^2 + \zeta_0^2}} + \frac{m_, M}{\sqrt{r_{,0}^2 + \zeta_{,0}^2}} + \frac{mm_,}{\sqrt{\{(\xi_{,0} - \xi_0)^2 + (\eta_{,0} - \eta_0)^2 + (\zeta_{,0} - \zeta_0)^2\}}}.$$

[These equations follow easily from (D^6.), p. 153.]

If we put

$$\xi_t = r_t \cos\theta_t, \quad \eta_t = r_t \sin\theta_t, \quad \xi_0 = r_0 \cos\theta_0, \quad \eta_0 = r_0 \sin\theta_0,$$

and similarly

$$\xi_{,t} = r_{,t} \cos\theta_{,t}, \quad \eta_{,t} = r_{,t} \sin\theta_{,t}, \quad \xi_{,0} = r_{,0} \sin\theta_{,0}, \quad \eta_{,0} = r_{,0} \sin\theta_{,0},$$

we have

$$\frac{\delta S}{\delta \xi_t} = \cos\theta_t \frac{\delta S}{\delta r_t} - \frac{\sin\theta_t}{r_t}\frac{\delta S}{\delta \theta_t}, \quad \frac{\delta S}{\delta \eta_t} = \sin\theta_t \frac{\delta S}{\delta r_t} + \frac{\cos\theta_t}{r_t}\frac{\delta S}{\delta \theta_t};$$

and therefore

$$\left(\frac{\delta S}{\delta \xi_t}\right)^2 + \left(\frac{\delta S}{\delta \eta_t}\right)^2 = \left(\frac{\delta S}{\delta r_t}\right)^2 + \frac{1}{r_t^2}\left(\frac{\delta S}{\delta \theta_t}\right)^2, \quad \left(\frac{\delta S}{\delta \xi_{,t}}\right)^2 + \left(\frac{\delta S}{\delta \eta_{,t}}\right)^2 = \left(\frac{\delta S}{\delta r_{,t}}\right)^2 + \frac{1}{r_{,t}^2}\left(\frac{\delta S}{\delta \theta_{,t}}\right)^2,$$

$$\frac{\delta S}{\delta \xi_t}\frac{\delta S}{\delta \xi_{,t}} + \frac{\delta S}{\delta \eta_t}\frac{\delta S}{\delta \eta_{,t}} = \left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}} + \frac{1}{r_t r_{,t}}\frac{\delta S}{\delta \theta_t}\frac{\delta S}{\delta \theta_{,t}}\right)\cos(\theta_{,t} - \theta_t) + \left(\frac{1}{r_t}\frac{\delta S}{\delta \theta_t}\frac{\delta S}{\delta r_{,t}} - \frac{1}{r_{,t}}\frac{\delta S}{\delta \theta_{,t}}\frac{\delta S}{\delta r_t}\right)\sin(\theta_{,t} - \theta_t).$$

The first of the above two partial differential equations gives

$$\begin{aligned} 0 = {} & \frac{\delta S}{\delta t} + \frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_t}\right)^2 + \left(\frac{\delta S}{r_t \delta \theta_t}\right)^2 + \left(\frac{\delta S}{\delta \zeta_t}\right)^2\right\} + \frac{M+m_{,}}{2Mm_{,}}\left\{\left(\frac{\delta S}{\delta r_{,t}}\right)^2 + \left(\frac{\delta S}{r_{,t}\delta \theta_{,t}}\right)^2 + \left(\frac{\delta S}{\delta \zeta_{,t}}\right)^2\right\} - \frac{Mm}{\sqrt{r_t^2 + \zeta_t^2}} \\ & - \frac{Mm_{,}}{\sqrt{r_{,t}^2 + \zeta_{,t}^2}} + \frac{\cos(\theta_{,t} - \theta_t)}{M}\left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}} + \frac{\delta S}{r_t \delta \theta_t}\frac{\delta S}{r_{,t}\delta \theta_{,t}}\right) + \frac{\sin(\theta_{,t} - \theta_t)}{M}\left(\frac{\delta S}{\delta r_{,t}}\frac{\delta S}{r_t \delta \theta_t} - \frac{\delta S}{\delta r_t}\frac{\delta S}{r_{,t}\delta \theta_{,t}}\right) \\ & + \frac{1}{M}\frac{\delta S}{\delta \zeta_t}\frac{\delta S}{\delta \zeta_{,t}} - \frac{mm_{,}}{\sqrt{r_t^2 + r_{,t}^2 - 2r_t r_{,t}\cos(\theta_{,t} - \theta_t) + (\zeta_{,t} - \zeta_t)^2}}, \end{aligned}$$

and the second may be similarly transformed. The first five terms may be made to vanish by employing elliptic values; the remaining terms give the perturbations.

We shall assume that the inclinations are neglected, and shall put $\zeta_t = \zeta_0 = 0$, $\zeta_{,t} = \zeta_{,0} = 0$. It is easy to perceive that in a system attracting according to Newton's law all the linear coordinates may be multiplied by any one common factor l (besides altering all the positions by any common motion of rotation), and all the masses may be multiplied by any other common factor $l_{,}$, provided that the time t is multiplied by the factor $l^{\frac{3}{2}} l_{,}^{-\frac{1}{2}}$. And then, in the general expression for the principal function of relative motion,

$$S = \int_0^t T_{,} dt + \int_0^t U \, dt,$$

the coefficient U will be multiplied by $l_{,}^2/l$ and $T_{,}$ will be multiplied by $l_{,} l^2/l^3 l_{,}^{-1} = l_{,}^2/l$, and therefore $U dt$ and $T_{,} dt$ and finally S itself will each be multiplied by $l_{,}^{\frac{3}{2}} l^{\frac{1}{2}}$. Hence, in the present system, in which the orbits are both in one plane,

$$S = M^{\frac{3}{2}} r_0^{\frac{1}{2}} \times \text{funct.}\left(\frac{m}{M}, \frac{m_{,}}{M}, \frac{r_t}{r_0}, \frac{r_{,0}}{r_0}, \frac{r_{,t}}{r_0}, \theta_t - \theta_0, \theta_{,0} - \theta_0, \theta_{,t} - \theta_0, nt\right)$$

rigorously, or, if we choose to express it so, then

$$S = Mm\sqrt{\frac{r_t + r_0}{2(M+m)}}\,\kappa + Mm_{,}\sqrt{\frac{r_{,t} + r_{,0}}{2(M+m_{,})}}\,\kappa_{,} + \frac{mm_{,}}{\sqrt{M}} W,$$

$$W = \sqrt{r_0 + r_{,0}}\,\text{funct.}\left(\lambda, \lambda_{,}, \iota, \iota_{,}, \frac{r_t - r_0}{r_t + r_0}, \frac{r_{,t} - r_{,0}}{r_{,t} + r_{,0}}, \theta_{,0} - \theta_0\right),$$

κ, λ, ι having the same significance as before for the first orbit and $\kappa_{,}$, $\lambda_{,}$, $\iota_{,}$ similar quantities for the second orbit. In rigour, W involves m/M, $m_{,}/M$, but these small ratios may be treated as equal to zero in W, if we wish only to deduce those perturbations which are of the first order with respect to the disturbing masses. For greater symmetry we might put

$$W=\sqrt{\frac{r_{,t}+r_t+r_{,0}+r_0}{2}}\,\text{funct.}\left(\frac{\theta_{,t}-\theta_{\iota}+\theta_{,0}-\theta_0}{2},\ \lambda,\ \lambda_{,},\ \iota,\ \iota_{,},\ \frac{r_t-r_0}{r_t+r_0},\ \frac{r_{,t}-r_{,0}}{r_{,t}+r_{,0}}\right).$$

Thus, if we put for abridgment, as before, $\alpha=\frac{1}{2}(r_{\iota}+r_0)$, $\mu=M+m$, and similarly

$$\alpha_{,}=\tfrac{1}{2}(r_{,t}+r_{,0}),\quad \mu_{,}=M+m_{,};$$

and if we put also

$$\beta=\frac{r_t-r_0}{r_t+r_0},\quad \beta_{,}=\frac{r_{,t}-r_{,0}}{r_{,t}+r_{,0}},\quad \vartheta=\frac{\theta_{,t}+\theta_{,0}}{2}-\frac{\theta_t+\theta_0}{2},$$

we may write

$$S=Mm\sqrt{\frac{\alpha}{\mu}}\kappa+Mm_{,}\sqrt{\frac{\alpha_{,}}{\mu_{,}}}\kappa_{,}+mm_{,}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\,\psi(\vartheta,\ \lambda,\ \lambda_{,},\ \iota,\ \iota_{,},\ \beta,\ \beta_{,});$$

and may form two partial differential equations relatively to the function ψ as follows.

S is explicitly a function of α, $\alpha_{,}$, λ, $\lambda_{,}$, β, $\beta_{,}$, ι, $\iota_{,}$ and ϑ, involving also the masses; its variation may therefore be thus expressed

$$\delta S=\frac{\delta S}{\delta\alpha}\delta\alpha+\frac{\delta S}{\delta\alpha_{,}}\delta\alpha_{,}+\frac{\delta S}{\delta\lambda}\delta\lambda+\frac{\delta S}{\delta\lambda_{,}}\delta\lambda_{,}+\frac{\delta S}{\delta\beta}\delta\beta+\frac{\delta S}{\delta\beta_{,}}\delta\beta_{,}+\frac{\delta S}{\delta\iota}\delta\iota+\frac{\delta S}{\delta\iota_{,}}\delta\iota_{,}+\frac{\delta S}{\delta\vartheta}\delta\vartheta.$$

Now

$$\sin\lambda^2=\left(\sin\frac{\theta_t-\theta_0}{2}\right)^2+\beta^2\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2,$$

and so

$$\delta\lambda=\frac{1-\beta^2}{2}\frac{\sin(\theta_t-\theta_0)}{\sin 2\lambda}(\delta\theta_t-\delta\theta_0)+\frac{1+\cos(\theta_t-\theta_0)}{\sin 2\lambda}\beta\,\delta\beta.$$

Also

$$\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2=\frac{\cos\lambda^2}{1-\beta^2},\quad \left(\sin\frac{\theta_t-\theta_0}{2}\right)^2=\frac{\sin\lambda^2-\beta^2}{1-\beta^2},$$

that is,

$$\frac{1+\cos(\theta_t-\theta_0)}{\sin 2\lambda}=\frac{\cotan\lambda}{1-\beta^2},\quad \frac{(1-\beta^2)\sin(\theta_t-\theta_0)}{\sin 2\lambda}=+\sqrt{1-\beta^2\operatorname{cosec}\lambda^2},$$

therefore

$$\delta\lambda=\tfrac{1}{2}\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\,(\delta\theta_t-\delta\theta_0)+\cotan\lambda\frac{\beta\,\delta\beta}{1-\beta^2}.$$

Now

$$\beta=\frac{r_t-r_0}{r_t+r_0},\quad \alpha=\tfrac{1}{2}(r_t+r_0),$$

and hence

$$\delta\beta=\frac{\delta r_t-\delta r_0}{2\alpha}-\frac{\beta}{2\alpha}(\delta r_t+\delta r_0)=\frac{1-\beta}{2\alpha}\delta r_t-\frac{1+\beta}{2\alpha}\delta r_0.$$

Also

$$-\lambda+\tfrac{1}{2}t\sqrt{\frac{\mu}{\alpha^3}},$$

therefore

$$\delta\iota = -\delta\lambda + \quad {}^{\mu}\,\delta t - {}^{3t}\quad /{}^{\mu}\ \delta\alpha = -\delta\lambda + \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\,\delta t - \frac{3(\lambda+\iota)}{2\alpha}\,\delta\alpha.$$

Hence

$$\frac{\delta S}{\delta\alpha}\delta\alpha+\frac{\delta S}{\delta\beta}\delta\beta+\frac{\delta S}{\delta\lambda}\delta\lambda+\frac{\delta S}{\delta\iota}\delta\iota=\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}\delta t+\left\{\frac{\delta S}{\delta\alpha}-\frac{3}{2}\frac{\lambda+\iota}{\alpha}\frac{\delta S}{\delta\iota}\right\}\frac{\delta r_t+\delta r_0}{2}$$
$$+\left\{\frac{\delta S}{\delta\beta}+\operatorname{cotan}\lambda\,\frac{\beta}{1-\beta^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\}\frac{(1-\beta)\,\delta r_t-(1+\beta)\,\delta r_0}{2\alpha}+\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\frac{\delta\theta_t-\delta\theta_0}{2},$$

that is,

$$\delta S=\frac{1}{2}\left(\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}+\sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\frac{\delta S}{\delta\iota_{,}}\right)\delta t+\left(\frac{\delta S}{\delta\alpha}-\frac{3}{2}\frac{\lambda+\iota}{\alpha}\frac{\delta S}{\delta\iota}\right)\frac{\delta r_t+\delta r_0}{2}+\left(\frac{\delta S}{\delta\alpha_{,}}-\frac{3}{2}\frac{\lambda_{,}+\iota_{,}}{\alpha_{,}}\frac{\delta S}{\delta\iota_{,}}\right)\frac{\delta r_{,t}+\delta r_{,0}}{2}$$
$$+\left\{\frac{\delta S}{\delta\beta}+\frac{\beta\operatorname{cotan}\lambda}{1-\beta^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\}\frac{(1-\beta)\,\delta r_t-(1+\beta)\,\delta r_0}{2\alpha}+\left\{\frac{\delta S}{\delta\beta_{,}}+\frac{\beta_{,}\operatorname{cotan}\lambda_{,}}{1-\beta_{,}^2}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right.\right.$$
$$\times\frac{(1-\beta_{,})\,\delta r_{,t}-(1+\beta_{,})\,\delta r_{,0}}{2\alpha_{,}}+\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\frac{\delta\theta_t-\delta\theta_0}{2}+\sqrt{1-\beta_{,}^2\operatorname{cosec}\lambda_{,}^2}$$
$$\times\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)\frac{\delta\theta_{,t}-\delta\theta_{,0}}{2}+\frac{\delta S}{\delta\vartheta}\frac{\delta\theta_{,t}-\delta\theta_t+\delta\theta_{,0}-\delta\theta_0}{2}.$$

From this equation we immediately deduce the partial derivatives of S with respect to the variables t, r_t, $r_{,t}$, θ_t, $\theta_{,t}$, r_0, $r_{,0}$, θ_0, and $\theta_{,0}$, and we have then to substitute these values in the two partial differential equations (for the case of null inclinations)

$$0=\frac{\delta S}{\delta t}+\frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_t}\right)^2+\left(\frac{\delta S}{r_t\delta\theta_t}\right)^2\right\}+\frac{M+m_{,}}{2Mm_{,}}\left\{\left(\frac{\delta S}{\delta r_{,t}}\right)^2+\left(\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)^2\right\}-\frac{Mm}{r_t}-\frac{Mm_{,}}{r_{,t}}$$
$$+\frac{\cos(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}}+\frac{\delta S}{r_t\delta\theta_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)+\frac{\sin(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_{,t}}\frac{\delta S}{r_t\delta\theta_t}-\frac{\delta S}{\delta r_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)$$
$$\frac{mm_{,}}{\sqrt{r_{,t}^2+r_t^2-2r_{,t}r_t\cos(\theta_{,t}-\theta_t)}}\cdot$$

and

$$0=\frac{\delta S}{\delta t}+\frac{M+m}{2Mm}\quad\left.\frac{\delta S}{\delta r_0}\right)^2+\left(\frac{\delta S}{r_0\delta\theta_0}\right)^2\right\}+\frac{M+m_{,}}{2Mm_{,}}\left\{\left(\frac{\delta S}{\delta r_{,0}}\right)^2+\left(\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right)^2\right\}-\frac{Mm}{r_0}-\frac{Mm_{,}}{r_{,0}}$$
$$+\frac{\cos(\theta_{,0}-\theta_0)}{M}\left(\frac{\delta S}{\delta r_0}\frac{\delta S}{\delta r_{,0}}+\frac{\delta S}{r_0\delta\theta_0}\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right)+\frac{\sin(\theta_{,0}-\theta_0)}{M}\left(\frac{\delta S}{\delta r_{,0}}\frac{\delta S}{r_0\delta\theta_0}-\frac{\delta S}{\delta r_0}\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right)$$
$$\frac{mm_{,}}{\sqrt{r_{,0}^2+r_0^2-2r_{,0}r_0\cos(\theta_{,0}-\theta_0)}}$$

We have

$$\theta_{,t}-\theta_t=\vartheta+\frac{\theta_{,t}-\theta_{,0}}{2}-\frac{\theta_t-\theta_0}{2},\quad \theta_{,0}-\theta_0=\vartheta-\frac{\theta_{,t}-\theta_{,0}}{2}+\frac{\theta_t-\theta}{2}$$

and

$$\cos\frac{\theta_t-\theta_0}{2}=\cos\lambda\qquad \sin\frac{\theta_t-\theta_0}{2}\quad\frac{\sin\lambda}{\sqrt{1-\beta^2}}\sqrt{1-\beta^2\operatorname{cosec}\lambda^2},$$
$$\cos\frac{\theta_{,t}-\theta_{,0}}{2}=\frac{\cos\lambda_{,}}{\sqrt{1-\beta_{,}^2}},\quad \sin\frac{\theta_{,t}-\theta_{,0}}{2}=\frac{\sin\lambda_{,}}{\sqrt{1-\beta_{,}^2}}\sqrt{1-\beta_{,}^2}$$

therefore

$$\cos(\theta_{,t}-\theta_{t})=K\cos\vartheta-L\sin\vartheta,\quad \sin(\theta_{,t}-\theta_{t})=K\cos\vartheta+L\sin\vartheta,$$

where we have put for abridgment

$$K=\cos\left(\frac{\theta_{,t}-\theta_{,0}}{2}-\frac{\theta_{t}-\theta_{0}}{2}\right)=\frac{\cos\lambda\cos\lambda_{,}+\sin\lambda\sin\lambda_{,}\sqrt{1-\beta^{2}\operatorname{cosec}\lambda^{2}}\sqrt{1-\beta_{,}^{2}\operatorname{cosec}\lambda_{,}^{2}}}{\sqrt{1-\beta^{2}}\sqrt{1-\beta_{,}^{2}}}$$

$$L=\sin\left(\frac{\theta_{,t}-\theta_{,0}}{2}-\frac{\theta_{t}-\theta_{0}}{2}\right)=\frac{\cos\lambda\sin\lambda_{,}\sqrt{1-\beta_{,}^{2}\operatorname{cosec}\lambda_{,}^{2}}-\sin\lambda\cos\lambda_{,}\sqrt{1-\beta^{2}\operatorname{cosec}\lambda^{2}}}{\sqrt{1-\beta^{2}}\sqrt{1-\beta_{,}^{2}}}.$$

Thus, making the above substitutions, the first partial differential equation becomes *rigorously* (for the case of null inclinations)

$$\begin{aligned}
0=&\frac{1}{2}\sqrt{\frac{\mu}{\alpha^{3}}}\frac{\delta S}{\delta\iota}+\frac{1}{2}\sqrt{\frac{\mu_{,}}{\alpha_{,}^{3}}}\frac{\delta S}{\delta\iota_{,}}-\frac{Mm}{\alpha(1+\beta)}-\frac{Mm_{,}}{\alpha_{,}(1+\beta_{,})}\\
&-\frac{mm_{,}}{\sqrt{\alpha^{2}(1+\beta)^{2}+\alpha_{,}^{2}(1+\beta_{,})^{2}-2\alpha\alpha_{,}(1+\beta)(1+\beta_{,})(K\cos\vartheta-L\sin\vartheta)}}\\
&+\frac{M+m}{8Mm\alpha^{2}}\left\{\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\operatorname{cotan}\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\}^{2}\\
&+\frac{M+m_{,}}{8Mm_{,}\alpha_{,}^{2}}\left\{\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+(1-\beta_{,})\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}+\frac{\beta_{,}\operatorname{cotan}\lambda_{,}}{1+\beta_{,}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)\right\}^{2}\\
&+\frac{M+m}{8Mm\alpha^{2}(1+\beta)^{2}}\left\{\sqrt{1-\beta^{2}\operatorname{cosec}\lambda^{2}}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right.\\
&+\frac{M+m_{,}}{8Mm_{,}\alpha_{,}^{2}(1+\beta_{,})^{2}}\left\{\sqrt{1-\beta_{,}^{2}\operatorname{cosec}\lambda_{,}^{2}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)+\frac{\delta S}{\delta\vartheta}\right.\\
&+\frac{K\cos\vartheta-L\sin\vartheta}{4M\alpha\alpha_{,}}\left\{\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\operatorname{cotan}\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right.\\
&\times\left\{\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+(1-\beta_{,})\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}+\frac{\beta_{,}\operatorname{cotan}\lambda_{,}}{1+\beta_{,}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)\right\}\\
&+\frac{K\cos\vartheta-L\sin\vartheta}{4M\alpha\alpha_{,}(1+\beta)(1+\beta_{,})}\left\{\sqrt{1-\beta^{2}\operatorname{cosec}\lambda^{2}}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right\}\\
&\times\left\{\sqrt{1-\beta_{,}^{2}\operatorname{cosec}\lambda_{,}^{2}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)+\frac{\delta S}{\delta\vartheta}\right\}+\frac{K\sin\vartheta+L\cos\vartheta}{4M\alpha\alpha_{,}(1+\beta)}\left\{\sqrt{1-\beta^{2}\operatorname{cosec}\lambda^{2}}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right.\\
&\times\left\{\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+(1-\beta_{,})\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}+\frac{\beta_{,}\operatorname{cotan}\lambda_{,}}{1+\beta_{,}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)\right\}\\
&-\frac{K\sin\vartheta+L\cos\vartheta}{4M\alpha\alpha_{,}(1+\beta_{,})}\left\{\sqrt{1-\beta_{,}^{2}\operatorname{cosec}\lambda_{,}^{2}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}\right)+\frac{\delta S}{\delta\vartheta}\right\}\\
&\qquad\qquad\left.\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\operatorname{cotan}\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\};
\end{aligned}$$

and the second partial differential equation may be *rigorously* formed from this by merely changing the signs of β, $\beta_{,}$, ϑ.

[6. *Approximations in the case of nearly circular orbits.*]

If we entirely neglect β and $\beta_{,}$, or rather suppose them rigorously to vanish, then the partial differential equations become

$$
\begin{aligned}
0 = {} & \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}+\frac{1}{2}\sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\frac{\delta S}{\delta\iota_{,}}-\frac{Mm}{\alpha}-\frac{Mm_{,}}{\alpha_{,}}-\frac{mm_{,}}{\sqrt{\alpha^2+\alpha_{,}^2-2\alpha\alpha_{,}\cos(\vartheta+\lambda_{,}-\lambda)}} \\
& +\frac{M+m}{8Mm\alpha^2}\left\{\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right\}^2+\frac{M+m_{,}}{8Mm_{,}\alpha_{,}^2}\left\{\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}\right\}^2 \\
& +\frac{M+m}{8Mm\alpha^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)^2+\frac{M+m_{,}}{8Mm_{,}\alpha_{,}^2}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}+\frac{\delta S}{\delta\vartheta}\right)^2 \\
& +\frac{\cos(\vartheta+\lambda_{,}-\lambda)}{4M\alpha\alpha_{,}}\left(\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right)\left(\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}\right) \\
& +\frac{\cos(\vartheta+\lambda_{,}-\lambda)}{4M\alpha\alpha_{,}}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}+\frac{\delta S}{\delta\vartheta}\right) \\
& +\frac{\sin(\vartheta+\lambda_{,}-\lambda)}{4M\alpha\alpha_{,}}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)\left(\alpha_{,}\frac{\delta S}{\delta\alpha_{,}}+\frac{\delta S}{\delta\beta_{,}}-\tfrac{3}{2}(\lambda_{,}+\iota_{,})\frac{\delta S}{\delta\iota_{,}}\right) \\
& -\frac{\sin(\vartheta+\lambda_{,}-\lambda)}{4M\alpha\alpha_{,}}\left(\frac{\delta S}{\delta\lambda_{,}}-\frac{\delta S}{\delta\iota_{,}}+\frac{\delta S}{\delta\vartheta}\right)\left(\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right)
\end{aligned}
$$

together with the other similar partial differential equation derived from this by changing the sign of ϑ and those of $\frac{\delta S}{\delta\vartheta}, \frac{\delta S}{\delta\beta}, \frac{\delta S}{\delta\beta_{,}}$. In these equations no power of any mass is neglected nor any of ι, $\iota_{,}$; but if we put

$$S = Mm\kappa\sqrt{\frac{\alpha}{\mu}}+Mm_{,}\kappa_{,}\sqrt{\frac{\alpha_{,}}{\mu_{,}}}+mm_{,}\psi\sqrt{\frac{\alpha+\alpha_{,}}{M}}$$

and neglect $m^2m_{,}$ and $mm_{,}^2$, the equations become, after being reduced by the partial differential equations of elliptic motion and divided by $mm_{,}/16$,

$$
\begin{aligned}
0 = {} & \sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\alpha\psi}{\alpha+\alpha_{,}}\pm 2\frac{\delta\psi}{\delta\beta}-3\lambda\frac{\delta\psi}{\delta\iota}-3\iota\frac{\delta\psi}{\delta\iota}\right) \\
& +\sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\kappa_{,}-3\lambda_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}-3\iota_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\left(\frac{\alpha_{,}\psi}{\alpha+\alpha_{,}}\pm 2\frac{\delta\psi}{\delta\beta_{,}}-3\lambda_{,}\frac{\delta\psi}{\delta\iota_{,}}-3\iota_{,}\frac{\delta\psi}{\delta\iota_{,}}\right) \\
& +4\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+4\sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\frac{\delta\kappa_{,}}{\delta\lambda_{,}}-\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\left(\frac{\delta\psi}{\delta\lambda_{,}}\pm\frac{\delta\psi}{\delta\vartheta}\right) \\
& -4\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}-2\right)\frac{\delta\psi}{\delta\iota}-4\sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\sqrt{\frac{\alpha+\alpha_{,}}{M}}\left(\frac{\delta\kappa_{,}}{\delta\lambda_{,}}-\frac{\delta\kappa_{,}}{\delta\iota_{,}}-2\right)\frac{\delta\psi}{\delta\iota_{,}} \\
& +\frac{M}{\sqrt{\mu\alpha}\sqrt{\mu_{,}\alpha_{,}}}\cos(\pm\vartheta+\lambda_{,}-\lambda)\left\{\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\kappa_{,}-3\lambda_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}-3\iota_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\right. \\
& \qquad\qquad\left.+4\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa_{,}}{\delta\lambda_{,}}-\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\right\} \\
& +\frac{}{\sqrt{\mu\alpha}\sqrt{\mu_{,}\alpha_{,}}}\sin(\pm\vartheta+\lambda_{,}-\lambda)\left\{\left(\kappa_{,}-3\lambda_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}-3\iota_{,}\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)-\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa_{,}}{\delta\lambda_{,}}-\frac{\delta\kappa_{,}}{\delta\iota_{,}}\right)\right\} \\
& \frac{16}{\sqrt{\alpha^2+\alpha_{,}^2-2\alpha\alpha_{,}\cos(\pm\vartheta+\lambda_{,}-\lambda)}}
\end{aligned}
$$

This double equation is rigorous so far as powers of ι are concerned, though $\beta, \beta_,$ and $mm_,^2$, $m^2m_,$ have been neglected. But if we change κ and $\kappa_,$ to their approximate values $3\lambda+\iota$ and $3\lambda_,+\iota_,$, and then neglect $\iota, \iota_,$, the double equation becomes

$$0=\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+\frac{1}{2}\sqrt{\frac{\mu_,}{\alpha_,^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\psi}{\delta\lambda_,}\pm\frac{\delta\psi}{\delta\vartheta}\right)+\frac{M\cos(\pm\vartheta+\lambda_,-\lambda)}{\sqrt{\mu\alpha}\sqrt{\mu_,\alpha_,}}-\frac{1}{\sqrt{\alpha^2+\alpha_,^2-2\alpha\alpha_,\cos(\pm\vartheta+\lambda_,-\lambda)}}.$$

If we now put

$$n=\sqrt{\frac{\mu}{\alpha^3}},\quad n_,=\sqrt{\frac{\mu_,}{\alpha_,^3}},$$

and therefore

$$\sqrt{\mu\alpha}=n\alpha^2,\quad \sqrt{\mu_,\alpha_,}=n_,\alpha_,^2,\quad \frac{\lambda}{n}=\frac{\lambda_,}{n_,},\quad \frac{\lambda\alpha^{\frac{3}{2}}}{\sqrt{\mu}}=\frac{\lambda_,\alpha_,^{\frac{3}{2}}}{\sqrt{\mu_,}}:$$

and make as a sufficient approximation $\mu/M=\mu_,/M=1$, we shall change the double equation to the following:

$$0=\sqrt{\lambda^{-\frac{2}{3}}+\lambda_,^{-\frac{2}{3}}}\left\{\lambda\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+\lambda_,\left(\frac{\delta\psi}{\delta\lambda_,}\pm\frac{\delta\psi}{\delta\vartheta}\right)\right\}+2\lambda^{\frac{1}{3}}\lambda_,^{\frac{1}{3}}\cos(\pm\vartheta+\lambda_,-\lambda)-\frac{2}{\sqrt{\lambda^{-\frac{4}{3}}+\lambda_,^{-\frac{4}{3}}-2\lambda^{-\frac{2}{3}}\lambda_,^{-\frac{2}{3}}\cos(\pm\vartheta+\lambda_,-\lambda)}},$$

that is,

$$0=\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}}\left(\frac{\delta\psi}{\delta\lambda_,}+\frac{\lambda}{\lambda_,}\frac{\delta\psi}{\delta\lambda}\right)+2\left(\frac{\lambda}{\lambda_,}\right)^{\frac{1}{3}}\cos(\lambda_,-\lambda)\cos\vartheta-\frac{1}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(\vartheta+\lambda_,-\lambda)}}$$

$1+($

and

$$0=\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}}\left(1-\frac{\lambda}{\lambda_,}\right)\frac{\delta\psi}{\delta\vartheta}-2\left(\frac{\lambda}{\lambda_,}\right)^{\frac{1}{3}}\sin(\lambda_,-\lambda)\sin\vartheta-\cdot$$

$$+\frac{}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(-\vartheta+\lambda_,-\lambda)}}$$

We have seen that $\frac{\lambda_,}{\lambda}=\frac{n_,}{n}=\left(\frac{\alpha}{\alpha_,}\right)^{\frac{3}{2}}$; if then we put for a moment $\alpha/\alpha_,=\alpha'$ and $\lambda_,-\lambda=\lambda'$, we shall have

$$\frac{\lambda_,}{\lambda}=\alpha'^{\frac{3}{2}},\quad \lambda=\frac{\lambda'}{\alpha'^{\frac{3}{2}}-1},\quad \lambda_,=\frac{\alpha'^{\frac{3}{2}}\lambda'}{\alpha'^{\frac{3}{2}}-1},$$

$$\frac{\delta\psi}{\delta\lambda_,}=\frac{\delta\psi}{\delta\lambda'}+\frac{2}{3}\frac{\delta\psi}{\delta\alpha'}\frac{\lambda_,^{-\frac{1}{3}}}{\lambda^{\frac{2}{3}}},\quad \frac{\delta\psi}{\delta\lambda}=-\frac{\delta\psi}{\delta\lambda'}-\frac{2}{3}\frac{\delta\psi}{\delta\alpha'}\frac{\lambda_,^{\frac{2}{3}}}{\lambda^{\frac{5}{3}}};$$

and hence

$$\frac{\delta\psi}{\delta\lambda_{,}}+\frac{\lambda}{\lambda_{,}}\frac{\delta\psi}{\delta\lambda}=\left(1-\frac{\lambda}{\lambda_{,}}\right)\frac{\delta\psi}{\delta\lambda'}.$$

Therefore

$$0=\sqrt{1+\alpha'}\,(1-\alpha'^{-\frac{3}{2}})\frac{\delta\psi}{\delta\lambda'}+\frac{2}{\sqrt{\alpha'}}\cos\lambda'\cos\vartheta-\frac{1}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(\vartheta+\lambda')}}-\frac{1}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(-\vartheta+\lambda')}},$$

and

$$0=\sqrt{1+\alpha'}\,(1-\alpha'^{-\frac{3}{2}})\frac{\delta\psi}{\delta\vartheta}-\frac{2}{\sqrt{\alpha'}}\sin\lambda'\sin\vartheta-\frac{1}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(\vartheta+\lambda')}}+\frac{1}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(-\vartheta+\lambda')}}.$$

Treating α' as constant, this gives

$$\sqrt{1+\alpha'}\,(1-\alpha'^{-\frac{3}{2}})\,\delta\psi=\frac{2}{\sqrt{\alpha'}}(\sin\lambda'\sin\vartheta\,\delta\vartheta-\cos\lambda'\cos\vartheta\,\delta\lambda')+\frac{\delta\vartheta+\delta\lambda'}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(\vartheta+\lambda')}}$$
$$+\frac{\delta\lambda'-\delta\vartheta}{\sqrt{1+\alpha'^{2}-2\alpha'\cos(-\vartheta+\lambda')}},$$

that is,

$$\sqrt{1+\alpha'}\,(1-\alpha'^{-\frac{3}{2}})\,\psi=\text{funct.}(\alpha')-\frac{2}{\sqrt{\alpha'}}\sin\lambda'\cos\vartheta+\int_{\vartheta-\lambda'}^{\vartheta+\lambda'}\frac{d\theta'}{\sqrt{1+\alpha'^{2}-2\alpha'\cos\theta'}}.$$

But the arbitrary function of α', introduced by this integration, must be identically zero because otherwise it would not vanish with λ', as it must do, since ψ does so, independently of α' and ϑ. Thus, finally

$$\sqrt{\alpha+\alpha_{,}}\,\psi=\sqrt{\alpha_{,}}\sqrt{1+\alpha'}\,\psi=-\frac{2\alpha\alpha_{,}}{\alpha^{\frac{3}{2}}-\alpha_{,}^{\frac{3}{2}}}\cos\vartheta\sin(\lambda_{,}-\lambda)+\frac{\alpha^{\frac{3}{2}}\alpha_{,}^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}-\alpha_{,}^{\frac{3}{2}}}\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\frac{d\theta'}{\sqrt{\alpha^{2}+\alpha_{,}^{2}-2\alpha\alpha_{,}\cos\theta'}},$$

which may also be written

$$\psi=\frac{(\lambda^{-\frac{2}{3}}+\lambda_{,}^{-\frac{2}{3}})^{-\frac{1}{2}}}{\lambda_{,}-\lambda}\left\{-2\lambda^{\frac{1}{3}}\lambda_{,}^{\frac{1}{3}}\cos\vartheta\sin(\lambda_{,}-\lambda)+\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\frac{d\theta'}{\sqrt{(\lambda^{-\frac{4}{3}}+\lambda_{,}^{-\frac{4}{3}}-2\lambda^{-\frac{2}{3}}\lambda_{,}^{-\frac{2}{3}}\cos\theta')}}\right\},$$

or in the alternate form

$$\boxed{\psi=\left\{\left(\frac{\lambda_{,}}{\lambda}\right)^{\frac{1}{3}}+\left(\frac{\lambda}{\lambda_{,}}\right)^{\frac{1}{3}}\right\}^{-\frac{1}{2}}\frac{\sqrt{\lambda\lambda_{,}}}{\lambda_{,}-\lambda}\left\{-2\cos\vartheta\sin(\lambda_{,}-\lambda)+\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\left\{\left(\frac{\lambda_{,}}{\lambda}\right)^{\frac{2}{3}}+\left(\frac{\lambda}{\lambda_{,}}\right)^{\frac{2}{3}}-2\cos\theta'\right\}^{-\frac{1}{2}}d\theta'\right\}.}$$

VI.

THEORY OF THE MOON*

[1837.]

[Note Book 48.]

Dunraven Castle, Sept^{r} 25, 1837.

Let x, y, z be the Moon's geocentric rectangular coordinates; m Moon's mass; $x_{,}, y_{,}, z_{,}$ those of the Sun; $m_{,}$ Sun's mass; $\mu =$ sum of masses of Moon and Earth. Then

$$(1)\quad \begin{cases} 0 = x'' + \mu x\,(x^2+y^2+z^2)^{-\frac{3}{2}} + m_{,} x_{,}\,(x_{,}^2+y_{,}^2+z_{,}^2)^{-\frac{3}{2}} + m_{,}\,(x-x_{,}) \\ \qquad\qquad \times \{(x-x_{,})^2+(y-y_{,})^2+(z-z_{,})^2\}^{-\frac{3}{2}}; \\ 0 = y'' + \\ 0 = z'' + \end{cases}$$

and

$$(2)\quad \begin{cases} 0 = x_{,}'' + (\mu - m + m_{,})\,x_{,}\,(x_{,}^2+y_{,}^2+z_{,}^2)^{-\frac{3}{2}} + mx\,(x^2+y^2+z^2)^{-\frac{3}{2}} + m\,(x_{,}-x) \\ \qquad\qquad \times \{(x-x_{,})^2+(y-y_{,})^2+(z-z_{,})^2\}^{-\frac{3}{2}}; \\ 0 = y_{,}'' + \\ 0 = z_{,}'' + \end{cases}$$

(Hence

$$0 = mx'' + m_{,}x_{,}'' + (\mu + m_{,})\,\{mx\,(x^2+y^2+z^2)^{-\frac{3}{2}} + m_{,}x_{,}\,(x_{,}^2+y_{,}^2+z_{,}^2)\}^{-\frac{3}{2}}, \qquad , \qquad .)$$

If m ($=$ Moon's mass) be extremely small, we shall have, very nearly, for the Sun's motion, the equations of elliptic motion,

$$(3)\qquad 0 = x_{,}'' + (\mu + m_{,})\,x_{,}\,(x_{,}^2+y_{,}^2+z_{,}^2)^{-\frac{3}{2}}, \quad 0 = \qquad , \quad 0 = \qquad ;$$

& may conceive $x_{,}, y_{,}, z_{,}$ expressed thereby as functions of the time t. (And as a mathematical problem we may propose to integrate the 3 equations (1), considering $x_{,}, y_{,}, z_{,}$ as explicit functions of t, which satisfy these 3 equations (3). Physically, this corresponds to seeking the limiting state of motion of the Moon, when the Moon's mass is considered to tend to the limiting value 0.)

By easy combination of (1) and (3) we get

$$(4)\quad 0 = x'' - \frac{m_{,}x_{,}''}{\mu + m_{,}} + \mu x\,(x^2+y^2+z^2)^{-\frac{3}{2}} + m_{,}\,(x-x_{,})\,\{(x-x_{,})^2+(y-y_{,})^2+(z-z_{,})^2\}^{-\frac{3}{2}};$$

$\because$, if $\delta t = 0$ & if

$$(5)\quad s = \int_0^t \left\{ \frac{1}{2}\left(x' - \frac{m_{,}x_{,}'}{\mu + m_{,}}\right)^2 + \quad + \quad + \mu\,(x^2+y^2+z^2)^{-\frac{1}{2}} + m_{,}\,(\overline{x-x_{,}}^{\,2} + \overline{y-y_{,}}^{\,2} + \overline{z-z_{,}}^{\,2})^{-\frac{1}{2}} \right\} dt,$$

$$(6)\qquad \delta s = \left(x' - \frac{m_{,}x_{,}'}{\mu + m_{,}}\right)\delta x + \quad + \quad - \left(x_0' - \frac{m_{,}x_{,0}'}{\mu + m_{,}}\right)\delta x_0 - \quad - \quad .$$

---

* [See also Correspondence with J. W. Lubbock, pp. 249 *et seq.*]

This last formula of variation would have resulted equally if instead of (5) we had put

$$(7)\qquad s=\int_0^t\left\{\frac{x'^2+y'^2+z'^2}{2}+\mu\,(x^2+y^2+z^2)^{-\frac{1}{2}}\right.$$
$$\left.+m_{,}\left(-\frac{x'x'_{,}+y'y'_{,}+z'z'_{,}}{\mu+m_{,}}+(\overline{x-x_{,}}^2+\overline{y-y_{,}}^2+\overline{z-z_{,}}^2)^{-\frac{1}{2}}\right)\right\}dt.$$

Besides, the part

$$-\frac{m_{,}}{\mu+m_{,}}\int_0^t(x'x'_{,}+y'y'_{,}+z'z'_{,})\,dt=-\frac{m_{,}}{\mu+m_{,}}(xx'_{,}+yy'_{,}+zz'_{,}-x_0x'_{,0}-y_0y'_{,0}-z_0z'_{,0})$$
$$+\frac{m_{,}}{\mu+m_{,}}\int_0^t(xx''_{,}+yy''_{,}+zz''_{,})\,dt=-\frac{m_{,}}{\mu+m_{,}}(xx'_{,}+\quad+\quad-x_0x'_{,0}-\quad-\quad)$$
$$-m_{,}\int_0^t\frac{xx_{,}+yy_{,}+zz_{,}}{(x_{,}^2+y_{,}^2+z_{,}^2)^{\frac{3}{2}}}\,dt.$$

If then we put*

$$(8)\qquad R=m_{,}\left\{\frac{xx_{,}+yy_{,}+zz_{,}}{(x_{,}^2+y_{,}^2+z_{,}^2)^{\frac{3}{2}}}-(\overline{x-x_{,}}\ \ +\overline{y-y_{,}}^2+\overline{z-z_{,}}^2)^{-\frac{1}{2}}\right\},$$

we shall have, by (7),

$$(9)\qquad s=\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\mu\,(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt$$
$$-\frac{m_{,}}{\mu+m_{,}}(xx'_{,}+yy'_{,}+zz'_{,}-x_0x'_{,0}-y_0y'_{,0}-z_0z'_{,0});$$

and hence by (6) we have

$$(\mathrm{A})\ldots\quad \delta\int\left(\frac{x'^2+y'^2+z'^2}{2}+\mu\,(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt=x'\delta x+y'\delta y+z'\delta z-x'_0\,\delta x_0-y'_0\,\delta y_0-z'_0\,\delta z_0.$$

The only thing neglected in this very simple formula is the mass of the Moon; by neglecting which we are able to treat the Sun's coordinates $x_{,}$, $y_{,}$, $z_{,}$ in R as explicit functions of the time t. Accordingly when the Moon's mass is thus neglected, & δt is made $=0$, the formula (A) results at once from the 3 known equations

$$(\mathrm{B})\ldots\qquad x''+\mu x\,(x^2+y^2+z^2)^{-\frac{3}{2}}+\frac{\delta R}{\delta x}=0,\qquad ,\qquad .$$

In fact if we multiply the three equations (B) by δx, δy, δz and add the products, we get

$$x''\delta x+y''\delta y+z''\delta z=\mu\delta\,(x^2+y^2+z^2)^{-\frac{1}{2}}-\delta R;$$

$\because$ because

$$x'\,(\delta x)'+y'\,(\delta y)'+z'\,(\delta z)'=\delta\,\frac{x'^2+y'^2+z'^2}{2},$$

we have

$$\underset{t=t_0}{\overset{t=t}{\Delta}}(x'\delta x+y'\delta y+z'\delta z)=\delta\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\mu\,(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt.$$

Let

$$x=r\cos\lambda,\quad y=r\sin\lambda,\quad z=0,$$
$$x_{,}=a_{,}\cos n_{,}t,\quad y_{,}=a_{,}\sin n_{,}t,\quad z_{,}=0,\quad a_{,}^2n_{,}^3=\mu+m_{,};$$

so that the Sun's orbit is now supposed to be circular and the inclination of the Moon's orbit is neglected.

Then the fundamental formula (A) becomes

$$(\mathrm{C})\ldots\qquad \delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}-R\right)dt=r'\delta r+r^2\lambda'\delta\lambda-r'_0\,\delta r_0-r_0^2\lambda'_0\,\delta\lambda_0;$$

---

* [See p. 51.]

also

$$\text{(D)}\ldots\qquad R=m_{,}\left\{\frac{r\cos(\lambda-n_{,}t)}{a_{,}^{2}}-\{a_{,}^{2}-2a_{,}r\cos(\lambda-n_{,}t)+r^{2}\}^{-\frac{1}{2}}\right\}.$$

At the same time the differential equations (B) become

$$\text{(E)}\ldots\qquad r''-r\lambda'^{2}+\frac{\mu}{r^{2}}+\frac{\delta R}{\delta r}=0;\quad (r^{2}\lambda')'+\frac{\delta R}{\delta\lambda}=0.$$

Besides we may put, with a great degree of approximation,* $m_{,}=a_{,}^{3}n_{,}^{2}$, neglecting Earth's mass in comparison with Sun's; and then if in general we make $(1-2\alpha p+\alpha^{2})^{-\frac{1}{2}}=1+\alpha P_{1}+\alpha^{2}P_{2}+$ &c., and put also $p=\cos\overline{\lambda-n_{,}t}$, we shall then have

$$-R=a_{,}^{2}n_{,}^{2}+n_{,}^{2}\left(r^{2}P_{2}+\frac{r^{3}}{a_{,}}P_{3}+\frac{r^{4}}{a_{,}^{2}}P_{4}+\text{\&c.}\right).$$

Hence,

$$\text{(F)}\ldots\qquad \delta\int_{0}^{t}\left(\frac{r'^{2}+r^{2}\lambda'^{2}}{2}+\frac{\mu}{r}\right)dt+n_{,}^{2}\delta\int_{0}^{t}\left(r^{2}P_{2}+\frac{r^{3}}{a_{,}}P_{3}+\frac{r^{4}}{a_{,}^{2}}P_{4}+\text{\&c.}\right)dt$$
$$=r'\delta r+r^{2}\lambda'\delta\lambda-r_{0}'\delta r_{0}-r_{0}^{2}\lambda_{0}'\delta\lambda_{0}.$$

As a first approximation we may neglect $n_{,}$; (which comes to treating the month as very small in comparison with the year;) & then we may take elliptic values for r and λ, (that is, for the moon's geocentric radius vector and geocentric longitude,) which values, if we retain only the 1st power of the excentricity e, will be

$$r=a-ae\cos(nt+\epsilon-\varpi)=a-ae\cos\xi,\quad(\text{if }\xi=nt+\epsilon-\varpi,)$$
$$\lambda=nt+\epsilon+2e\sin(nt+\epsilon-\varpi)=nt+\epsilon+2e\sin\xi;$$

also $\mu=a^{3}n^{2}$.

Accordingly these expressions give, when we put

$$k=e\cos(\epsilon-\varpi),\quad l=e\sin(\epsilon-\varpi),$$
$$r=a(1-k\cos nt+l\sin nt),\quad \lambda=nt+\epsilon+2k\sin nt+2l\cos nt,$$
$$\lambda'=n(1+2k\cos nt-2l\sin nt),\quad \tfrac{1}{2}r^{2}\lambda'^{2}=a^{2}n^{2}(\tfrac{1}{2}+k\cos nt-l\sin nt),$$

& $\because$

$$\delta\int_{0}^{t}\left(\frac{r'^{2}+r^{2}\lambda'^{2}}{2}+\frac{\mu}{r}\right)dt=\delta\{\tfrac{3}{2}a^{2}n^{2}t+2a^{2}n(k\sin nt+l\,\overline{\cos nt-1})\}$$
$$=a^{2}n(t\delta n+2\delta k\sin nt+2\delta l\,\overline{\cos nt-1})=a^{2}n\,\delta(\lambda-\lambda_{0})=a^{2}n\,\delta\lambda-a^{2}n\,\delta\lambda_{0}$$
$$=r^{2}\lambda'\delta\lambda-r_{0}^{2}\lambda_{0}'\delta\lambda_{0}=r'\delta r+r^{2}\lambda'\delta\lambda-r_{0}'\delta r_{0}-r_{0}^{2}\lambda_{0}'\delta\lambda_{0}.$$

In these last equations it has been supposed that k and l vanish after the act of variation δ has been performed, but if we even retain terms of the 1st dimension with respect to k & l, we find that the differential equations

$$r''-r\lambda'^{2}+\frac{\mu}{r^{2}}=0\quad\text{and}\quad(r^{2}\lambda')'=0\text{ are satisfied.}$$

---

* $\left[a_{,}^{3}n_{,}^{2}=m_{,}\left(1+\frac{\mu}{m_{,}}\right)\right.$. We have the following approximate values: $\frac{\mu}{m_{,}}=\frac{1}{330{,}000}$; $\frac{n_{,}}{n}=\frac{1}{13}$; $e=\frac{1}{20}$; $\frac{a}{a_{,}}=\frac{1}{400}$ the tangent of the inclination of the Moon's orbit $=\frac{1}{11}$ and the excentricity of the Sun's orbit $=\frac{1}{60}$. Brown, *Lunar Theory*, pp. 42, 80.]

As a second approximation we may take the differential equations

$$0=r''-r\lambda'^2+\frac{\mu}{r^2}+\frac{\delta R}{\delta r}, \quad 0=(r^2\lambda')'+\frac{\delta R}{\delta \lambda},$$

retaining in R only the term

$$-n_{,}^2r^2P_2=-\frac{n_{,}^2r^2}{}(1+3\cos 2\lambda-2n_{,}t).$$

At the same time,

$$\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{}+\frac{\mu}{r}\right)dt+\tfrac{1}{4}n_{,}^2\,\delta\int_0 r^2\,(1+3\cos\overline{2\lambda-2n_{,}t})\,dt=r'\delta r+r^2\lambda'\delta\lambda-r_0'\delta r_0-r_0^2\lambda_0'\delta\lambda_0.$$

We are now to employ the expressions

$$r=a\,(1-k\cos nt+l\sin nt), \quad \lambda=nt+\epsilon+2k\sin nt+2l\cos nt;$$

but are no longer to consider a, n, ϵ, k, l as constant. We are however to suppose $r_0=a\,(1-k)$, $\lambda_0=\epsilon+2l$; but r', λ', r_0', λ_0' will now have new values, at least in the 2nd member of the formula $\delta\int$ &c. $=r'\delta r+$ &c., though in the 1st member of that formula the values of r' and λ' remain unaltered. We may ∵ make, in the 2nd member of that formula, (if we neglect $n_{,}^2k$ and $n_{,}^2l$)

$$r'=\Delta r', \quad \lambda'=n+\Delta\lambda', \quad r_0'=\Delta r_0', \quad \lambda_0'=n+\Delta\lambda_0', \quad r=r_0=a,$$

$$\delta r=\delta a-a\,\delta k\,.\cos nt+a\,\delta l\,.\sin nt, \quad \delta\lambda=t\,\delta n+\delta\epsilon+2\delta k\,.\sin nt+2\delta l\,.\cos nt,$$

$$\delta r_0=\delta a-a\,\delta k, \quad \delta\lambda_0=\delta\epsilon+2\delta l,$$

& we may suppress in it the part $a^2n\,(\delta\lambda-\delta\lambda_0)$, if we suppress in the 1st member the part

$$\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}\right)dt;$$

by which means it becomes*

$$\Delta r'\,.\,(\delta a-a\,\delta k\,.\cos nt+a\,\delta l\,.\sin nt)+a^2\Delta\lambda'\,.\,(t\,\delta n+\delta\epsilon+2\delta k\,.\sin nt+2\delta l\,.\cos nt)$$
$$-\Delta r_0'\,.\,(\delta a-a\,\delta k)-a^2\Delta\lambda_0'\,.\,(\delta\epsilon+2\delta l)=\tfrac{1}{4}n_{,}^2\,\delta\int_0^t r^2\,(1+3\cos\overline{2\lambda-2n_{,}t})\,dt=\delta s;$$

that is, after expressing $\quad s=\tfrac{1}{4}n_{,}^2\int_0^t r^2\,(1+3\cos\overline{2\lambda-2n_{,}t})\,dt$

as a function of a, n, ϵ, k, l, t & taking its variation relatively to n, ϵ, k, l, (observing that $a=(\mu/n^2)^{-\frac{1}{3}}$ & ∵ $\delta a=-\frac{2}{3}\frac{a}{n}\delta n$,)

$$\frac{\delta s}{\delta n}=-\frac{2}{3}\frac{a}{n}(\Delta r'-\Delta r_0')+a^2t\,\Delta\lambda'; \quad \frac{\delta s}{\delta\epsilon}=a^2\,(\Delta\lambda'-\Delta\lambda_0');$$

$$\frac{\delta s}{\delta k}=-a\,(\Delta r'\,.\cos nt-\Delta r_0')+2a^2\Delta\lambda'\,.\sin nt; \quad \frac{\delta s}{\delta l}=a\,\Delta r'\,.\sin nt+2a^2\,(\Delta\lambda'\,.\cos nt-\Delta\lambda_0').$$

These four equations will give, by elimination, expressions for $\Delta r_0'$ and $\Delta\lambda_0'$ of the forms

$$R_1\frac{\delta s}{\delta n}+R_2\frac{\delta s}{\delta\epsilon}+R_3\frac{\delta s}{\delta k}+R_4\frac{\delta s}{\delta l} \quad \text{and} \quad L_1\frac{\delta s}{\delta n}+\ldots+L_4\frac{\delta s}{\delta l}.$$

R_1, ... L_4 being functions of n & t; while $\frac{\delta s}{\delta n}$, ... $\frac{\delta s}{\delta l}$ are functions of n, t and $n_{,}$.

---

* [See First Essay, p. 161, (G^8.).]

Besides, if we neglect the products of n, k, l, we may retain the expressions thus found for $\Delta r'_0$ and $\Delta\lambda'_0$ and substitute them in these new equations

$$r'_0 = anl + \Delta r'_0, \quad \lambda'_0 = n(1+2k) + \Delta\lambda'_0;$$

with which we are then to combine the 2 equations

$$r_0 = a(1-k), \quad \lambda_0 = \epsilon + 2l, \quad (\text{and } a^3n^2 = \mu,)$$

in order to get n, ϵ, k, l (and a) as functions of r_0, λ_0, r'_0, λ'_0, t and n, : which functions are then to be substituted in the expressions

$$r = a(1 - k\cos nt + l\sin nt), \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt.$$

To effect this substitution, it is convenient* to change a and n, ϵ, k, l to $a+\Delta a$, $n+\Delta n$, $\epsilon+\Delta\epsilon$, $k+\Delta k$, $l+\Delta l$, and to establish the equations

$$r_0 = a(1-k), \quad \lambda_0 = \epsilon + 2l, \quad r'_0 = anl, \quad \lambda'_0 = n(1+2k), \quad a^3n^2 = \mu,$$

and

$$0 = \Delta a - a\Delta k, \quad 0 = \Delta\epsilon + 2\Delta l, \quad 0 = an\Delta l + \Delta r'_0,$$

$$0 = \Delta n + 2n\Delta k + \Delta\lambda'_0, \quad \Delta a = -2a\Delta n/3n;$$

after which we shall have

$$\Delta r = \Delta a - a\Delta k \,.\, \cos nt + a\Delta l \,.\, \sin nt; \quad \Delta\lambda = t\Delta n + \Delta\epsilon + 2\Delta k \,.\, \sin nt + 2\Delta l \,.\, \cos nt,$$

and finally

$$r = a(1 - k\cos nt + l\sin nt) + \Delta r, \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt + \Delta\lambda;$$

in which expressions a and n, ϵ, k, l are constants independent of the time t.

In this manner we have

$$\Delta a = a\Delta k; \quad \Delta n = -\frac{3n}{2}\Delta k; \quad \Delta k = -\frac{2}{n}\Delta\lambda'_0;$$

$$\Delta n = 3\Delta\lambda'_0, \quad \Delta a = -\frac{2a}{n}\Delta\lambda'_0; \quad \text{and} \quad \Delta l = -\frac{1}{an}\Delta r'_0, \quad \Delta\epsilon = \frac{2}{an}\Delta r'_0;$$

therefore

$$\Delta r = -\frac{2a}{n}(1-\cos nt)\Delta\lambda'_0 - \frac{1}{n}\sin nt \,.\, \Delta r'_0;$$

$$\Delta\lambda = \left(3t - \frac{4\sin nt}{n}\right)\Delta\lambda'_0 + \frac{2}{an}(1-\cos nt)\Delta r'_0.$$

Besides, by the equations near the foot of page 241, we have

$$\Delta\lambda' = \Delta\lambda'_0 + \frac{1}{a^2}\frac{\delta s}{\delta\epsilon}, \quad \Delta r' = \Delta r'_0 + \tfrac{3}{2}ant\left(\Delta\lambda'_0 + \frac{1}{a^2}\frac{\delta s}{\delta\epsilon}\right) - \frac{3n}{2a}\frac{\delta s}{\delta n};$$

---

* [Six variable parameters are introduced: $\Delta r'_0$, $\Delta\lambda'_0$, Δn (or Δa), $\Delta\epsilon$, Δk, Δl. It is possible therefore to prescribe *four* relations between them. The four given above are not the usual relations employed. Cf. Appendix, Note 7, p. 628.]

also, $$\sin nt\,.\,\frac{\delta s}{\delta k}+\cos nt\,.\,\frac{\delta s}{\delta l}=2a^2(\Delta\lambda'-\cos nt\,\Delta\lambda_0')+a\,\Delta r_0'\sin nt$$
$$=2a^2(1-\cos nt)\,\Delta\lambda_0'+a\sin nt\,\Delta r_0'+2\frac{\delta s}{\delta\epsilon}=-an\,\Delta r+2\frac{\delta s}{\delta\epsilon},$$

and

$$\cos nt\,.\,\frac{\delta s}{\delta k}-\sin nt\,.\,\frac{\delta s}{\delta l}=-a\,\Delta r'+a\,\Delta r_0'\cos nt+2a^2\Delta\lambda_0'\sin nt$$
$$=-a(1-\cos nt)\,\Delta r_0'+a^2(-\tfrac{3}{2}nt+2\sin nt)\,\Delta\lambda_0'-\tfrac{3}{2}nt\frac{\delta s}{\delta\epsilon}+\frac{3n}{2}\frac{\delta s}{\delta n}$$
$$=-\frac{a^2n}{2}\Delta\lambda-\frac{3nt}{2}\frac{\delta s}{\delta\epsilon}+\frac{3n}{2}\frac{\delta s}{\delta n};$$

therefore

$$\Delta r=\frac{2}{an}\frac{\delta s}{\delta\epsilon}-\frac{1}{an}\left(\sin nt\frac{\delta s}{\delta k}+\cos nt\frac{\delta s}{\delta l}\right),$$
$$\Delta\lambda=\frac{3}{a^2}\left(\frac{\delta s}{\delta n}-t\frac{\delta s}{\delta\epsilon}\right)-\frac{2}{a^2n}\left(\cos nt\frac{\delta s}{\delta k}-\sin nt\frac{\delta s}{\delta l}\right).$$

To calculate s, we have

$$r^2=a^2(1-2k\cos nt+2l\sin nt),$$
$$2\lambda-2n_,t=2(n-n_,)t+2\epsilon+4k\sin nt+4l\cos nt;$$
$$1+3\cos(2\lambda-2n_,t)=1+3\cos(2nt-2n_,t+2\epsilon)-12(k\sin nt+l\cos nt)\sin(2nt-2n_,t+2\epsilon);$$
$$\frac{4s'}{a^2n_,^2}=\{1+3\cos(2nt-2n_,t+2\epsilon)\}\{1-2k\cos nt-2l\sin nt\}$$
$$-12(k\sin nt+l\cos nt)\sin(2nt-2n_,t+2\epsilon)$$
$$=1+3\cos(2nt-2n_,t+2\epsilon)-2k\cos nt+2l\sin nt$$
$$+3k\{\cos(3nt-2n_,t+2\epsilon)-3\cos(nt-2n_,t+2\epsilon)\}$$
$$-3l\{\sin(3nt-2n_,t+2\epsilon)+3\sin(nt-2n_,t+2\epsilon)\};$$
$$\frac{4s}{n_,^2}=a^2t+\frac{3a^2}{2n-2n_,}\{\sin(2nt-2n_,t+2\epsilon)-\sin 2\epsilon\}$$
$$+ka^2\left\{-\frac{2}{n}\sin nt+\frac{3}{3n-2n_,}\{\sin(3nt-2n_,t+2\epsilon)-\sin 2\epsilon\}\right.$$
$$-\frac{9}{n-2n_,}\{\sin(nt-2n_,t+2\epsilon)-\sin 2\epsilon\}$$
$$+la^2\left\{\frac{2}{n}-\frac{2}{n}\cos nt+\frac{}{3n-2n}\{\cos(3nt-2n_,t+2\epsilon)-\cos 2\epsilon\}\right.$$
$$\left.-\frac{9}{n-2n_,}\{\cos(nt-2n_,t+2\epsilon)-\cos 2\epsilon\}\right\};$$
$$\frac{4n}{an_,^2}\Delta r=\frac{6}{n-n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos 2\epsilon)+\frac{4}{n}(1-\cos nt)$$
$$-\frac{}{3n-2n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt-2\epsilon})$$
$$-\frac{9}{n-2n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt+2\epsilon});$$

and
$$\frac{4n}{n_{,}^2}\Delta\lambda = -4\left\{t+\frac{3}{2n-2n_{,}}(\sin\overline{2nt-2n_{,}t+2\epsilon}-\sin 2\epsilon)\right\}+\frac{9nt\cos 2\epsilon}{n-n_{,}}$$
$$\frac{9n}{2(n-n_{,})^2}(\sin\overline{2nt-2n_{,}t+2\epsilon}-\sin 2\epsilon)+\frac{4}{n}\sin nt$$
$$-\frac{}{3n-2n}(\sin\overline{2nt-2n_{,}t+2\epsilon}+\sin\overline{nt-2\epsilon})$$
$$+\frac{18}{n-2n_{,}}(\sin\overline{2nt-2n_{,}t+2\epsilon}-\sin\overline{nt+2\epsilon}).$$

(As verifications, these expressions for Δr and $\Delta\lambda$ should not only vanish themselves when $t=0$, which they evidently do, but also their differential coefficients, taken with respect to t, should vanish at the same time; we ought ∵ to have
$$0=-12+3+9 \quad \& \quad 0=-4+4+\cos 2\epsilon(-12-6+18);$$
& so we have.)

It results then from the foregoing calculations that if squares & products of $n_{,}^2$, k, l be neglected, the two differential equations of the 2nd order
$$0=r''-r\lambda'^2+\frac{\mu}{r^2}-\frac{n_{,}^2 r}{2}(1+3\cos\overline{2\lambda-2n_{,}t}), \qquad 0=(r^2\lambda')'+\frac{3n_{,}^2r^2}{2}\sin\overline{2\lambda-2n_{,}t},$$
(in which μ and $n_{,}$ are constant,) admit of having their integrals expressed as follows:
$$r=a-ak\cos nt+al\sin nt+\frac{a\,n_{,}^2}{2\,n^2}(1-\cos nt)$$
$$+\frac{3an_{,}^2}{4n}\left\{\frac{2}{n-n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos 2\epsilon)-\frac{1}{3n-2n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos\overline{nt-2\epsilon})\right.$$
$$\left.-\frac{3}{n-2n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos\overline{nt+2\epsilon})\right\};$$
$$\lambda=nt+\epsilon+2k\sin nt+2l\cos nt-\frac{n_{,}^2t}{n}\left(1-\frac{9n\cos 2\epsilon}{4(n-n_{,})}\right)+\frac{n_{,}^2}{n^2}\sin nt$$
$$-\frac{3n_{,}^2}{4n}\left\{\left(\frac{2}{n-n_{,}}+\frac{3n}{2(n-n_{,})^2}\right)(\sin\overline{2nt-2n_{,}t+2\epsilon}-\sin 2\epsilon)\right.$$
$$\left.+\frac{}{3n-2n_{,}}(\sin\overline{2nt-2n_{,}t+2\epsilon}+\sin\overline{nt-2\epsilon})-\frac{6}{n-2n_{,}}(\sin\overline{2nt-2n_{,}t+2\epsilon}-\sin\overline{nt+2\epsilon})\right\};$$
in which a and n, ϵ, k, l are 5 arbitrary constants determinable by the 5 conditions
$$r_0=a-ak, \quad \lambda_0=\epsilon+2l, \quad r_0'=anl, \quad \lambda_0'=n+2nk, \quad a^3n^2=\mu.$$

These expressions may be put under the forms
$$r=a\left\{1+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n\cos 2\epsilon}{n-n_{,}}\right)\right\}$$
$$-a\cos nt\left\{k+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_{,}}+\frac{3}{n-2n_{,}}\right)\cos 2\epsilon\right)\right\}$$
$$+a\sin nt\left\{l+\frac{3n_{,}^2}{4n}\left(\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\sin 2\epsilon\right\}$$
$$+\frac{3an_{,}^2}{4n}\left(\frac{2}{n-n_{,}}-\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\cos(2nt-2n_{,}t+2\epsilon);$$

$$\begin{aligned}\lambda = nt &\left\{1-\frac{n_{,}^2}{n^2}\left(1-\frac{9n\cos 2\epsilon}{4\,(n-n_{,})}\right)\right. \\ &+\epsilon+\quad\left(\frac{2}{n-n_{,}}+\frac{3n}{2\,(n-n_{,})^2}\right)\sin 2\epsilon \\ &+2\sin nt\left\{k+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_{,}}+\frac{3}{n-2n_{,}}\right)\cos 2\epsilon\right)\right\} \\ &+2\cos nt\left\{l+\frac{3n_{,}^2}{4n}\left(\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\sin 2\epsilon\right\} \\ &-\frac{3n_{,}^2}{4n}\left(\frac{2}{n-n_{,}}+\frac{3n}{2\,(n-n_{,})^2}+\frac{2}{3n-2n_{,}}-\frac{6}{n-2n_{,}}\right)\sin\,(2nt-2n_{,}t+2\epsilon).\end{aligned}$$

In the same order of approximation, if we put

$$\mathrm{n}=n-\frac{n_{,}^2}{n}\left(1-\frac{9n\cos 2\epsilon}{4\,(n-n_{,})}\right);\quad \mathrm{a}=\sqrt[3]{\frac{\mu}{\mathrm{n}^2}}\left\{=a+\frac{2n_{,}^2a}{3n^2}\left(1-\frac{9n\cos 2\epsilon}{4\,(n-n_{,})}\right)\right\};$$

$$\mathrm{k}=k+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_{,}}+\frac{3}{n-2n_{,}}\right)\cos 2\epsilon\right);\quad \mathrm{l}=l+\frac{3n_{,}^2}{4n}\left(\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\sin 2\epsilon;$$

$$\mathrm{e}=\epsilon+\frac{3n_{,}^2}{2n\,(n-n_{,})}\left(1+\frac{3n}{4\,(n-n_{,})}\right)\sin 2\epsilon;$$

we shall have

$$\begin{aligned}r=\mathrm{a}&\left\{1-\frac{n_{,}^2}{6\mathrm{n}^2}-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t\right\} \\ &\qquad+\frac{3\mathrm{a}n_{,}^2}{4\mathrm{n}}\left(\frac{2}{\mathrm{n}-n_{,}}-\frac{1}{3\mathrm{n}-2n_{,}}-\frac{3}{\mathrm{n}-2n_{,}}\right)\cos\,(2\mathrm{n}t-2n_{,}t+2\mathrm{e});\end{aligned}$$

$$\begin{aligned}\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin\mathrm{n}t+2\mathrm{l}\cos\mathrm{n}t& \\ -\frac{3n_{,}^2}{4\mathrm{n}}\left(\frac{2}{\mathrm{n}-n_{,}}+\frac{3\mathrm{n}}{2\,(\mathrm{n}-n_{,})^2}+\frac{}{3\mathrm{n}-2n_{,}}-\frac{}{\mathrm{n}-2n_{,}}\right)&\sin\,(2\mathrm{n}t-2n_{,}t+2\mathrm{e}).\end{aligned}$$

Developing, we have

$$\frac{2}{\mathrm{n}-n_{,}}=\frac{2}{\mathrm{n}}(1+m);\quad -\frac{1}{3\mathrm{n}-2n_{,}}=-\frac{1}{3\mathrm{n}}(1+\tfrac{2}{3}m);\quad -\frac{3}{\mathrm{n}-2n_{,}}=-\frac{3}{\mathrm{n}}(1+2m);$$

if we put for abbreviation $m=\frac{n_{,}}{\mathrm{n}}$, & neglect m^2; $\because$ sum $=-\frac{4}{3\mathrm{n}}-\frac{38m}{9\mathrm{n}}$; and similarly

$$\frac{2}{\mathrm{n}-n_{,}}=\frac{2}{\mathrm{n}}(1+m);\quad \frac{3\mathrm{n}}{2\,(\mathrm{n}-n_{,})^2}=\frac{3}{2\mathrm{n}}(1+2m);\quad \frac{2}{3\mathrm{n}-2n_{,}}=\frac{2}{3\mathrm{n}}(1+\tfrac{2}{3}m);\quad -\frac{6}{\mathrm{n}-2n_{,}}=-\frac{6}{\mathrm{n}}(1+2m);$$

$\because$ sum $=-\frac{11}{6\mathrm{n}}-\frac{59m}{9\mathrm{n}}$; $\because$ putting for abbreviation $\tau=\mathrm{n}t-n_{,}t+\mathrm{e}$, we have

$$r=\mathrm{a}\left\{1-\frac{m^2}{6}-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t-m^2\left(1+\frac{19m}{6}\right)\cos 2\tau\right\},$$

$$\lambda=\mathrm{n}t+\mathrm{e}+2\,(\mathrm{k}\sin\mathrm{n}t+\mathrm{l}\cos\mathrm{n}t)+m^2\left(\frac{11}{8}+\frac{59m}{12}\right)\sin 2\tau;$$

and accordingly these agree, so far as they go, with the expressions of M. Plana, as cited by Mr Lubbock,* in the Appendix to the 1st Part of his Theory of the Moon.

We might have commenced our 2nd approximation by retaining in R the two terms $-n_{,}^{2}r^{2}P_{2}$ and $-\frac{n_{,}^{2}}{a_{,}}r^{3}P_{3}$; in which

$$P_{3}=\frac{1}{6}\left(\frac{d}{dp}\right)^{3}\left(\frac{p^{2}-1}{2}\right)^{3}=\tfrac{1}{48}(p^{6}-3p^{4}+3p^{2}-1)'''=\frac{5p^{3}-3p}{2}=\tfrac{5}{2}\cos\overline{\lambda-n_{,}t}^{\,3}-\tfrac{3}{2}\cos\overline{\lambda-n_{,}t}$$

$$\frac{5\cos\overline{3\lambda-3n_{,}t}+3\cos\overline{\lambda-n_{,}t}}{8};$$

so that we should thus have had

$$s=\frac{n_{,}^{2}}{4}\int_{0}^{t}r^{2}(1+3\cos\overline{2\lambda-2n_{,}t})\,dt+\frac{n_{,}^{2}}{8a_{,}}\int_{0}^{t}r^{3}(3\cos\overline{\lambda-n_{,}t}+5\cos\overline{3\lambda-3n_{,}t})\,dt,$$

while Δr and $\Delta\lambda$ would still have been given by the formulae of page 243,

$$\Delta r=\frac{1}{an}\left(2\frac{\delta s}{\delta\epsilon}-\sin nt\frac{\delta s}{\delta k}-\cos nt\frac{\delta s}{\delta l}\right),\quad \Delta\lambda=\frac{1}{a^{2}n}\left(3n\frac{\delta s}{\delta n}-3nt\frac{\delta s}{\delta\epsilon}-2\cos nt\frac{\delta s}{\delta k}+2\sin nt\frac{\delta s}{\delta l}\right).$$

If then we put

$$s^{\backprime}=\int_{0}^{t}r^{3}(3\cos\overline{\lambda-n_{,}t}+5\cos\overline{3\lambda-3n_{,}t})\,dt,$$

and

$$r^{\backprime}=2\frac{\delta s^{\backprime}}{\delta\epsilon}-\sin nt\frac{\delta s^{\backprime}}{\delta k}-\cos nt\frac{\delta s^{\backprime}}{\delta l},\quad \lambda^{\backprime}=3n\frac{\delta s^{\backprime}}{\delta n}-3nt\frac{\delta s^{\backprime}}{\delta\epsilon}-2\cos nt\frac{\delta s^{\backprime}}{\delta k}+2\sin nt\frac{{}^{\backprime}}{\delta l},$$

we shall only have to add $\frac{n_{,}^{2}r^{\backprime}}{8aa_{,}n}$ and $\frac{n_{,}^{2}\lambda^{\backprime}}{8a^{2}a_{,}n}$ to the values already found for r and λ.

In developing $s^{\backprime}$ we are to use for r and λ their 1st approximate values

$$r=a-ak\cos nt+al\sin nt,\quad \lambda=nt+\epsilon+2k\sin nt+2l\cos nt,$$

which give

$$r^{3}=a^{3}(1-3k\cos nt+3l\sin nt),$$

$$3\cos\overline{\lambda-n_{,}t}=3\cos\overline{nt-n_{,}t+\epsilon}-6(k\sin nt+l\cos nt)\sin\overline{nt-n_{,}t+\epsilon},$$

$$5\cos\overline{3\lambda-3n_{,}t}=5\cos\overline{3nt-3n_{,}t+3\epsilon}-30(k\sin nt+l\cos nt)\sin\overline{3nt-3n_{,}t+3\epsilon},$$

$$\frac{2s^{\backprime\prime}}{a^{3}}=\frac{2r^{3}}{a^{3}}(3\cos\overline{\lambda-n_{,}t}+5\cos\overline{3\lambda-3n_{,}t})=6\cos\overline{nt-n_{,}t+\epsilon}+10\cos\overline{3nt-3n_{,}t+3\epsilon}$$

$$-6(k\cos nt-l\sin nt)(3\cos\overline{nt-n_{,}t+\epsilon}+5\cos\overline{3nt-3n_{,}t+3\epsilon})$$

$$-12(k\sin nt+l\cos nt)(\sin\overline{nt-n_{,}t+\epsilon}+5\sin\overline{3nt-3n_{,}t+3\epsilon});$$

---

* [Lubbock, *Theory of the Moon* (1834), Appendix, pp. i, viii [1]. Brown, *Lunar Theory*, p. 110.]

$$\because \quad s' = \frac{3a^3}{n-n_{,}}(\sin\overline{nt-n_{,}t+\epsilon}-\sin\epsilon)+\frac{5a^3}{3\,(n-n_{,})}(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)$$
$$-\frac{3ka^3}{2}\left\{\frac{5}{n_{,}}(\sin\overline{n_{,}t-\epsilon}+\sin\epsilon)+\frac{1}{2n-n_{,}}(\sin\overline{2nt-n_{,}t+\epsilon}-\sin\epsilon)\right.$$
$$\left.+\frac{15}{2n-3n_{,}}(\sin\overline{2nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)-\frac{5}{4n-3n_{,}}(\sin\overline{4nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)\right.$$
$$-\frac{3la^3}{2}\left\{\frac{5}{n_{,}}(\cos\overline{n_{,}t-\epsilon}-\cos\epsilon)+\frac{1}{2n-n_{,}}(\cos\overline{2nt-n_{,}t+\epsilon}-\cos\epsilon)\right.$$
$$-\frac{15}{2n-3n_{,}}(\cos 2nt-3n_{,}t+3\epsilon-\cos 3\epsilon)-\frac{5}{4n-3n_{,}}(\cos 4nt-3n_{,}t+3\epsilon-\cos 3\epsilon)$$

& consequently,

$$r' = \frac{6a^3}{n-n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\epsilon)+\frac{10a^3}{n-n_{,}}(\cos 3nt-3n_{,}t+3\epsilon-\cos 3\epsilon)$$
$$+\frac{3a^3}{2}\left\{\frac{5}{n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\overline{nt+\epsilon})+\frac{1}{2n-n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\overline{nt-\epsilon})\right.$$
$$-\frac{15}{2n-3n_{,}}(\cos 3nt-3n_{,}t+3\epsilon-\cos nt+3\epsilon)-\frac{5}{4n-3n_{,}}(\cos 3nt-3n_{,}t+3\epsilon-\cos nt-3\epsilon)$$

and

$$\lambda' = -\frac{a^3\,(3n-2n_{,})}{(n-n_{,})^2}\{9\,(\sin\overline{nt-n_{,}t+\epsilon}-\sin\epsilon)+5\,(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)\}$$
$$+\frac{3a^3nt}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon)$$
$$-3a^3\left\{\frac{5}{n_{,}}(\sin\overline{nt-n_{,}t+\epsilon}-\sin\overline{nt+\epsilon})-\frac{1}{2n-n_{,}}(\sin nt-n_{,}t+\epsilon+\sin nt-\epsilon)\right.$$
$$\left.-\frac{15}{2n-3n_{,}}(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin\overline{nt+3\epsilon})+\frac{5}{4n-3n_{,}}(\sin\overline{3nt-3n_{,}t+3\epsilon}+\sin nt-3\epsilon)\right\}.$$

We have only to add

to k the terms

$$\frac{3a}{16a_{,}}\frac{n_{,}}{n}\left\{\left(5+\frac{n_{,}}{2n-n_{,}}\right)\cos\epsilon-5\left(\frac{3n_{,}}{2n-3n_{,}}+\frac{n_{,}}{4n-3n_{,}}\right)\cos 3\epsilon\right\};$$

to l the terms

$$\frac{3a}{16a_{,}}\frac{n_{,}}{n}\left\{\left(5-\frac{n_{,}}{2n-n_{,}}\right)\sin\epsilon-5\left(\frac{3n_{,}}{2n-3n_{,}}-\frac{n_{,}}{4n-3n_{,}}\right)\sin 3\epsilon\right\};$$

to e the terms

$$\frac{1}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n}\frac{3n-2n_{,}}{(n-n_{,})^2}(9\sin\epsilon+5\sin 3\epsilon);$$

to n the terms

$$\frac{3}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon);$$

and consequently (because $a^3n^2=\mu$)

to a the terms

$$-\frac{1}{4}\frac{a^2}{a_{,}}\frac{n_{,}}{n}\frac{n_{,}}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon):$$

& we shall have $\left(\text{making } \frac{n_{,}}{\mathrm{n}}=m,\ \mathrm{n}t-n_{,}t+\mathrm{e}=\tau\right)$

$$\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin\mathrm{n}t+2\mathrm{l}\cos\mathrm{n}t-\frac{3m^2}{4}\left\{\frac{2}{1-m}+\frac{3}{2(1-m)^2}+\frac{2}{3-2m}-\frac{6}{1-2m}\right\}\sin 2\tau$$

$$-\frac{3m}{8}\frac{\mathrm{a}}{a_{,}}\left\{5-\frac{m}{2-m}+\frac{3m(3-2m)}{(1-m)^2}\right\}\sin\tau+\frac{5m^2}{8}\frac{\mathrm{a}}{a_{,}}\left\{\frac{}{2-3m}-\frac{}{4-3m}-\frac{3-2m}{(1-m)^2}\right\}\sin 3\tau;$$

$$\frac{r}{\mathrm{a}}=1-\frac{m^2}{6}-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t+\frac{3m^2}{4}\left\{\frac{}{1-m}-\frac{}{3-2m}-\frac{}{1-2m}\right\}\cos 2\tau$$

$$+\frac{3m}{16}\frac{\mathrm{a}}{a_{,}}\left\{5+\frac{m}{2-m}+\frac{4m}{1-m}\right\}\cos\tau-\frac{5m^2}{16}\frac{\mathrm{a}}{a_{,}}\left\{\frac{9}{2-3m}+\frac{}{4-3m}-\frac{}{1-m}\right\}\cos 3\tau.$$

Developing,

$$\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin\mathrm{n}t+2\mathrm{l}\cos\mathrm{n}t+m^2\left(\frac{11}{8}+\frac{59m}{12}\right)\sin 2\tau-\frac{3m\mathrm{a}}{8a_{,}}\left(5+\frac{17m}{2}+\frac{47m^2}{4}\right)\sin\tau$$

$$+\frac{5m^2\mathrm{a}}{32a_{,}}\left(3+\frac{35m}{4}\right)\sin 3\tau;$$

$$\frac{r}{\mathrm{a}}=1-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t-\frac{m^2}{6}-m^2\left(1+\frac{19m}{6}\right)\cos 2\tau+\frac{3m\mathrm{a}}{16a_{,}}\left(5+\frac{9m}{2}+\frac{17m^2}{4}\right)\cos\tau$$

$$-\frac{5m^2\mathrm{a}}{64a_{,}}\left(5+\frac{53m}{}\right)\cos 3\tau.$$

And accordingly these equations are integrals, in the present order of approximation, of the following system of differential equations of the 2nd order:

$$r''-r\lambda'^2+\frac{\mu}{r^2}=\frac{n_{,}^2r}{2}(1+3\cos\overline{2\lambda-2n_{,}t})+\frac{3n_{,}^2r^2}{8a}(3\cos\overline{\lambda-n_{,}t}+5\cos\overline{3\lambda-3n_{,}t});$$

$$(r^2\lambda')'=-\frac{3n_{,}^2r^2}{2}\sin\overline{2\lambda-2n_{,}t}-\frac{3n_{,}^2r^3}{8a_{,}}(\sin\overline{\lambda-n_{,}t}+5\sin\overline{3\lambda-3n_{,}t}).$$

Yet the coefficients of $\sin\tau$ and $\cos\tau$ do not agree except in their first terms with the expressions of Plana and Lubbock.*

---

* [In the remainder of this manuscript Hamilton attempts to verify that these equations for λ and r satisfy the differential equations of motion to the present order of approximation. Actually he recognised later (see page 275) that the approximations of Plana and Lubbock were correct. Lubbock, *loc. cit.* pp. vi, xviii [101]. Brown, *loc. cit.* p. 241. See also Appendix, Note 6, p. 627.]

VII.

CORRESPONDENCE WITH J. W. LUBBOCK

[1837.]

[In this correspondence Hamilton numbered his own letters I, II, III and IV. Lubbock's letters are not numbered. We mark them here A, B, C, D.

CONTENTS

PAGE

A. Lubbock asks for information as to Hamilton's Dynamical Methods and their utility for calculating Lunar Tables. 249

I. This is a sketch of the Method of Principal Relations. 250

B. A request for the application of this method to Lunar Theory. 256

II. This gives the application of Hamilton's method to Lunar approximations. . . . 257

C. Lubbock finds a difficulty in seeing a difference between the varying parameters of Lagrange and Hamilton. 263

III. Hamilton gives more details of Letter II. 266

D. A request for further details. 276

IV. (Incomplete.) Hamilton brings his part of the correspondence to an end by discussing at length an interesting example.] 278

A. *J. W. Lubbock*[*] *to Sir W. R. Hamilton.*

London, 24th *July*, 1837.

MY DEAR SIR,

You are so familiar with all questions in Physical Optics that I take the liberty of troubling you with a few remarks upon a point of some importance and I shall be much obliged to you to tell me whether you find them correct and if not if you will look at them with indulgence.[†]

Have you thought more about the Lunar Theory and have you pursued at all the application of your general views to this question? I want to draw up some report for the Association about the construction of Tables. Airy appears to think that Sir W. Hamilton...[part of letter missing]...the labour of getting out terms in this manner would be even greater than that of pursuing theory to the requisite extent. In the latter channel my patience is exhausted and so also I fear is that of M. Pontécoulant who is not easily frightened. But we continue ever and anon to find terms differing from M. Plana's figures and there can be no doubt that until they have all been carefully verified they are not to be implicitly relied on. No one has yet succeeded in obtaining by *the method of the variation of constants* the exact numerical coefficient of the term in R multiplied by m^3 for the angle $2gt-2ct$ Arg. 77. Can you find time to put this matter straight? I think it might throw light upon the manner of employing the equations referred to in *Comptes Rendus*.

---

[*] [Sir J. W. Lubbock, banker & mathematician, afterwards the third baronet, had been vice-president and treasurer of the Royal Society. He was the father of the first Lord Lubbock.]

[†] [There is no trace of the statement of this optical problem among the correspondence.]

Pray let me hear from you & in the mean time

I remain, dear Sir,
Yours very faithfully
J. W. LUBBOCK.

I. *Sir W. R. Hamilton to J. W. Lubbock.*

Letter to Mr. Lubbock on the Calculus of Principal Relations.

Observatory, Aug^{t}. 29, 1837.

MY DEAR SIR,

I am desirous to write to you upon a subject which has some connexion with the theory of the Moon, though not so close an one as to make my letter an answer to yours. Having lately resumed the consideration of my Method of Principal Relations, and brought it a little nearer to a state in which it may be practically applied, I wish to give you some sort of sketch of the way in which I hope, at a future but perhaps not distant time, to make such practical application.

(1.) To take one of the very simplest instances that can be employed to illustrate my method, let us consider the ordinary differential equation of the second order

$$x_t''=\alpha x_t; \tag{1}$$

in which x_t is a function of t and x_t'' is its differential coefficient of the second order, while α is a small real constant, positive or negative. You see at once that the integral of this equation is, in finite terms,

$$x_t=\tfrac{1}{2}(e^{t\sqrt{\alpha}}+e^{-t\sqrt{\alpha}})\,x_0+\frac{1}{2\sqrt{\alpha}}(e^{t\sqrt{\alpha}}-e^{-t\sqrt{\alpha}})\,x_0', \tag{2}$$

or, when developed according to ascending powers of α,

$$x_t=x_0\left(1+\alpha\frac{t^2}{2}+\alpha^2\frac{t^4}{24}+\text{etc.}\right)+x_0'\left(t+\alpha\frac{t^3}{6}+\alpha^2\frac{t^5}{120}+\text{etc.}\right). \tag{3}$$

And if we had supposed the finite form (2) unknown and had begun by assuming the expression

$$x_t=X_t+\alpha\overset{1}{X}_t+\alpha^2\overset{2}{X}_t+\alpha^3\overset{3}{X}_t+\text{etc.}, \tag{4}$$

in which X_t, $\overset{1}{X}_t$, $\overset{2}{X}_t$, etc. are unknown functions of t, we should then have had the series of differential equations

$$X_t''=0,\quad \overset{1}{X}{}_t''=X_t,\quad \overset{2}{X}{}_t''=\overset{1}{X}_t,\quad \text{etc.}, \tag{5}$$

which would have given, successively,

$$\begin{aligned} X_t&=a+bt,\\ \overset{1}{X}_t&=\overset{1}{a}+\overset{1}{b}t+\tfrac{1}{2}at^2+\tfrac{1}{6}bt^3,\\ \overset{2}{X}_t&=\overset{2}{a}+\overset{2}{b}t+\tfrac{1}{2}\overset{1}{a}t^2+\tfrac{1}{6}\overset{1}{b}t^3+\tfrac{1}{24}at^4+\tfrac{1}{120}bt^5,\\ &\text{etc.,}\end{aligned} \tag{6}$$

and therefore

$$\begin{aligned} x_t = a + bt &+ \alpha\,(\overset{1}{a} + \overset{1}{b}t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3) \\ &+ \alpha^2\,(\overset{2}{a} + \overset{2}{b}t + \tfrac{1}{2}\overset{1}{a}t^2 + \tfrac{1}{6}\overset{1}{b}t^3 + \tfrac{1}{24}at^4 + \tfrac{1}{120}bt^5) \\ &+ \text{etc.}: \end{aligned} \tag{7}$$

an expression which gives

$$\begin{aligned} x'_t &= b + \alpha\,(\overset{1}{b} + at + \tfrac{1}{2}bt^2) + \alpha^2\,(\overset{2}{b} + \overset{1}{a}t + \tfrac{1}{2}\overset{1}{b}t^2 + \tfrac{1}{6}at^3 + \tfrac{1}{24}bt^4) + \text{etc.}, \\ x'_0 &= b + \alpha\overset{1}{b} + \alpha^2\overset{2}{b} + \text{etc.}, \quad x_0 = a + \alpha\overset{1}{a} + \alpha^2\overset{2}{a} + \text{etc.}, \end{aligned} \tag{8}$$

and therefore, as before,

$$x_t = x_0\left(1 + \alpha\frac{t^2}{2} + \alpha^2\frac{t^4}{24} + \text{etc.}\right) + x'_0\left(t + \alpha\frac{t^3}{6} + \alpha^2\frac{t^5}{120} + \text{etc.}\right). \tag{3}$$

All these things are perceived without the slightest difficulty, but for that reason they are only the more proper to illustrate the following new mode of proceeding.

(**2.**) Let s denote the definite integral

$$s = \int^t \frac{x'^2_t + \alpha x^2_t}{2}\,dt; \tag{9}$$

and let its variation be taken without varying t or dt: we find, by the usual rules of the Calculus of Variations,

$$\delta s = x'_t\,\delta x_t - x'_0\,\delta x_0, \tag{10}$$

if the function x_t be such as to satisfy the differential equation (1); and, if this last condition be not satisfied, then

$$\delta s = x'_t\,\delta x_t - x'_0\,\delta x_0 + \int_0^t (\alpha x_{t_,} - x''_{t_,})\,\delta x_{t_,}\,dt_,; \tag{11}$$

$t_,$ being here marked with a lower accent merely to assist us in remembering that it flows, or varies differentially, from 0 to t, which latter quantity may be viewed for the present as an arbitrary constant. If we put, for abridgment,

$$b = \frac{x_t - x_0}{t} \tag{12}$$

it is evident that $x_0 + bt_,$ is an approximate expression for the function $x_{t_,}$, because, by (1), it would be an exact expression for that function if the small quantity α were to vanish; treating, therefore, α as infinitely small of the first order, we may calculate the definite integral s with only an error of the second order (in virtue of a principle contained in the equation (11)), by simply making in (9)

$$x_{t_,} = x_0 + bt_,, \tag{13}$$

which gives

$$x'_{t_,} = b, \tag{14}$$

and finally

$$\begin{aligned} s &= \int_0^t \frac{b^2 + \alpha\,(x_0 + bt_,)^2}{2}\,dt_, \\ &= \tfrac{1}{2}b^2\left(t + \alpha\frac{t^3}{3}\right) + \tfrac{1}{2}\alpha b x_0 t^2 + \tfrac{1}{2}\alpha x_0^2 t. \end{aligned} \tag{15}$$

In the same order of approximation we may employ the expression (10) for the variation of this integral s; and therefore, by observing that the equation (12) gives the rigorous relation

$$x_t = x_0 + bt, \tag{16}$$

we find

$$(x'_t - x'_0)\,\delta x_0 + t x'_t \,\delta b = \alpha\,(x_0 t + \tfrac{1}{2} b t^2)\,\delta x_0 + \left\{\tfrac{1}{2}\alpha x_0 t^2 + b\left(t + \alpha\frac{t^3}{3}\right)\right\}\delta b, \tag{17}$$

an equation which resolves itself into the two following, on account of the independence of δx_0 and δb,

$$x'_t - x'_0 = \alpha\,(x_0 t + \tfrac{1}{2} b t^2), \quad t x'_t = b\,(t + \tfrac{1}{3}\alpha t^3) + \tfrac{1}{2}\alpha x_0 t^2; \tag{18}$$

which gives, by elimination of x'_t,

$$x'_0 = b\,(1 - \tfrac{1}{6}\alpha t^2) - \tfrac{1}{2}\alpha x_0 t, \tag{19}$$

and therefore

$$\frac{x_t - x_0}{t} = b = x'_0\,(1 + \tfrac{1}{6}\alpha t^2) + \tfrac{1}{2}\alpha x_0 t, \tag{20}$$

that is,

$$x_t = x_0\,(1 + \tfrac{1}{2}\alpha t^2) + x'_0\,(t + \tfrac{1}{6}\alpha t^3). \tag{21}$$

And, on account of the arbitrariness of t, this must be, to the accuracy of the first order inclusive, an expression for the sought function x_t, or an approximate expression for the sought integral of the proposed differential equation of the second order, namely,

$$x''_t = \alpha x_t. \tag{1}$$

Accordingly, when α^2 is neglected, the expression (21) with the two arbitrary constants x_0 and x'_0 satisfies that differential equation (1) and agrees with the expressions (2) and (3).

(3.) The process thus gone through is certainly less simple, as well as less obvious, than that which consists in passing from the first approximate expression

$$x_t = X_t = a + bt \tag{22}$$

to the improved expression

$$x_t = X_t + \alpha \overset{1}{X}_t = a + bt + \alpha\,(\overset{1}{a} + \overset{1}{b}t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3); \tag{23}$$

which may, because α^2 is neglected, be written thus,

$$x_t = (a + \alpha\overset{1}{a})\left(1 + \frac{\alpha}{2}t^2\right) + (b + \alpha\overset{1}{b})\left(t + \frac{\alpha}{6}t^3\right), \tag{24}$$

or finally

$$x_t = x_0\left(1 + \frac{\alpha}{2}t^2\right) + x'_0\left(t + \frac{\alpha}{6}t^3\right): \tag{21}$$

the passage from (22) to (23) being made by integrating the very simple differential equation of the 2nd order

$$\overset{1}{X}{}''_t = X_t, \tag{25}$$

namely the 2nd equation of the series (5). Yet it is important to observe that the very different process of the article (2.), which conducts to the same improved approximate expression (21), attains that end without the integration of any equation of the second order, and indeed (to speak properly) *without the integration of any differential equation whatever*, through the medium of a single *definite integral* (9) and of the expression (10) for the *variation* of that integral. Before I comment further on this new process (2.), I shall extend it so as to obtain, by a definite integral alone and without the integration of any differential equation, an improved expression for x_t, correct to the accuracy of terms proportional to the cube of α inclusive, namely,

$$x_t = x_0\left(1 + \alpha\frac{t^2}{2} + \alpha^2\frac{t^4}{24} + \alpha^3\frac{t^6}{720}\right) + x'_0\left(t + \alpha\frac{t^3}{6} + \alpha^2\frac{t^5}{120} + \alpha^3\frac{t^7}{5040}\right), \tag{26}$$

which would, in the ordinary course of procedure, be obtained by integrating the two differential equations

$$\overset{2}{X}{}''_t = \overset{1}{X}_t, \quad \overset{3}{X}{}''_t = \overset{2}{X}_t, \tag{27}$$

after the form of $\overset{1}{X}_t$ had been deduced from that of the first approximate expression X_t by integration of the equation (25).

(4.) In this extension which we are about to make of the process of the article (2.), we are to start with the two equations

$$x_t = x_0(1 + \tfrac{1}{2}\alpha t^2) + b(t + \tfrac{1}{6}\alpha t^3), \tag{28}$$

$$x_{t,} = x_0(1 + \tfrac{1}{2}\alpha t_{,}^2) + b(t_{,} + \tfrac{1}{6}\alpha t_{,}^3), \tag{29}$$

of which the first is to be considered as rigorous for some assumed value of t, while the second is not rigorous but has its error very small and of the same order as α^2 for every value t intermediate between 0 and t. We are next to calculate, by means of this approximate expression (29), the definite integral (9), which we know must thus be found with only an error of the order of α^4; and in this way we obtain

$$x'_{t,} = b(1 + \tfrac{1}{2}\alpha t_{,}^2) + \alpha x_0 t_{,}, \tag{30}$$

$$\begin{aligned} s = \int_0^t \{&\tfrac{1}{2}b^2(1 + 2\alpha t_{,}^2 + \tfrac{7}{12}\alpha^2 t_{,}^4 + \tfrac{1}{36}\alpha^3 t_{,}^6) + bx_0(2\alpha t_{,} + \tfrac{7}{6}\alpha^2 t_{,}^3 + \tfrac{1}{12}\alpha^3 t_{,}^5) \\ &+ \tfrac{1}{2}x_0^2(\alpha + 2\alpha^2 t_{,}^2 + \tfrac{1}{4}\alpha^3 t_{,}^4)\}\,dt, \\ = \tfrac{1}{2}b^2&(t + \tfrac{2}{3}\alpha t^3 + \tfrac{7}{60}\alpha^2 t^5 + \tfrac{1}{252}\alpha^3 t^7) + bx_0(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6) \\ &+ \tfrac{1}{2}x_0^2(\alpha t + \tfrac{2}{3}\alpha^2 t^3 + \tfrac{1}{20}\alpha^3 t^5), \end{aligned} \tag{31}$$

$$\begin{aligned} \delta s = \{&x_0(\alpha t + \tfrac{2}{3}\alpha^2 t^3 + \tfrac{1}{20}\alpha^3 t^5) + b(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6)\}\,\delta x_0 \\ &+ \{x_0(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6) + b(t + \tfrac{2}{3}\alpha t^3 + \tfrac{7}{60}\alpha^2 t^5 + \tfrac{1}{252}\alpha^3 t^7)\}\,\delta b; \end{aligned} \tag{32}$$

and since this variation δs must coincide by (10) with the following,

$$x'_t\,\delta x_t - x'_0\,\delta x_0 = \{x'_t(1 + \tfrac{1}{2}\alpha t^2) - x'_0\}\,\delta x_0 + x'_t(t + \tfrac{1}{6}\alpha t^3)\,\delta b, \tag{33}$$

we have, by comparing the coefficients of δx_0 and δb, the two equations following,

$$\left.\begin{aligned} x'_t(1 + \tfrac{1}{2}\alpha t^2) - x'_0 &= x_0(\alpha t + \tfrac{2}{3}\alpha^2 t^3 + \tfrac{1}{20}\alpha^3 t^5) + b(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6), \\ x'_t(t + \tfrac{1}{6}\alpha t^3) &= x_0(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6) + b(t + \tfrac{2}{3}\alpha t^3 + \tfrac{7}{60}\alpha^2 t^5 + \tfrac{1}{252}\alpha^3 t^7); \end{aligned}\right\} \tag{34}$$

between which and (28) we are to eliminate x'_t and b, that we may obtain an expression of the sought kind for x_t as a function of t and of the two initial data x_0 and x'_0, which shall err only by a quantity of the same order as α^4. The elimination may conveniently be conducted thus:—multiplying the approximate expression (30) for $x'_{t,}$, by the coefficient of x'_t in the second equation (34), after changing t to $t_{,}$, we find

$$x'_{t,}(t_{,} + \tfrac{1}{6}\alpha t_{,}^3) = x_0(\alpha t_{,}^2 + \tfrac{1}{6}\alpha^2 t_{,}^4) + b(t_{,} + \tfrac{2}{3}\alpha t_{,}^3 + \tfrac{1}{12}\alpha^2 t_{,}^5), \tag{35}$$

so that this 2$^{\text{nd}}$ equation (34) may be transformed as follows,

$$x'_t = b\,(1+\tfrac{1}{2}\alpha t^2) + \alpha x_0 t + \frac{1}{t+\frac{1}{6}\alpha t^3}\{x_0\,(\tfrac{1}{8}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6) + b\,(\tfrac{1}{30}\alpha^2 t^5 + \tfrac{1}{252}\alpha^3 t^7)\}, \tag{36}$$

and therefore the 1$^{\text{st}}$ equation (34) gives

$$\begin{aligned} x'_0 = b\,(1+\alpha t^2+\tfrac{1}{4}\alpha^2 t^4) + x_0\,(\alpha t + \tfrac{1}{2}\alpha^2 t^3) + \frac{1+\frac{1}{2}\alpha t^2}{1+\frac{1}{6}\alpha t^2}\{x_0\,(\tfrac{1}{8}\alpha^2 t^3 + \tfrac{1}{72}\alpha^3 t^5) + b\,(\tfrac{1}{30}\alpha^2 t^4 + \tfrac{1}{252}\alpha^3 t^6)\} \\ - b\,(\alpha t^2 + \tfrac{7}{24}\alpha^2 t^4 + \tfrac{1}{72}\alpha^3 t^6) - x_0\,(\alpha t + \tfrac{2}{3}\alpha^2 t^3 + \tfrac{1}{20}\alpha^3 t^5), \end{aligned} \tag{37}$$

that is, neglecting α^4,

$$\begin{aligned} x'_0 &= b\,(1-\tfrac{1}{24}\alpha^2 t^4 - \tfrac{1}{72}\alpha^3 t^6) - x_0\,(\tfrac{1}{6}\alpha^2 t^3 + \tfrac{1}{20}\alpha^3 t^5) \\ &\quad + (1+\tfrac{1}{3}\alpha t^2)\{b\,(\tfrac{1}{30}\alpha^2 t^4 + \tfrac{1}{252}\alpha^3 t^6) + x_0\,(\tfrac{1}{8}\alpha^2 t^3 + \tfrac{1}{72}\alpha^3 t^5)\} \\ &= b\,(1-\tfrac{1}{120}\alpha^2 t^4 + \tfrac{1}{840}\alpha^3 t^6) + x_0\,(-\tfrac{1}{24}\alpha^2 t^3 + \tfrac{1}{180}\alpha^3 t^5), \end{aligned} \tag{38}$$

an equation which gives

$$b = x'_0\,(1+\tfrac{1}{120}\alpha^2 t^4 - \tfrac{1}{840}\alpha^3 t^6) + x_0\,(\tfrac{1}{24}\alpha^2 t^3 - \tfrac{1}{180}\alpha^3 t^5), \tag{39}$$

and therefore, by (28), the expression (26) results, namely,

$$x_t = x_0\,(1+\tfrac{1}{2}\alpha t^2 + \tfrac{1}{24}\alpha^2 t^4 + \tfrac{1}{720}\alpha^3 t^6) + x'_0\,(t + \tfrac{1}{6}\alpha t^3 + \tfrac{1}{120}\alpha^2 t^5 + \tfrac{1}{5040}\alpha^3 t^7). \tag{26}$$

And a repetition of the same process of correction would give an expression for x_t which should err only by quantities of the same order as α^8 and so on for ever, the accuracy being *doubled* by every successive repetition.

(**5.**) On reviewing the foregoing process of successive and indefinite approximation, it will be perceived to be deduced, chiefly, from these two principles: First, that if the definite integral s be rigorously calculated by the formula (9), the function x_t on which it depends being such as to satisfy rigorously the differential equation (1), then the coefficients of the variation of that definite integral will be assigned rigorously by the formula (10); and second, that if, in calculating s, we employ a function which coincides rigorously with x_t at the limits of the integration and which errs only by a quantity of the order of α^n between those limits, the resulting value of s will err only by a quantity of the order of α^{2n}. The first of these principles, generalised, has long since conducted me to finite expressions for the integrals of the most important systems of differential equations in Optics and Dynamics; and a generalisation of the second of the same principles will lead, I hope, to processes of approximation practically useful. I have indeed already exemplified the theoretical possibility of employing a principle closely connected with it, so as to obtain approximate results of which the accuracy goes on increasing in geometrical progression, (namely, in the 26th and 27th articles* of my Second Essay on a General Method in Dynamics, *Phil. Trans.* Part I for 1835, pages 122, 123, 124); but because the extreme values of the variables were, in that former process of approximation, introduced *explicitly* into the expression of the Principal Function, an introduction which it would be almost impracticable to effect in the theory of the perturbations of the Moon or Planets, it was (in a manner) necessary to adopt a more convenient mode of proceeding: and the mode which has been sketched in the present letter is one of the most convenient among those which have since occurred to me.

* [Pp. 189–192 of this volume.]

(6.) Whatever form may be chosen of my Method of the Principal Function, when it is employed to improve an approximation, it bears a close analogy to the Method of the Variation of Parameters. Thus from the expression

$$x_t = x_0 + bt, \tag{16}$$

which is subject to an error of the 1st order when b is treated as equal to the initial datum x'_0, I passed, in the present letter, to the more exact expression

$$x_t = x_0(1+\tfrac{1}{2}\alpha t^2) + x'_0(t+\tfrac{1}{6}\alpha t^3), \tag{21}$$

by finding that the expression (16) might be retained with only an error of the 2nd order, provided that b was treated as variable and was expressed as follows:

$$b = x'_0(1+\tfrac{1}{6}\alpha t^2) + \tfrac{1}{2}\alpha x_0 t. \tag{20}$$

And again from the expression

$$x_t = x_0(1+\tfrac{1}{2}\alpha t^2) + b(t+\tfrac{1}{6}\alpha t^3), \tag{28}$$

in which b was known to be nearly equal to the constant x'_0, the more exact expression (26) was deduced by showing that b was more exactly equal to the function (39). But there is this essential difference between Lagrange's Method and mine, that *he* determined his varying parameter so as to make his expressions true for two moments infinitely near, *I* for two moments separated by a finite interval. Thus, while I keep the parameter x_0 unchanged and only alter b in (16) from the constant x'_0 to the function (20), Lagrange would have altered *both* x_0 and b or would have employed the system of the two expressions

$$x_t = a + bt, \quad x'_t = b, \tag{40}$$

which give

$$a = x_t - tx'_t; \tag{41}$$

and thus he would have decomposed the differential equation of the 2nd order

$$x''_t = \alpha x_t \tag{1}$$

into the system of the two following equations of the first order,

$$b' = \alpha x_t, \quad a' = -\alpha t x_t, \tag{42}$$

that is,

$$b' = \alpha(a+bt), \quad a' = -\alpha t(a+bt); \tag{43}$$

from which system both a and b should be deduced as functions of t. His method is unquestionably elegant in the highest degree and has the very great advantage of having been developed and employed by himself and by other eminent analysts, yet I do not despair of mine also being hereafter thought worthy to be developed and applied to the chief problems of dynamics by persons able to do it justice and disposed to try what can be done by taking a new track. For instance, I have some little hope that you, who confess your patience almost exhausted in working at the Theory of the Moon as it presents itself to other methods, including probably improvements of your own, may feel some curiosity to see developed, and may give your valuable aid to develope, the general and rigorous integrals which I have assigned: provided that I do not drag you through too wide a maze of abstract speculation, but reduce the problem nearly to its least terms and depart as little as the nature of my method will permit from those processes which have already been (to so great a degree) productive of success to yourself and others. At the very least there would be some utility in employing a new method as a *verification* of the results of an old one; but my hopes aspire somewhat higher.

I regret that I have so little to say on the subject of your important proposition respecting the construction of Empirical Lunar Tables. I should certainly like to see such Tables carefully constructed, and think they would be useful both practically for the calculation of Ephemerides and theoretically for comparison with formulae; but I am not prepared to join in pressing on the Association at this moment the going to the necessary expense; especially as Airy, whose opinion on such a subject must be admitted to possess great weight, appears to be against the doing so. But probably you would not be disinclined to wait another year, without pressing for a final decision; and before the meeting in 1838, if we all live, we are likely to have all formed more decided and concurrent opinions.

I regret also to have to mention that I have as yet not thought at all on the subject of your communication on Optics. Perhaps even before the approaching meeting of the Association I may find time to do so, and to put in writing the result of my thoughts.

Looking forward with great pleasure to our soon seeing each other again, I remain, my dear sir, with best remembrances to Mrs. Lubbock,

Very truly yours

WILLIAM R. HAMILTON.

B. *J. W. Lubbock to Sir W. R. Hamilton.*

London, Sept^{r}., 1837.

MY DEAR SIR,

There is another* great objection to your method which I forgot to mention. I apprehend if you employ it to determine the constants, the resulting equations of condition between which it will be necessary to eliminate will not be linear. Moreover if you obtain the values the trouble of the substitution required to produce the longitude or latitude will be immense. Pray write soon and say whether I may communicate to M. de Pontécoulant the example given in your last letter to me & whatever you may send me in future about the Moon. I am very anxious the question should be settled as soon as possible for no one will think of carrying forward my methods while there is any probability of their being superseded by others more feasible and the announcement at Liverpool of your intention to try to apply your method to the Theory of the Moon will make those hesitate who know your great power of analysis. I shall be very agreeably surprized if you ever get out a single term in the longitude or latitude which I have not already obtained & more recently M. de Pontécoulant.

The quantities ticked in the Appendix enclosed are right, there are others undecided where I have obtained a term differing from that of M. de Pontécoulant.

I think the method of your last letter to me might be applied to find the equations which I have found for the perturbations according to powers of the time explicitly p. 39 of my Tract on Comets, and would furnish an excellent example of your method. Pray write to me soon and if you can give a glance at the optical paper I sent you.

Meanwhile, I remain, dear Sir,

Yours very truly

J. W. LUBBOCK.

---

* [Probably Lubbock had in the meantime discussed the matter with Hamilton at the British Association Meeting in Liverpool.]

II. *Sir W. R. Hamilton to J. W. Lubbock.*

(*II^d Letter to Mr Lubbock on the Theory of the Moon.*)

Observatory, Oct. 4, 1837.

MY DEAR SIR,

I am about to write to you, a little more in detail than I have yet done, though still only in a very undeveloped way, upon the Theory of the Moon. And because my present remarks will be found to have a close connexion with those contained in the letter which I wrote to you before the Liverpool Meeting, I shall (though the expedient is not an elegant one) continue to number my equations as they come, without reference to their importance or affinities, considering them as forming one unbroken series from the beginning of the former letter, & therefore calling the first of the new equations number 44, because the last of the old ones was number 43. I shall also call this letter itself II, supposing the former to have been called I; & shall retain the same plan of numbering in others which may follow these.

Adopting your fundamental equations

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\left(\frac{dR}{dx}\right)=0, \quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+ \ \frac{dR}{dy} \ =0, \quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}+\left(\frac{dR}{dz}\right)=0, \qquad (44)$$

which I shall write as follows

$$x''+\frac{\mu x}{r^3}+\frac{\delta R}{\delta x}=0, \qquad +\frac{\mu y}{r^3}+\frac{\delta R}{\delta y}=0, \quad z''+\frac{\mu z}{r^3}+\frac{\delta R}{\delta z}=0, \qquad (45)$$

and in which, according to your notation, x, y, z, r are the Moon's geocentric rectangular coordinates and geocentric distance, while μ is the sum of the masses of Moon and Earth, & R is the disturbing function,

$$R=m_,\left\{\frac{xx_,+yy_,+zz_,}{r_,^3} \qquad \frac{1}{\sqrt{\{r_,^2-2\,(xx_,+yy_,+zz_,)+r^2\}}}\right\} \qquad (46)$$

if $m_,$ be the Sun's mass, & $x_,$, $y_,$, $z_,$, $r_,$ its geocentric coordinates and distance; I shall suppose the 4 last mentioned quantities to be explicit functions of the time t, namely those corresponding to that elliptic relative motion which the Sun would have if the mass of the Moon were to vanish. This supposition is equivalent to treating the Moon as exerting no sensible attraction on Earth or Sun, although it is attracted by them, and they by each other. How far this supposition must fall short, when fully developed, of conducting to results as accurate as those obtained by the methods of yourself and others, you are a much better judge than I can at present be; but I think that the error must be small, & am sure that it will be smaller than other errors which I propose at the outset to neglect. Yet I should be glad to be honoured with your opinion on the degree of error which may in this way be occasioned.

I add the 3 equations (45) multiplied respectively by δx, δy, δz and thus obtain

$$x''\delta x+y''\delta y+z''\delta z+\mu\frac{\delta r}{r^2}+\delta R=0; \qquad (47)$$

δt being here treated as $=0$, & only x, y, z (& of course r) as subject to the variation δ. To (47), I add the identical equation

$$x'\,(\delta x)'+y'\,(\delta y)'+z'\,(\delta z)'=\delta\frac{x'^2+y'^2+z'^2}{2}; \qquad (48)$$

& thus obtain

$$(x'\delta x+y'\delta y+z'\delta z)'=\delta\left(\frac{x'^2+y'^2+z'^2}{2}+\frac{\mu}{r}-R\right), \tag{49}$$

& consequently, by integration,

$$x'\delta x+y'\delta y+z'\delta z-x_0'\delta x_0-y_0'\delta y_0-z_0'\delta z_0=\delta\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\frac{\mu}{r}-R\right)dt: \tag{50}$$

the quantities marked with 0 as a lower index being here initial values. This equation would be rigorously true if the Moon's mass were rigorously null; at some future time I may discuss the correction which must be introduced when that small mass is allowed for.

In the next place, I neglect the inclination of the Moon's orbit; & thus obtain, (the ecliptic being taken for the plane of xy,)

$$x'\delta x+y'\delta y-x_0'\delta x_0-y_0'\delta y_0=\delta\int_0^t\left(\frac{x'^2+y'^2}{2}+\frac{\mu}{r}-R\right)dt; \tag{51}$$

in which

$$R=m_{,}\left\{\frac{xx_{,}+yy_{,}}{r_{,}^3}-\frac{\;}{\sqrt{r_{,}^2-2\,(xx_{,}+yy_{,})+r^2}}\right\}, \tag{52}$$

$$r=\sqrt{x^2+y^2}, \tag{53}$$

$$r_{,}=\sqrt{x_{,}^2+y_{,}^2}: \tag{54}$$

so that if we introduce the longitudes λ, $\lambda_{,}$, putting

$$x=r\cos\lambda,\quad y=r\sin\lambda, \tag{55}$$

and

$$x_{,}=r_{,}\cos\lambda_{,},\quad y_{,}=r_{,}\sin\lambda_{,}, \tag{56}$$

we have

$$r'\delta r+r^2\lambda'\delta\lambda-r_0'\delta r_0-r_0^2\lambda_0'\delta\lambda_0=\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}-R\right)dt, \tag{57}$$

and

$$R=m_{,}\{rr_{,}^{-2}\cos\overline{\lambda-\lambda_{,}}-(r_{,}^2-2rr_{,}\cos\overline{\lambda-\lambda_{,}}+r^2)^{-\frac{1}{2}}\}. \tag{58}$$

I neglect the excentricity of the Solar orbit, and thus obtain

$$r_{,}=a_{,},\quad \lambda_{,}=n_{,}t,\quad m_{,}=a_{,}^3n_{,}^2; \tag{59}$$

and consequently

$$R=-n_{,}^2a_{,}^2-n_{,}^2\left(r^2P_2+\frac{r^3}{a_{,}}P_3+\frac{r^4}{a_{,}^2}P_4+\&c.\right), \tag{60}$$

if we denote by P_2, P_3, P_4, &c. the coefficients of the development

$$(1-2\alpha p+\alpha^2)^{-\frac{1}{2}}=1+\alpha p+\alpha^2P_2+\alpha^3P_3+\alpha^4P_4+\&c., \tag{61}$$

in which

$$\alpha=\frac{r}{a_{,}},\quad p=\cos\overline{\lambda-n_{,}t}. \tag{62}$$

The general expression for these coefficients being

$$P_i=\frac{1}{1\,.\,2\,.\,3\,.\,4\ldots i}\left(\frac{d}{dp}\right)^i\left(\frac{p^2-1}{2}\right)^i, \tag{63}$$

we have, in particular,

$$P_2=\frac{1}{2}\left(\frac{d}{dp}\right)^2\left(\frac{p^2-1}{2}\right)^2=\frac{3p^2-1}{2}=\tfrac{3}{4}(2p^2-1)+\tfrac{1}{4}=\tfrac{1}{4}(1+3\cos\overline{2\lambda-2n_{,}t}); \tag{64}$$

I shall therefore put, for the present,

$$R = -\frac{n_,^2}{4} r^2 (1 + 3\cos \overline{2\lambda - 2n_, t}), \tag{65}$$

suppressing the term $-n_,^2 a_,^2$ as being constant, (and as consequently contributing nothing to the variation $\delta \int_0^t R\,dt$,) and neglecting the terms multiplied by $n_,^2 \frac{r}{a_,}$, on account of the smallness of that factor.

If we entirely neglected the disturbing function R, we should have elliptic expressions for the Moon's polar coordinates r & λ, namely

$$r = a - ak\cos nt + al\sin nt, \tag{66}$$

$$\lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt; \tag{67}$$

in which k and l are arbitrary but small constants, of which the squares and products are here neglected; (the first power only of the excentricity of the Moon's orbit being retained;) ϵ and n are other arbitrary but not small constants; and a depends on n by the relation

$$a^3 n^2 = \mu. \tag{68}$$

With these expressions we could verify the relation (57); for, making, after the variation δ, k & l each $= 0$, we have

$$\begin{aligned} r'\delta r + r^2\lambda'\delta\lambda - r_0'\delta r_0 - r_0^2\lambda_0'\delta\lambda_0 &= a^2 n\,\delta(\lambda - \lambda_0) \\ &= a^2 n\,(t\,\delta n + 2\sin nt\,\delta k + 2\,\overline{\cos nt - 1}\,\delta l); \end{aligned} \tag{69}$$

and also

$$\begin{aligned} \delta \int_0^t \left(\frac{r'^2 + r^2\lambda'^2}{2} + \frac{\mu}{r}\right) dt &= \delta \int_0^t a^2 n^2 \left(\tfrac{3}{2} + 2k\cos nt - 2l\sin nt\right) dt \\ &= \delta \Big\{ a^2 n \Big(\frac{3nt}{\,} + 2k\sin nt + 2l\cos nt - 1 \\ &= a^2 n\,(t\,\delta n + 2\sin nt\,\delta k + 2\cos nt - 1\,\delta l). \end{aligned} \tag{70}$$

But when we take account of R, we must consider the expressions (66) & (67) as being defective, & as requiring to be corrected by the addition of terms proportional to $n_,^2$, if n, ϵ, k, l be still considered as constant. However, by considering n, ϵ, k, l $\left(\text{\& consequently } a = \sqrt[3]{\frac{\mu}{n^2}}\right)$, as suitable functions of the time, we may still retain the equations (66) and (67), & the analogous initial equations

$$r_0 = a - ak, \tag{71}$$

$$\lambda_0 = \epsilon + 2l; \tag{72}$$

though we cannot then retain also the differentiated expressions

$$r' = an\,(k\sin nt + l\cos nt), \tag{73}$$

$$\lambda' = n\,(1 + 2k\cos nt - 2l\sin nt), \tag{74}$$

and

$$r_0' = anl, \tag{75}$$

$$\lambda_0' = n\,(1 + 2k). \tag{76}$$

And to find, to the accuracy of terms proportional to $n_,^2$ inclusive, (the error being proportional to $n_,^4 = (n_,^2)^2$,) the new expressions analogous to (73), (74), (75), (76), which are to be combined with (66), (67), (68), (71), (72), we are, by my general method, to develope the 2nd member of (57) as a linear function of δn, $\delta\epsilon$, δk, δl, employing for R the expression (65) & using also the expressions (66), (67), (68), (73), (74) & making, after having taken the variation δ, the products $n_{,}^2 k$ & $n_{,}^2 l$ vanish and indeed k & l themselves.

In this manner I obtain the formula

$$\Delta r' \delta r + a^2 \Delta\lambda' \delta\lambda - \Delta r_0' \delta r_0 - a^2 \Delta\lambda_0' \delta\lambda_0 = -\delta \int_0^t R\,dt \tag{77}$$

in which $\Delta r'$, $\Delta\lambda'$, $\Delta r_0'$, $\Delta\lambda_0'$ are corrections proportional to $n_{,}^2$ (and perhaps partly to $n_{,}^3$), which are to be added to the 2nd members of the equations (73), (74), (75), (76), in order to make those equations consistent with (66), (67), (68), (71), (72). And because we are to make in the 1st member of (77)

$$\delta r = \delta a - a \cos nt\, \delta k + a \sin nt\, \delta l, \tag{78}$$

$$\delta\lambda = t\,\delta n + \delta\epsilon + 2 \sin nt\, \delta k + 2 \cos nt\, \delta l, \tag{79}$$

$$\delta r_0 = \delta a - a\,\delta k, \tag{80}$$

$$\delta\lambda_0 = \delta\epsilon + 2\delta l, \tag{81}$$

and

$$\delta a = -\frac{2}{3}\frac{a}{n}\,\delta n, \tag{82}$$

we have, by the mutual independence of δn, $\delta\epsilon$, δk, δl, the 4 following equations

$$-\frac{\delta}{\delta n}\int_0^t R\,dt = -\frac{2}{3}\frac{a}{n}(\Delta r' - \Delta r_0') + a^2 t\,\Delta\lambda'; \tag{83}$$

$$-\frac{\delta}{\delta\epsilon}\int_0^t R\,dt = a^2(\Delta\lambda' - \Delta\lambda_0'); \tag{84}$$

$$-\frac{\delta}{\delta k}\int_0^t R\,dt = -a(\cos nt\,\Delta r' - \Delta r_0') + 2a^2 \sin nt\,\Delta\lambda'; \tag{85}$$

$$-\frac{\delta}{\delta l}\int_0^t R\,dt = a \sin nt\,\Delta r' + 2a^2(\cos nt\,\Delta\lambda' - \Delta\lambda_0'). \tag{86}$$

There would be little difficulty in resolving these 4 linear equations so as to deduce $\Delta r'$, $\Delta\lambda'$, $\Delta r_0'$, $\Delta\lambda_0'$ as functions of n and t & of the 4 partial differential coefficients of $\int_0^t R\,dt$, which coefficients themselves may be regarded as functions of $n_{,}$, n and t. And after correcting the equations (75) and (76) by the values thus found for $\Delta r_0'$ & $\Delta\lambda_0'$, we might then eliminate the 5 quantities a, n, ϵ, k, l between the 7 equations (66), (67), (68), (71), (72), (75), (76), so as to get expressions for r and λ as functions of t, $n_{,}$, r_0, λ_0, r_0', λ_0'. But because the last elimination would not be linear, it is convenient to proceed as follows.

By altering the symbols r, λ, a, n, ϵ, k, l to the following, $r + \Delta r$, $\lambda + \Delta\lambda$, $a + \Delta a$, $n + \Delta n$, $\epsilon + \Delta\epsilon$, $k + \Delta k$, $l + \Delta l$, we may suppose that the 7 new quantities r, λ, a, n, ϵ, k, l satisfy the 7 old conditions (66), (67), (68), (71), (72), (75), (76), provided that we determine the 7 corrections Δr, $\Delta\lambda$, Δa, Δn, $\Delta\epsilon$, Δk, Δl so as to satisfy these other 7 conditions

$$\Delta r = \Delta a - a \cos nt\,\Delta k + a \sin nt\,\Delta l, \tag{87}$$

$$\Delta\lambda = t\,\Delta n + \Delta\epsilon + 2 \sin nt\,\Delta k + 2 \cos nt\,\Delta l, \tag{88}$$

$$\Delta a = -\frac{2}{3}\frac{a}{n}\,\Delta n, \tag{89}$$

$$0 = \Delta a - a\,\Delta k, \tag{90}$$

$$0 = \Delta\epsilon + 2\Delta l, \tag{91}$$

$$0 = an\,\Delta l + \Delta r_0', \tag{92}$$

$$0 = \Delta n + 2n\,\Delta k + \Delta\lambda_0'; \tag{93}$$

and thus Δr and $\Delta\lambda$ will be the perturbational parts of the Moon's distance and longitude, & will be expressed as linear functions of the 4 partial differential coefficients of $\int_0^t R\,dt$. This new elimination is by no means difficult,* and it gives

$$\Delta r = -\frac{2}{an}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt + \frac{\sin nt}{an}\frac{\delta}{\delta k}\int_0^t R\,dt + \frac{\cos nt}{an}\frac{\delta}{\delta l}\int_0^t R\,dt; \tag{94}$$

$$\Delta\lambda = \frac{3t}{a^2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt - \frac{3}{a^2}\frac{\delta}{\delta n}\int_0^t R\,dt + \frac{2\cos nt}{a^2 n}\frac{\delta}{\delta k}\int_0^t R\,dt - \frac{2\sin nt}{a^2 n}\frac{\delta}{\delta l}\int_0^t R\,dt; \tag{95}$$

expressions which may be considered as remarkable for their form & for their simplicity. Besides, we have, by (65),

$$\begin{aligned}\int_0^t R\,dt = -\frac{n_{,}^2}{4}\Big\{&a^2 t + \frac{3a^2}{2n-2n}\left(\sin\overline{2nt-2n_{,}t+2\epsilon} - \sin 2\epsilon\right)\\ &+\frac{n_{,}^2 a^2 k}{4}\Big\{\frac{2}{n}\sin nt - \frac{3}{3n-2n_{,}}\left(\sin\overline{3nt-2n_{,}t+2\epsilon} - \sin 2\epsilon\right)\\ &+\frac{9}{n-2n_{,}}\left(\sin\overline{nt-2n_{,}t+2\epsilon} - \sin 2\epsilon\right)\Big\}\\ &-\frac{n_{,}^2 a^2 l}{4}\Big\{\frac{2}{n}(1-\cos nt) + \frac{3}{3n-2n_{,}}\left(\cos\overline{3nt-2n_{,}t+2\epsilon} - \cos 2\epsilon\right)\\ &+\frac{9}{n-2n_{,}}\left(\cos\overline{nt-2n_{,}t+2\epsilon} - \cos 2\epsilon\right)\Big\};\end{aligned} \tag{96}$$

so that the expressions for Δr and $\Delta\lambda$ become

$$\begin{aligned}\Delta r = \frac{n_{,}^2 a}{2n}&\Big\{\frac{3}{n-n_{,}}\left(\cos\overline{2nt-2n_{,}t+2\epsilon} - \cos 2\epsilon\right) + \frac{1}{n}(1-\cos nt)\Big\}\\ &-\frac{3n_{,}^2 a}{4n(3n-2n_{,})}\left(\cos\overline{2nt-2n_{,}t+2\epsilon} - \cos\overline{nt-2\epsilon}\right)\\ &-\frac{9n_{,}^2 a}{4n(n-2n_{,})}\left(\cos\overline{2nt-2n_{,}t+2\epsilon} - \cos\overline{nt+2\epsilon}\right);\end{aligned} \tag{97}$$

and

$$\begin{aligned}\Delta\lambda = -\frac{n_{,}^2 t}{n} &- \frac{3n_{,}^2}{2n(n-n_{,})}\left(\sin\overline{2nt-2n_{,}t+2\epsilon} - \sin 2\epsilon\right) + \frac{9n_{,}^2 t\cos 2\epsilon}{4(n-n_{,})}\\ &-\frac{9n_{,}^2}{8(n-n_{,})^2}\left(\sin\overline{2nt-2n_{,}t+2\epsilon} - \sin 2\epsilon\right) + \frac{n_{,}^2}{n^2}\sin nt\\ &-\frac{3n_{,}^2}{2n(3n-2n_{,})}\left(\sin\overline{2nt-2n_{,}t+2\epsilon} + \sin\overline{nt-2\epsilon}\right)\\ &+\frac{9n_{,}^2}{2n(n-2n_{,})}\left(\sin\overline{2nt-2n_{,}t+2\epsilon} - \sin\overline{nt+2\epsilon}\right);\end{aligned} \tag{98}$$

---

* [See pp. 241–243, $s = -\int_0^t R\,dt$.]

or, in the same order of approximation,

$$\begin{aligned}\Delta r = {} & \left(\frac{n_{,}}{n}\right)^2 \frac{a}{2}\left(1 - 3\cos 2\epsilon - 3\frac{n_{,}}{n}\cos 2\epsilon\right) \\ & - \left(\frac{n_{,}}{n}\right)^2 \frac{a}{2}\left(1 - 5\cos 2\epsilon - \frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right)\cos nt \\ & - \left(\frac{n_{,}}{n}\right)^2 a\left(2 + \frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon \sin nt \\ & - \left(\frac{n_{,}}{n}\right)^2 a \quad 1 + \frac{19}{6}\frac{n_{,}}{n}\Big)\cos \overline{2nt - 2n_{,}t + 2\epsilon};\end{aligned} \tag{99}$$

and

$$\begin{aligned}\Delta\lambda = - \qquad {}^{2} nt\left(1 - \frac{9}{4}\cos 2\epsilon - \frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right) + \frac{3}{2}\left(\frac{n_{,}}{n}\right)^2\left(\frac{7}{4} + \frac{5}{2}\frac{n_{,}}{n}\right)\sin 2\epsilon \\ + \left(\frac{n_{,}}{n}\right)^2\left(1 - 5\cos 2\epsilon - \frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right)\sin nt \\ - 2\left(\frac{n_{,}}{n}\right)^2\left(2 + \frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon\cos nt \\ + \left(\frac{n_{,}}{n}\right)^2\left(\frac{11}{8} + \frac{59}{12}\frac{n_{,}}{n}\right)\sin \overline{2nt - 2n_{,}t + 2\epsilon}.\end{aligned} \tag{100}$$

These, then, are, as results of the present method and in the present order of approximation, expressions for those parts of the perturbations of the Moon's distance and longitude which are independent of the excentricities & inclinations; and by the nature of the process they are so constructed that both themselves and their first differential coefficients vanish at the origin of the time: which is in some respects advantageous. We may however dispense with this last condition, and introduce, instead, an incorporation of certain terms of the expressions (99) & (100) with the corresponding terms of the elliptic expressions (66) & (67). With this latter view we may put

$$\mathrm{n} = n - \frac{n_{,}^2}{n}\left(1 - \frac{9}{4}\cos 2\epsilon - \frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{101}$$

$$\mathrm{a} = \sqrt[3]{\frac{\mu}{\mathrm{n}^2}} = a + \frac{2}{3}\left(\frac{n_{,}}{n}\right)^2 a\left(1 - \frac{9}{4}\cos 2\epsilon - \frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{102}$$

$$\mathrm{e} = \epsilon + \frac{3}{8}\left(\frac{n_{,}}{n}\right)^2\left(7 + 10\frac{n_{,}}{n}\right)\sin 2\epsilon; \tag{103}$$

$$\mathrm{k} = k + \frac{1}{2}\left(\frac{n_{,}}{n}\right)^2\left(1 - 5\cos 2\epsilon - \frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{104}$$

$$= l - \left(\frac{n_{,}}{n}\right)^2\left(2 + \frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon; \tag{105}$$

and if we put, besides,

$$m = \frac{n_{,}}{\mathrm{n}}, \tag{106}$$

and

$$\tau = \mathrm{n}t - n_{,}t + \epsilon, \tag{107}$$

we find, by (99), (100) and (66), (67),

$$\frac{r+\Delta r}{\mathrm{a}} = 1 - \frac{m^2}{6} - \mathrm{k}\cos \mathrm{n}t + \mathrm{l}\sin \mathrm{n}t - m^2 \left(1 + \frac{19}{\ } m\right) \cos 2\tau; \tag{108}$$

and

$$\lambda + \Delta\lambda = \mathrm{n}t + \mathrm{e} + 2\mathrm{k}\sin \mathrm{n}t + 2\mathrm{l}\cos \mathrm{n}t + m^2\left(\frac{11}{8} + \frac{59}{12}m\right)\sin 2\tau : \tag{109}$$

results which perfectly agree with those of yourself and of Plana.

Even this slight sketch of an actual application of my method to the Theory of the Moon may make my meaning clearer than it was before, and may prepare for future & more laborious & subtle applications.

I am, my dear Sir,
very truly yours
WILLIAM R. HAMILTON.

J. W. LUBBOCK, Esq$^{\mathrm{r}}$.

C. *J. W. Lubbock to Sir W. R. Hamilton.*

29 EATON PLACE.
Tues., 10th *Oct*r., 1837.

MY DEAR SIR,

I have received your highly interesting letter of the 4th, the example you have given is very instructive and is worthy of your great mathematical talent. Even if your method should prove impracticable, as an example of analysis every mathematician will wish to see it carried to some other terms besides those contained in your last letter.

I do not understand why you "make after the variation δ the products $n_{,}^2 k$ and $n_{,}^2 l$ vanish and indeed k and l themselves."

I suspect your method as modified by the introduction of the quantities Δa, Δn, $\Delta\epsilon$, Δk and Δl amounts only to *another* way of finding Lagrange's values of these quantities. Thus for example I see that you get

$$\Delta l = \frac{1}{an}\frac{\delta}{\delta k}\int_0^t R\,dt$$

which is only another way of writing

$$dl = \frac{1}{an}\frac{\delta R}{\delta k}dt$$

and this last I believe is Lagrange's value. I suspect therefore that if you write Lagrange's values for $\Delta a = \int da$, $\Delta k = \int dk$, &c. in your equations (87) and (88) you will get precisely the values you have written down for Δr and $\Delta\lambda$ in equations (94) and (95).

If you take the trouble to calculate in my way your equations (108) and (109), you will judge which method gives less trouble. But a great difficulty begins in your method when you introduce the perturbational terms in R in order to get a further approximation because you must either develope

$$\frac{\delta}{\delta n}\int R\,dt, \quad \frac{\delta}{\delta t}\int R\,dt, \quad \frac{\delta}{\delta k}\int R\,dt, \quad \frac{\delta}{\delta l}\int R\,dt$$

separately or you must do what will be equally troublesome.

I look upon this method of the variation of the constants as quite impracticable except for the easy terms and this is also the opinion of M. de Pontécoulant.

There are very few sensible terms depending upon the Moon's mass. I have calculated[*] one of them as an example of my methods and found a term differing from M. Poisson, but M. Pontécoulant tells me that he has found a numerical slip in M. Poisson's work which accounts for it.

The equation to the fluid surface of the ocean being

$$(X-u')\,dx+(Y-v')\,dy+(Z-w')\,dz=0,$$

u', v', w' the complete differentials of $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$, cannot we integrate and say

$$\int\{X\,dx+Y\,dy+Z\,dz\}-\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}-\frac{dz^2}{dt^2}=\text{constant?}$$ [†]

The Theory of the Tides in the *Mécanique Céleste* is the only portion which has not been commented on by Lagrange, by Poisson & by others. I wish you had time to examine it. I think much might be done in solving the problem at least in settling what are its real difficulties.

Will you oblige me by writing out according to your method the steps which would belong to the equation

$$x''+x+\frac{dR}{dx}=0,$$

beginning from

$$x'\delta x-x_0'\delta x_0=\delta\int_0^t\left\{\frac{x'^2}{2}-\frac{x^2}{2}-R\right\}dt? \tag{50}$$

The approximate integrals are

$$x=a\cos(t+b),\quad x'=-a\sin(t+b).$$

Lagrange's equations give

$$-\frac{dR}{da}\,dt=\left\{\frac{dx'}{db}\frac{dx}{da}-\frac{dx}{db}\frac{dx'}{da}\right\}db=-a\,db,$$

$$-\frac{dR}{db}\,dt=\left\{\frac{dx'}{da}\frac{dx}{db}-\frac{dx}{da}\frac{dx'}{db}\right\}da=a\,da.$$

M. Poisson's equations give, directly,

$$db=-\frac{1}{a}\frac{dR}{da}\,dt,\quad da=-\frac{1}{a}\frac{dR}{db}\,dt.$$

You must arrive by your methods at these equations and I should like to see how you proceed. I think you hardly appreciate yet sufficiently the difficulties which will arise even in your method from the *depression* by integration, in the theory of the Moon and in similar problems.

I shall be delighted to hear from you whenever you have time to write and it will give me the greatest pleasure to find that your methods will lead to more rapid approximations than those which have been hitherto employed, including my own.

Ever most truly yours

J. W. LUBBOCK.

---

[*] [Lubbock, *Tracts on Lunar Theory*, p. 289.]

[†] [This, with the obvious numerical slip corrected, is true if the $\int(u\,dx+v\,dy+w\,dz)$, taken between any two points of the liquid and moving with it, is independent of the time.]

P.S. Allow me to venture a few remarks upon your method, which I do with great diffidence for I am not sure yet if I quite understand it.

For simplicity I will take the differential equation

$$x'' + x + \frac{dR}{dx} = 0.$$

The approximate integrals of this equation are

$$x = a\cos(t+b), \quad x' = -a\sin(t+b).$$

Your equation (50) is

$$x'\delta x - x_0'\delta x_0 = \delta\int_0^t\left(\frac{x'^2}{2} - \frac{x^2}{2} - R\right)dt.$$

Your equation (77) is

$$\Delta x'\delta x - \Delta x_0'\delta x_0 = -\delta\int_0^t R\,dt,$$

$$x' = f'(a,b) + \Delta x', \quad x = f(a,b) + \Delta x,$$

where

$f'(a,b)$ is in this example $-a\sin(t+b)$,
$f(a,b)$ $a\cos(t+b)$.

If I write $\bar{x}'$ for $f'(a,b)$ and $\bar{x}$ for $f(a,b)$ for convenience merely,

$$\delta x = \frac{d\bar{x}}{da}\delta a + \frac{d\bar{x}}{db}\delta b, \tag{51}$$

$$\delta x_0 = \frac{d\bar{x}_0}{da}\delta a + \frac{d\bar{x}_0}{db}\delta b. \tag{52}$$

The condition that $f(a,b)$ represents the value of x in the disturbed motion gives $\Delta x = 0$,

$$\frac{d\bar{x}}{da}\Delta a + \frac{d\bar{x}}{db}\Delta b + \frac{1}{1\,.\,2}\frac{d^2\bar{x}}{da^2}\Delta a^2 + \frac{1}{1\,.\,2}\frac{d^2\bar{x}}{db^2}\Delta b^2 + \&c. = 0, \tag{53}$$

$$\frac{d\bar{x}_0}{da}(\Delta a)_0 + \frac{d\bar{x}_0}{db}(\Delta b)_0 + \frac{1}{1\,.\,2}\frac{d^2\bar{x}_0}{da^2}(\Delta a)_0^2 + \frac{1}{1\,.\,2}\frac{d^2\bar{x}_0}{db^2}(\Delta b)_0^2 + \&c. = 0, \tag{54}$$

$$\Delta x' = -\frac{d\bar{x}'}{da}\Delta a - \frac{d\bar{x}'}{db}\Delta b - \frac{1}{1\,.\,2}\frac{d^2\bar{x}'}{da^2}\Delta a^2 - \&c., \tag{55}$$

$$\Delta x_0' = -\frac{d\bar{x}_0'}{da}(\Delta a)_0 - \frac{d\bar{x}_0'}{db}(\Delta b)_0 - \&c. \tag{56}$$

Equation (50) gives

$$\Delta x'\frac{d\bar{x}}{da} - \Delta x_0'\frac{d\bar{x}_0}{da} = -\int_0^t\frac{dR}{da}dt, \tag{57}$$

$$\Delta x'\frac{d\bar{x}}{db} - \Delta x_0'\frac{d\bar{x}_0}{db} = -\int_0^t\frac{dR}{db}dt. \tag{58}$$

These equations correspond to the equations (83), (84), (85), and (86) of your IInd Letter.

Substituting in (57) and (58) the values of $\Delta x'$ and $\Delta x_0'$ from (55) and (56) I get, *neglecting* Δa^2, Δb^2, &c.,

$$-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{da}\Delta a + \frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{da}\Delta a_0 - \frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\Delta b + \frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{db}\Delta b_0 = -\int_0^t\frac{dR}{da}dt.$$

Multiply (53) by $\dfrac{d\bar{x}'}{da}$ and add it to the last equation,

(54) by $\dfrac{d\bar{x}_0'}{da}$ and subtract it from the last equation,

and we get, the coefficients of Δa and $(\Delta a)_0$ vanishing,

$$\left\{\frac{d\bar{x}}{db}\frac{d\bar{x}'}{da}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\right\}\Delta b-\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}_0'}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{db}\right\}\Delta b_0=-\int_0^t\frac{dR}{da}\,dt.$$

But Lagrange has shown generally that the quantity which is the coefficient of Δb is constant; hence we may write

$$\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}_0'}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{db}\right\}\{\Delta b-\Delta b_0\}=-\int_0^t\frac{dR}{da}\,dt$$

and, differentiating,

$$-\frac{dR}{da}\,dt=\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}_0'}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{db}\right\}db,$$

which is Lagrange's equation.

It would have been as easy to take a more complicated example, but the reasoning would have been exactly the same.

It seems to me that you ought to distinguish between Δa and Δa_0,* Δb and Δb_0. For the equation

$$\left\{\frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{da}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{da}\right\}\Delta a+\left\{\frac{d\bar{x}_0}{da}\frac{d\bar{x}_0'}{db}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\right\}\Delta b=-\int\frac{dR}{da}\,dt,$$

which your methods would lead to (?) if Δa is not distinguished from Δa_0, is incorrect.

I have written out this since I closed my letter & if I am wrong I am sure you will not be severe.

III. *Sir W. R. Hamilton to J. W. Lubbock.*

Observatory, Oct. 12th, 1837.

My dear Sir,

Your letter of the 10th has just arrived & I hasten to make some remarks in reply to it, & in continuation of my own last letter.

The sentence respecting the making k and l vanish was certainly obscure; I felt it to be so at the time of writing it, but trusted that my meaning would be fully caught when I should have proceeded farther than I have yet done in the development & application of my method; meantime I may remind you of what you yourself foresaw, namely that I am obliged to *drop an excentricity* in passing from one approximation to another; so that starting only with expressions for the coordinates r & λ which are accurate to the 1st power of the Moon's excentricity inclusive, I can only expect to get the circular parts of the perturbations, or the parts independent of excentricity. For this reason I omitted as useless (because liable to be changed by combination with unknown terms of the same order) the terms which were multiplied by $n_{,}^{2}k$ and $n_{,}^{2}l$;—the words "and indeed k and l themselves" were added in a hurry afterwards, & though I do not exactly admit them to be erroneous, I feel that they require more commentary & justification than I think it worth while to give, considering that I intend to resume the subject soon, in a more full & explicit manner.

---

* [In the Hamilton ellipse, the initial velocity is different from the orbital velocity. See Appendix, Note 7, p. 628.]

You say that the equations (89), (90), (91), (92), (93) are not obvious, & this is so far true, that in my first attempt (which I showed to you at Mr Turner's house in Liverpool) I used wrong signs for $\Delta r_0'$ and $\Delta\lambda_0'$ through an error of reasoning rather than through a mere slip of the pen, and so obtained expressions for the perturbations which were themselves affected with wrong signs, though I soon found out the mistake. It may therefore be proper to develope the corrected reasoning now.—The equations (68), (71), (72) may be said to be true by definition; but if we combine with them the equations (66), (67), *defining* these also to be true & considering r & λ as the Moon's *disturbed* coordinates, we then establish meanings for a, n, ϵ, k, l in virtue of which they are no longer constants but rather are parameters made variable so that they no longer satisfy the equations (75), (76) but rather satisfy these other equations

$$r_0' = anl + \Delta r_0', \tag{110}$$

$$\lambda_0' = n(1+2k) + \Delta\lambda_0'; \tag{111}$$

r_0' and λ_0' being still constants, namely the initial rates of increase of the coordinates r & λ but $\Delta r_0'$ and $\Delta\lambda_0'$ being those (small and perturbational) functions of the time which may be found by elimination from the formulae (83), (84), (85), (86). One mode, then, of expressing the Moon's disturbed coordinates is to denote them by r and λ and to retain, as definitions, the expressions (66) & (67), but to substitute in these expressions instead of a, n, ϵ, k, l those values as functions of the time which may be obtained by elimination from the equations (68), (71), (72), (110), (111) and (83), (84), (85), (86). To facilitate this elimination, we may denote the undisturbed or constant parts of a, n, ϵ, k, l by $\grave{a}$, $\grave{n}$, $\grave{\epsilon}$, $\grave{k}$, $\grave{l}$ and the remaining or perturbational parts by Δa, Δn, $\Delta\epsilon$, Δk, Δl; so that

$$a = \grave{a} + \Delta a, \quad n = \grave{n} + \Delta n, \quad \epsilon = \grave{\epsilon} + \Delta\epsilon, \quad k = \grave{k} + \Delta k, \quad l = \grave{l} + \Delta l; \tag{112}$$

and we may neglect the squares and products of Δa, Δn, $\Delta\epsilon$, Δk, Δl. In this manner the 5 equations (68), (71), (72), (110), (111) resolve themselves into ten others, namely,

$$\left.\begin{aligned} \mu &= \grave{a}^3\grave{n}^2, \\ r_0 &= \grave{a} - \grave{a}\grave{k}, \\ \lambda_0 &= \grave{\epsilon} + 2\grave{l}, \\ r_0' &= \grave{a}\grave{n}\grave{l}, \\ \lambda_0' &= \grave{n}(1+2\grave{k}), \end{aligned}\right\} \tag{113}$$

and

$$\left.\begin{aligned} 0 &= 3\grave{n}\Delta a + 2\grave{a}\Delta n, \\ 0 &= \Delta a - \grave{a}\Delta k, \\ 0 &= \Delta\epsilon + 2\Delta l, \\ 0 &= \grave{a}\grave{n}\Delta l + \Delta r_0', \\ 0 &= \Delta n + 2\grave{n}\Delta k + \Delta\lambda_0'; \end{aligned}\right\} \tag{114}$$

which latter are, as you see, equivalent to the five equations (89)...(93), only made a little fuller & (perhaps) more clear by the introduction of the grave accents over a and n, which are here designed to distinguish elliptic from disturbed parameters. The same grave accents might have been introduced (because squares and products of perturbations were neglected) into the second members of the equations (83), (84), (85), (86); and then the two expressions (66) & (67) might have been resolved each into two parts, as follows,

$$r = r^{\backprime} + \Delta r, \quad \lambda = \lambda^{\backprime} + \Delta\lambda, \tag{115}$$

in which

$$\left.\begin{aligned} r^{\backprime} &= a^{\backprime} - a^{\backprime}k^{\backprime}\cos n^{\backprime}t + a^{\backprime}l^{\backprime}\sin n^{\backprime}t, \\ \lambda^{\backprime} &= n^{\backprime}t + \epsilon^{\backprime} + 2k^{\backprime}\sin n^{\backprime}t + 2l^{\backprime}\cos n^{\backprime}t, \end{aligned}\right\} \tag{116}$$

and

$$\left.\begin{aligned} \Delta r &= \Delta a - a^{\backprime}\cos n^{\backprime}t\,\Delta k + a^{\backprime}\sin n^{\backprime}t\,\Delta l, \\ \Delta\lambda &= t\Delta n + \Delta\epsilon + 2\sin n^{\backprime}t\,\Delta k + 2\cos n^{\backprime}t\,\Delta l; \end{aligned}\right\} \tag{117}$$

the two latter equations corresponding to (87) & (88). But I grudged the expense of so many new accents & new equations; nor do I think that on consideration you will desire that I should complicate my method by continuing to use them explicitly.

You seem to think that a few words more might have been expended on the elimination by which the expressions (94), (95) were deduced from the equations (83), (84), (85), (86), (87), (88), (89), (90), (91), (92), (93). I proceeded nearly as follows:—The equations (89) ... (93) give, very easily,

$$\Delta a = -\frac{2a}{n}\Delta\lambda_0'; \quad \Delta n = 3\Delta\lambda_0'; \quad \Delta\epsilon = \frac{2}{an}\Delta r_0'; \quad \Delta k = -\frac{2}{n}\Delta\lambda_0'; \quad \Delta l = -\frac{1}{an}\Delta r_0'; \tag{118}$$

and therefore, by (87) and (88),

$$\left.\begin{aligned} \Delta r &= -\frac{2a}{n}(1-\cos nt)\,\Delta\lambda_0' - \frac{1}{n}\sin nt\,\Delta r_0', \\ \Delta\lambda &= \left(3t - \frac{4}{n}\sin nt\right)\Delta\lambda_0' + \frac{2}{an}(1-\cos nt)\,\Delta r_0'. \end{aligned}\right\} \tag{119}$$

At the same time the 2 equations (83), (84) give

$$\left.\begin{aligned} \Delta\lambda' &= \Delta\lambda_0' - \frac{1}{a^2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt, \\ \Delta r' &= \Delta r_0' + \tfrac{3}{2}ant\,\Delta\lambda_0' - \frac{3nt}{2a}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt + \frac{3n}{2a}\frac{\delta}{\delta n}\int_0^t R\,dt; \end{aligned}\right\} \tag{120}$$

and therefore the 2 equations (85), (86) give

$$\begin{aligned} \cos nt\frac{\delta}{\delta k}\int_0^t R\,dt - \sin nt\frac{\delta}{\delta l}\int_0^t R\,dt &= a(\Delta r' - \cos nt\,\Delta r_0') - 2a^2\sin nt\,\Delta\lambda_0' \\ &= a(1-\cos nt)\,\Delta r_0' + a^2(\tfrac{3}{2}nt - 2\sin nt)\,\Delta\lambda_0' - \frac{3nt}{2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt + \frac{3n}{2}\frac{\delta}{\delta n}\int_0^t R\,dt \\ &= \frac{a^2n}{2}\Delta\lambda - \frac{3nt}{2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt + \frac{3n}{2}\frac{\delta}{\delta n}\int_0^t R\,dt, \end{aligned} \tag{121}$$

and

$$\begin{aligned} \sin nt\frac{\delta}{\delta k}\int_0^t R\,dt + \cos nt\frac{\delta}{\delta l}\int_0^t R\,dt &= -a\sin nt\,\Delta r_0' - 2a^2(\Delta\lambda' - \cos nt\,\Delta\lambda_0') \\ &= -a\sin nt\,\Delta r_0' - 2a^2(1-\cos nt)\,\Delta\lambda_0' + 2\frac{\delta}{\delta\epsilon}\int_0^t R\,dt \\ &= an\Delta r + 2\frac{\delta}{\delta\epsilon}\int_0^t R\,dt; \end{aligned} \tag{122}$$

results agreeing evidently with the expressions (94), (95). Perhaps you rather meant to enquire how I propose to extend this process of elimination to the more complicated systems of equations which must arise hereafter; and this I hope to make clear at some future stage of our correspondence.

With respect generally to the comparative facility of my own & other methods, your candour of course will lead you to think that it would be unfair to judge of this by contrasting the perfected and published works of yourself and other mathematicians with the rude sketches which these letters contain; and which are written down *very* nearly as they first present themselves to my mind, scarcely any development being suppressed and scarcely any alteration being made. I feel little or no doubt that the same *principle* and (in a large sense) the same *method* as that which is here employed will conduct me or some one else hereafter to *processes* of actual calculation, more convenient and more elegant than those which are here deduced. Even my two published Essays (on a general method in dynamics) contain results, I think, which would be useful in this way, though I prefer at present to suppose that you have not read them, & to take up the subject from the beginning. For instance, if you take the trouble to turn to the last page but two of my First Essay* (*Phil. Trans.* 1834, Part II, Page 306), you will find two systems of expressions for the variations of the parameters of a system of attracting or repelling particles, & a remark analogous to that which you have made in your last letter, respecting the possibility of deducing Lagrange's formulae & mine reciprocally from each other; and some of the theorems of my Second Essay might doubtless have been used with advantage; as probably I shall make use of them, whenever I come to write for publication. Meanwhile it is essential to remember that although the equations (94), (95) of my last letter *might* have been deduced from (66), (67) by supposing a, n, ϵ, k, l to receive the perturbational increments

$$\begin{gathered}\Delta a = -\frac{2}{an}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt, \quad \Delta n = \frac{3}{a^2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt, \quad \Delta\epsilon = -\frac{3}{a^2}\frac{\delta}{\delta n}\int_0^t R\,dt, \\ \Delta k = -\frac{1}{a^2n}\frac{\delta}{\delta l}\int_0^t R\,dt, \quad \Delta l = \frac{1}{a^2n}\frac{\delta}{\delta k}\int_0^t R\,dt,\end{gathered} \tag{123}$$

which would correspond to Lagrange's conception of a varying and tangent ellipse, yet they *were* in fact deduced by a very different set of increments (118); & that the set last mentioned corresponded to the geometrical conception of a variable ellipse† which *cuts*, not touches, the geocentric orbit of the Moon, but which has over the tangent ellipse at least one advantage in simplicity, namely that it passes through one *fixed initial point* determined by the Moon's fixed initial coordinates r_0 and λ_0. You say that Lagrange's method has been found in practice far from convenient, or rather practically useless; & *that* is doubtless a strong presumption against the practical success of my method: but I cannot give it up without a further trial and especially without being convinced of its inutility as applied to terms depending on the higher powers of the Sun's disturbing force.

I confess however, that when I wrote (in haste) my II$^{\text{d}}$ Letter of this series & made (in equal haste) the calculations contained in it, I did not remember that the expressions (94), (95) might be obtained by Lagrange's method; and I was about to verify them by submitting them to the following transformation:—R being originally a function of r and λ, let its two partial differential coefficients $\frac{\delta R}{\delta r}$ and $\frac{\delta R}{\delta \lambda}$ be conceived to take values denoted by $R^{\backprime}$ and $R^{\backprime\backprime}$ when r and λ reduce themselves to the circular values a and $nt+\epsilon$; we shall then have, by making k and l each $=0$,

---

* [Pp. 159, 160 of this volume.]

† [See Appendix, Note 7, p.

$$\frac{\delta R}{\delta n}=-\frac{2}{3}\frac{a}{n}R^{\backprime}+tR^{\backprime\backprime},\quad \frac{\delta R}{\delta \epsilon}=R^{\backprime\backprime},$$

$$\frac{\delta R}{\delta k}=-aR^{\backprime}\cos nt+2R^{\backprime\backprime}\sin nt,\quad \frac{\delta R}{\delta t}=aR^{\backprime}\sin nt+2R^{\backprime\backprime}\cos nt; \tag{124}$$

and therefore

$$\frac{\delta}{\delta n}\int_0^t R\,dt=-\frac{2a}{3n}\int_0^t R^{\backprime}dt+\int_0^t tR^{\backprime\backprime}dt,\quad \frac{\delta}{\delta\epsilon}\int_0^t R\,dt=\int_0^t R^{\backprime\backprime}dt,$$

$$\frac{\delta}{\delta k}\int_0^t R\,dt=-a\int_0^t R^{\backprime}\cos nt\,dt+2\int_0^t R^{\backprime\backprime}\sin nt\,dt, \tag{125}$$

$$\frac{\delta}{\delta l}\int_0^t R\,dt=a\int_0^t R^{\backprime}\sin nt\,dt+2\int_0^t R^{\backprime\backprime}\cos nt\,dt;$$

so that the expressions (94) and (95) become

$$\Delta r=\frac{1}{n}\left(\cos nt\int_0^t R^{\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime}\cos nt\,dt\right)$$

$$an\left(\sin nt\int_0^t R^{\backprime\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime\backprime}\cos nt\,dt-\int_0^t R^{\backprime\backprime}dt\right), \tag{126}$$

and

$$\Delta\lambda=-\frac{2}{an}\left(\sin nt\int_0^t R^{\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime}\cos nt\,dt-\int_0^t R^{\backprime}dt\right)$$

$$+\frac{4}{a^2n}\left(\cos nt\int_0^t R^{\backprime\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime\backprime}\cos nt\,dt\right)$$

$$+\frac{3}{a^2}\left(t\int_0^t R^{\backprime\backprime}dt-\int_0^t R^{\backprime\backprime}t\,dt\right). \tag{127}$$

These give, by a first differentiation,

$$(\Delta r)'=-\left(\sin nt\int_0^t R^{\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime}\cos nt\,dt\right)$$

$$+\frac{2}{a}\left(\cos nt\int_0^t R^{\backprime\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime\backprime}\cos nt\,dt\right), \tag{128}$$

$$(\Delta\lambda)'=-\frac{2}{a}\left(\cos nt\int_0^t R^{\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime}\cos nt\,dt\right)$$

$$-\frac{4}{a^2}\left(\sin nt\int_0^t R^{\backprime\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime\backprime}\cos nt\,dt\right)+\frac{3}{a^2}\int_0^t R^{\backprime\backprime}dt; \tag{129}$$

and, by a second,

$$(\Delta r)''=-R^{\backprime}-n\left(\cos nt\int_0^t R^{\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime}\cos nt\,dt\right)$$

$$-\frac{2n}{a}\left(\sin nt\int_0^t R^{\backprime\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime\backprime}\cos nt\,dt\right), \tag{130}$$

$$(\Delta\lambda)''=\frac{2n}{a}\left(\sin nt\int_0^t R^{\backprime}\sin nt\,dt+\cos nt\int_0^t R^{\backprime}\cos nt\,dt\right)$$

$$-\frac{4n}{a^2}\left(\cos nt\int_0^t R^{\backprime\backprime}\sin nt\,dt-\sin nt\int_0^t R^{\backprime\backprime}\cos nt\,dt\right)-\frac{R^{\backprime\backprime}}{a^2}; \tag{131}$$

so that the functions Δr & $\Delta\lambda$, besides having the property that they and their first differential coefficients $(\Delta r)'$ & $(\Delta\lambda)'$ vanish when $t=0$, have also the property that they satisfy the following pair of differential equations of the 2nd order,

$$(\Delta r)'' - 3n^2\Delta r - 2an\,(\Delta\lambda)' + R^{\backprime} = 0, \tag{132}$$

$$a^2\,(\Delta\lambda)'' + 2an\,(\Delta r)' + R^{\backprime\backprime} = 0. \tag{133}$$

Now these last two equations are precisely the conditions necessary, in order that the functions

$$r = a + \Delta r, \quad \lambda = nt + \epsilon + \Delta\lambda, \tag{134}$$

may satisfy (when the square of the disturbing force is neglected & when a and n are connected by the relation

$$a^3n^2 = \mu) \tag{68}$$

the two differential equations, deduced from (48),

$$r'' - r\lambda'^2 + \frac{\mu}{r^2} + \frac{\delta R}{\delta r} = 0, \quad (r^2\lambda')' + \frac{\delta R}{\delta\lambda} = 0. \tag{135}$$

And if, reciprocally, without thinking of my own method, we had set about to integrate the equations (132), (133), we should have had, first, from the equation (133),

$$a^2\,(\Delta\lambda)' + 2an\,\Delta r + \int_0 R^{\backprime\backprime}\,dt = 0, \tag{136}$$

(observing that Δr, $\Delta\lambda$ and $(\Delta r)'$, $(\Delta\lambda)'$ are supposed to vanish when $t=0$,) and, next, by (132),

$$(\Delta r)'' + n^2\Delta r + R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt = 0; \tag{137}$$

which would have given, by well-known methods,

$$\begin{aligned}\Delta r = {} & \frac{\cos nt}{n}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\sin nt\,dt \\ & - \frac{\sin nt}{n}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\cos nt\,dt;\end{aligned} \tag{138}$$

and also

$$\Delta\lambda = -\frac{2n}{a}\int_0^t \Delta r\,dt - \frac{1}{a^2}\int_0^t\left(\int_0^t R^{\backprime\backprime}\,dt\right)dt, \tag{139}$$

that is

$$\begin{aligned}\Delta\lambda = {} & -\frac{2\sin nt}{an}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\sin nt\,dt \\ & - \frac{2\cos nt}{an}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\cos nt\,dt \\ & + \frac{1}{an}\int_0^t\left(2R^{\backprime} + \frac{3n}{a}\int_0^t R^{\backprime\backprime}dt\right)dt;\end{aligned} \tag{140}$$

in which expressions the double integrals might have been reduced by observing that

$$\begin{aligned}\int_0^t\left(\sin nt\int_0^t R^{\backprime\backprime}dt\right)dt &= -\frac{\cos nt}{n}\int_0^t R^{\backprime\backprime}dt + \frac{1}{n}\int_0^t R^{\backprime\backprime}\cos nt\,dt, \\ \int_0^t\left(\cos nt\int_0^t R^{\backprime\backprime}dt\right)dt &= \frac{\sin nt}{n}\int_0^t R^{\backprime\backprime}\,dt - \frac{1}{n}\int_0^t R^{\backprime\backprime}\sin nt\,dt, \\ \int_0^t\left(\int_0^t R^{\backprime\backprime}\,dt\right)dt &= t\int_0^t R^{\backprime\backprime}dt - \int_0^t R^{\backprime\backprime}t\,dt;\end{aligned} \tag{141}$$

and then the expressions (126) and (127) would have resulted.

I intended also to remark that, whichever of these methods we employ, if we introduce into the expression of R the additional term

$$-\frac{n_{,}^2 r^3}{} P_3 = -\frac{n_{,}^2 r^3}{6a_{,}}\left(\frac{d}{dp}\right)^3\left(\frac{p^2-1}{2}\right)^3 = -\frac{n_{,}^2 r^3}{48a_{,}}\left(\frac{d}{dp}\right)^3(p^6-3p^4+3p^2-1)$$
$$= -\frac{n_{,}^2 r^3}{2a_{,}}(5p^3-3p) = -\frac{n_{,}^2 r^3}{8a_{,}}\{5\cos\overline{3\lambda-3n_{,}t}+3\cos\overline{\lambda-n_{,}t}\}, \qquad (142)$$

we obtain, in the expressions of the Moon's disturbed coordinates, the following additional terms:

$$\Delta r = -\frac{2}{3}\frac{a}{n}\Delta' n - a\cos nt\,\Delta' k + a\sin nt\,\Delta' l$$
$$+\frac{3}{16}\frac{n_{,}}{n}\frac{a^2}{a_{,}}\left(5+\frac{n_{,}}{2n-n_{,}}+\frac{4n_{,}}{n-n_{,}}\right)\cos(nt-n_{,}t+\epsilon)$$
$$-\frac{5}{16}\left(\frac{n_{,}}{n}\right)^2\frac{a^2}{a_{,}}\left(\frac{9n}{2n-3n_{,}}+\frac{3n}{4n-3n_{,}}-\frac{4n}{n-n_{,}}\right)\cos(3nt-3n_{,}t+3\epsilon); \qquad (143)$$

$$\Delta\lambda = t\Delta' n + \Delta'\epsilon + 2\sin nt\,\Delta' k + 2\cos nt\,\Delta' l$$
$$-\frac{3}{8}\frac{n_{,}}{n}\frac{a}{a_{,}}\left\{5-\frac{n}{2n-n_{,}}+\frac{3n_{,}(3n-2n_{,})}{(n-n_{,})^2}\right\}\sin(nt-n_{,}t+\epsilon)$$
$$+\frac{5}{8}\left(\frac{n_{,}}{n}\right)^2\frac{a}{a_{,}}\left\{\frac{9n}{2n-3n_{,}}-\frac{3n}{4n-3n_{,}}-\frac{(3n-2n_{,})n}{(n-n_{,})^2}\right\}\sin(3nt-3n_{,}t+3\epsilon); \qquad (144)$$

in which I have put, for abridgment,

$$\Delta' n = \frac{3}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon);$$
$$\Delta'\epsilon = \frac{1}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n}\frac{3n-2n_{,}}{(n-n_{,})^2}(9\sin\epsilon+5\sin 3\epsilon);$$
$$\Delta' k = \frac{3}{16}\frac{a}{a_{,}}\frac{n_{,}}{n}\left\{\left(5+\frac{n_{,}}{2n-n_{,}}\right)\cos\epsilon - 5\left(\frac{3n_{,}}{2n-3n_{,}}+\frac{n_{,}}{4n-3n_{,}}\right)\cos 3\epsilon\right. \qquad (145)$$
$$\Delta' l = \frac{3}{16}\frac{a}{a_{,}}\frac{n_{,}}{n}\left\{\left(5-\frac{n_{,}}{2n-n_{,}}\right)\sin\epsilon - 5\left(\frac{3n_{,}}{2n-3n_{,}}-\frac{n_{,}}{4n-3n_{,}}\right)\sin 3\epsilon\right.$$

It is true that these results would add, to the second members of the equations (108) & (109) respectively, the terms

$$\frac{15\mathrm{a}}{16a_{,}}\left(m+\frac{9m^2}{10}+\frac{17m^3}{20}\right)\cos\tau-\frac{25\mathrm{a}}{64a_{,}}\left(m^2+\frac{53m^3}{20}\right)\cos 3\tau, \qquad (146)$$
$$-\frac{15\mathrm{a}}{8a_{,}}\left(m+\frac{17m^2}{10}+\frac{47m^3}{20}\right)\sin\tau+\frac{15\mathrm{a}}{32a_{,}}\left(m^2+\frac{35m^3}{12}\right)\sin 3\tau, \qquad (147)$$

and that the coefficients of

$$\frac{\mathrm{a}m^2\cos\tau}{a_{,}},\quad \frac{\mathrm{a}m^2\sin\tau}{a_{,}},\quad \frac{\mathrm{a}m^3\cos\tau}{a_{,}},\quad \frac{\mathrm{a}m^3\sin\tau}{a_{,}},\quad \frac{\mathrm{a}m^3\cos 3\tau}{a_{,}},\quad \frac{\mathrm{a}m^3\sin 3\tau}{a_{,}},$$

thus obtained, do not agree with those deduced by you & Plana:* I must therefore suppose that an improved approximation would correct the subordinate (though not the principal) numerical coefficients of the expressions last obtained by me. But, in the meantime, I incline to think that *so far as the mere ordering according to powers of* m (*or of* $n_,$) *is concerned,* (*abstraction being made of the indefinite increase which the variable* t *may and must receive in the course of time,*) my expressions for the Moon's coordinates r & λ are accurate, *as far as the cube of* m *and the* 1st *power of the reciprocal of* $a_,$ *inclusive*. However, even the appearance of differing from you and Plana must naturally make me cautious: and I intend to consider this point also with a more close attention hereafter. The terms proportional to $\frac{m^3}{a_,}\cos\tau$ and $\frac{m^3}{a_,}\sin\tau$ seem to me at present to be the most doubtful of those in which the difference apparently exists.

---

* [See p. 248 and Appendix, Note 6, p. 627.]

It may be useful to observe that in deducing the perturbational terms (143) & (144) from the part (142) of R in the way most immediately suggested by my method I was led to calculate the following part of the integral $\int_0^t R\,dt$:

$$-\frac{n_,^2}{8a_,}\int_0^t r^3\{5\cos\overline{3\lambda-3n_,t}+3\cos\overline{\lambda-n_,t}\}\,dt=-\frac{3}{8}\frac{a^3n_,^2}{a_,(n-n_,)}(\sin\overline{nt-n_,t+\epsilon}-\sin\epsilon)$$

$$-\frac{5a^3n_,^2}{24a_,(n-n_,)}(\sin 3nt-3n_,t+3\epsilon-\sin 3\epsilon)+\frac{[illegible]}{16a_,}\left\{\frac{5}{n_,}(\sin n_,t-\epsilon+\sin\epsilon)\right.$$

$$+\frac{1}{2n-n_,}(\sin 2nt-n_,t+\epsilon-\sin\epsilon)+\frac{15}{2n-3n_,}(\sin 2nt-3n_,t+3\epsilon-\sin 3\epsilon)$$

$$\left.\frac{5}{4n-3n}(\sin 4nt-3n_,t+3\epsilon-\sin 3\epsilon)\right\}+\frac{3la^3n_,^2}{16a}\left\{\frac{5}{n_,}(\cos n_,t-\epsilon-\cos\epsilon)\right.$$

$$+\frac{1}{2n-n_,}(\cos\overline{2nt-n_,t+\epsilon}-\cos\epsilon)-\frac{15}{2n-3n_,}(\cos 2nt-3n_,t+3\epsilon-\cos 3\epsilon)$$

$$\left.{}_{4n-3n_,}(\cos\overline{4nt-3n_,t+3\epsilon}-\cos 3\epsilon)\right\}; \qquad (148)$$

in which, for the first time, a lowering of the order of a small term occurred, so far as the powers of the small quantity $n_,$ are concerned; namely by the introduction of that small quantity $n_,$ as divisor in passing from the nearly constant (or rather slowly varying) terms

$$\frac{15a^3n_,^2}{16a_,}(k\cos\overline{n_,t-\epsilon}-l\sin n_,t-\epsilon), \qquad (149)$$

which occur in the development of R itself, to the corresponding terms

$$\frac{15a^3n_,}{16a_,}\{k(\sin\overline{n_,t-\epsilon}+\sin\epsilon)+l(\cos\overline{n_,t-\epsilon}-\cos\epsilon)\}, \qquad (150)$$

in the integral $\int_0^t R\,dt$. Reciprocally, so far as the mere ordering according to the powers of $n_,$ is concerned, these terms (150) are really, though not apparently, of the same order as $n_,^2$, or rather as $\frac{kn_,^2}{\sim}$; and in like manner the corresponding terms in Δr and $\Delta\lambda$, namely (by (94) & (95))

$$\frac{15a^2n_,}{16a_,n}(\cos\overline{nt-n_,t+\epsilon}-\cos\overline{nt+\epsilon}), \qquad (151)$$

and

$$-\frac{15an_,}{8a_,n}(\sin\overline{nt-n_,t+\epsilon}-\sin\overline{nt+\epsilon}), \qquad (152)$$

are really of the same order as $\frac{n_{,}}{a_{,}}$. This remark appears to me not useless because, from the form of the differential equations (135) and from the circumstance that R contains $n_{,}^2$ as a factor, it might have been foreseen that the perturbations Δr and $\Delta\lambda$ ought also to contain that small square $n_{,}^2$ or m^2 as a factor; and therefore the occurrence of such terms as

$$\frac{15\mathrm{a}^2 m}{16a_{,}}\cos\tau \quad \text{and} \quad -\frac{15\mathrm{a}m}{8a_{,}}\sin\tau, \tag{153}$$

in the expressions of those perturbations, may seem at first paradoxical.—But though *some* depression of order, by introduction of small divisors, will certainly occur in my method, namely in the calculation of that single definite integral to which the problem is reduced, it is still not obvious how *so great* a depression can arise as in that double integration which is required by other methods; on this point also I therefore await the conviction of what may be called experiment, before I give up all hope of being useful in the new track which I am attempting to pursue. I can only write by snatches, but expect to make & to transmit some calculations soon, which will present my views in a still more definite shape. Meantime believe me to remain, my dear Sir,

very truly yours,

William R. Hamilton.

J. W. Lubbock, Esq^r^.

P.S. To prevent any misunderstanding with respect to the nature of the results to which I have arrived, so far, in investigating the coordinates of the Moon by my own method, it may be right to recapitulate them thus.—I have been led to the 2 equations

$$\begin{aligned} r = \mathrm{a}\Big\{1 - \mathrm{k}\cos \mathrm{n}t + \mathrm{l}\sin \mathrm{n}t - \frac{m^2}{6} - m^2\left(1+\frac{19m}{6}\right)\cos 2\tau \\ + \frac{15}{16}\frac{\mathrm{a}}{a_{,}}m\left(1+\frac{9m}{10}+\frac{17m^2}{20}\right)\cos\tau - \frac{25}{64}\frac{\mathrm{a}}{a_{,}}m^2\left(1+\frac{53m}{20}\right)\cos 3\tau \end{aligned} \tag{154}$$

and

$$\begin{aligned} \lambda = \mathrm{n}t + \mathrm{e} + 2\mathrm{k}\sin \mathrm{n}t + 2\mathrm{l}\cos \mathrm{n}t + \frac{11m^2}{8}\left(1+\frac{118m}{33}\right)\sin 2\tau \\ - \frac{15}{8}\frac{\mathrm{a}}{a_{,}}m\left(1+\frac{17m}{10}+\frac{47m^2}{20}\right)\sin\tau + \frac{15}{32}\frac{\mathrm{a}}{a_{,}}m^2\left(1+\frac{35m}{12}\right)\sin 3\tau, \end{aligned} \tag{155}$$

(in which a, m, & τ denote respectively $\sqrt[3]{\frac{\mu}{\mathrm{n}^2}}$, $\frac{n_{,}}{\mathrm{n}}$, & $\mathrm{n}t - n_{,}t + \mathrm{e}$, & k and l are small) as integrals, with 4 arbitrary constants n, e, k, l, of the two differential equations of the second order

$$\left.\begin{aligned} r'' - r\lambda'^2 + \frac{\mu}{r^2} = \frac{n_{,}^2 r}{2}(1 + 3\cos\overline{2\lambda - 2n_{,}t}) + \frac{3n_{,}^2 r^2}{8a_{,}}(3\cos\overline{\lambda - n_{,}t} + 5\cos\overline{3\lambda - 3n_{,}t}), \\ (r^2\lambda')' = -\frac{3n_{,}^2 r^2}{2}\sin\overline{2\lambda - 2n_{,}t} - \frac{3n_{,}^2 r^3}{8a_{,}}(\sin\overline{\lambda - n_{,}t} + 5\sin\overline{3\lambda - 3n_{,}t}); \end{aligned}\right\} \tag{156}$$

in which μ, $n_{,}$ and $a_{,}$ are constants, and $n_{,}$ & $\frac{1}{a_{,}}$ are small, and the upper accents correspond to differentiations with respect to t: also the 4th power of $n_{,}$ and the square of $\frac{1}{a_{,}}$ are neglected;

and so are the squares and product of k and l, and the products of these small quantities (k & l) with others. These expressions (154) & (155) were obtained through the medium of the formulae (94), (95), and the development of $\int_0^t R\,dt$; but I have verified them by actual differentiation, & have reobtained them, on principles much more elementary, by integrating the equations (132), (133) after assigning for R' and R'' their values. Yet the subordinate coefficients of the sine and cosine of τ & 3τ do not agree with yours and Plana's & will, therefore, I am sure, be altered by a subsequent approximation:—the ground of alteration being, I suppose, the indefinite increase of t and the consequent practical necessity of keeping that symbol under the signs of periodicity, except in the one term nt; which latter necessity has not yet shown itself in the operation of my method and indeed is not recognised in the first conception of that method—certain quantities being treated as small but none thought of as large. I mean in my *present* method of successive & indefinite *approximation* for my *rigorous and finite integrals* of the differential equations of motion of any system of attracting or repelling points extend to any intervals of time, however great.

Since writing the above I have perceived that my results can be reconciled with yours & Plana's in such a way as I conjectured; at least as far as the coefficients of $\frac{m^2a}{a_,}$ inclusive beyond which you have marked Plana's numbers as erroneous without stating (at least in your appendix) what your own numbers are. But I am unwilling to delay this letter or to add to its length. (Observatory, Oct. 17th, 1837.)

Sir W. R. Hamilton to J. W. Lubbock.

Oct. 18th, 1837.

MY DEAR SIR,

The enclosed letter having been delayed, I add a word or two. You suggested that I should compare my own process of obtaining my equations (108) and (109) with yours. I wish to do so: would it be quite unreasonable to ask you to take the trouble to write out just so much of your own method as is absolutely necessary for deducing those two equations? At least you might point out the pages & parts of pages most requisite for me to read with that particular view—but I fear that I am likely to postpone the reading of your printed work until I have advanced much farther in the development of my own views unless you stimulate me and set me a-going by doing some such thing as I just now ventured to request—which, after all, would not perhaps give you much trouble, & might not be uninstructive to other students hereafter. An *elementary* treatise on the theory of the Moon, going about as far as Airy's tract but drawn up on *your* plan, would doubtless be useful & acceptable to a large class of readers; and I, for one, should hail its appearance with great pleasure.

All that I am *now* doing is (you see) quite piecemeal & preparatory. My *hope* is that a nearly perfect theory of the Moon may be obtained by my method, through the medium of *two* successive developments; namely, first, the development of $\int R\,dt$ with elliptic values of the coordinates, and secondly the development of the same integral with improved expressions for those coordinates. In fact, I think that after the second of these developments a new set of expressions for the coordinates must result by differentiation and elimination, which will err only by quantities of the 8th order relatively to n, and of the 4th order relatively to $\frac{1}{a}$, abstraction being made of the indefinite increase of t. But there are, I own, two practical dangers. One is this very increase which may ultimately vitiate the expressions, or at least prevent them from being accurate to so high an order as I have named. I hope however that this will not depress the final order so much as to prevent the results from being useful. The other danger is that the second development of $\int R dt$ may be intolerably laborious. I have some hope of your co-operation in that task, & with it I do not despair.

Once more, very truly yours

W. R. H.

D. *J. W. Lubbock to Sir W. R. Hamilton.*

EATON PLACE.
Oct^r. 21, 1837.

MY DEAR SIR,

I was very glad to receive your letter of the 12th, in which you have calculated the terms belonging to τ to 3τ or my arguments [101] and [116]. I hope you will not calculate r (radius vector) but its reciprocal which will I apprehend be quite the same to you and by so doing your radius vector results will be directly comparable with Plana's and mine but not otherwise.

I still think that your expressions are to be considered as identical with those of Lagrange as far as our purpose is concerned (although I admit that your method of arriving at them is very remarkable) so that I think your views conduct to precisely the same calculations as those developed by M. Poisson in his paper. In the *Phil. Trans.* (1832), p. 229, I showed how these equations lead to precisely the same results generally at least when the squares of the excentricities are neglected as those given by the other method which I may call mine as I believe no one ever made use of it before. But I think Poisson in his paper about the Moon first enunciated the manner in which the equations are to be employed in further approximations and in one instance of the simplest kind applied them.

M. Poisson says, p. 50:* "La substitution des elements elliptiques et de ξ ainsi augmenté pourra se faire soit dans les formules (10) après avoir effectué les differentiations relatives à ces mêmes elements soit dans la fonction R &c." The calculations which are indicated in this paragraph are precisely those which seem to me to be practically beyond the reach of human patience. And I believe that you will not be able to evade this difficulty.† I consider your equations (94) and (95) as precisely those which would arise from making use of Lagrange's expressions for the variation of the constants and therefore although your proof of Lagrange's expressions be new yet your method as explained in your last letter is precisely, if I am not mistaken, the same as that proposed by Poisson in his "Mémoire sur le mouvement de la Lune", *Mémoires de l'Institut*, Tome XIII, and of which he gave some examples choosing however those cases in which that method is seen to the best advantage. This method has never been carried further than by myself in the *Phil. Trans.* (1832), p. 229, & by Poisson in the Memoir alluded to, so that Lagrange's method can hardly yet be said to have been found in *practice* inconvenient. After writing the paper however to which I have alluded in the *Phil. Trans.* I was able better to estimate its relative advantage, & I reflected much upon them before I finally adopted the methods which I have employed. The analytical difficulties even are not trifling as may be seen by looking at the communications from Plana & Pontécoulant in the *Comptes Rendus* about the value of R_{77}. M. Cauchy has just published in Liouville's journal a new method of deducing Lagrange's expressions, viz. from the integral of vis viva, but in other respects it has no similarity with yours. I sent you some time back a short paper by Pontécoulant, you will see what he thinks of Poisson's method (p. 22) & he has expressed himself quite as strongly in his letter to me. I shall be glad to know what you think about his controversy with Poisson. I think Poisson quite right. I think particularly that the introduction of the quantities σ and σ' (p. 53) & the interpretation assigned to them by Poisson completely obviates Pontécoulant's objections.*

---

* *Mémoires de l'Institut,* Tome XIII.

† We shall be able to judge of this when you have got out some terms depending on the next approximation.

However Pontécoulant is very positive and he ought to know what he is about. (I enclose Pontécoulant's pamphlet, if you find the other copy you can return it.)

The weakest point analytically about the theory of the Moon seems to me to be that I mentioned to you about the form which has been given to the acceleration of the mean motion & I hope you will consider this difficult question.

The reason why your term in the longitude argument [101] or τ does not agree with mine is I think because your de† and $d\varpi$ contain argument [102] or $\tau-\xi$ which will always be lowered by integration, & this by combination with ξ gives the term sought in the longitude argument [101]. The same applies to argument [116].

I make

$$\left.\begin{matrix}15m & 81m^2 & 3021m^3\\ 16 & 16 & 128\end{matrix}\right\}\frac{a}{a_{,}}\cos\tau \qquad [101], \text{ p. } 276‡$$

$$+\left.\frac{25m^2}{64}-\frac{115m^3}{128}\right\}\frac{a}{a_{,}}\cos 3\tau, \qquad [116], \text{ p. } 274$$

$$\lambda=\left\{-\frac{15m}{\frown}-\frac{93m^2}{8}-\frac{889m^3}{16}\right\}\frac{a}{a_{,}}\sin\tau \qquad [101], \text{ p. } 277$$

$$+\left\{\frac{15m^2}{32}-\frac{5m^3}{8}\right\}\frac{a}{a_{,}}\sin 3\tau. \qquad [116]$$

Unless you write your results in this form we shall have much trouble hereafter in comparing them with mine & Plana's. I mean without putting your first coefficient *outside* the bracket. By employing other constants than mine such as l and k you will give yourself an immense deal of unnecessary labour because you will not be able to employ my development of R (p. 30) which would otherwise serve your purpose. You can hardly think how much trouble I have spent upon this development in order to get it free from error and with a few exceptions all the terms have been calculated by two distinct and independent methods.§ If I tried to apply the method of variation of constants I think I should use the expressions which I have given at the foot of p. 75. I now close this long letter hoping to hear from you again soon,

& remain, yours very faithfully

J. W. LUBBOCK.

---

* [For a general account of the different methods, see Brown, *Lunar Theory*, Cambridge University Press, Chap. XII.]

† dl and dk come to precisely the same.

‡ [These page references are to Lubbock's *Theory of the Moon and the Perturbations of the Planets*, Part II (1836).]

§ And have been verified by Pontécoulant by another method.

IV. *Sir W. R. Hamilton to J. W. Lubbock.*

[Four pages of this letter are missing. At the top of page 5 is "written subsequently to the meeting of the British Association at Liverpool in 1837."

It deals with the integration of the equation

$$x'' = -\frac{\delta R}{\delta x},$$

where R is a perturbing function having as multiplier a small quantity α. Neglecting R we have

$$x = a + bt. \tag{23}$$

If the actual initial and final values of x' are denoted by x'_0 and x'_t, we have then by Hamilton's method

$$x'_t \delta x_t - x'_0 \delta x_0 = \delta \int_0^t (\tfrac{1}{2} x'^2_t - R_{x_t})\, dt, \tag{160}$$

which gives

$$(x'_t - b)\, \delta x_t - (x'_0 - b)\, \delta x_0 = -\delta \int R\, dt \tag{161}$$

or

$$(x'_t - b)\, (\delta a + t \delta b) - (x'_0 - b)\, \delta a = -\delta \int R\, dt, \tag{162}$$

that is,

$$x'_t - x'_0 = -\frac{\delta}{\delta a} \int R\, dt, \quad x'_t - b = -\frac{\delta}{t \delta b} \int R\, dt{:}]$$

therefore, by elimination of x'_t,

$$x'_0 = b + \frac{\delta}{\delta a} \int_0^t R\, dt - \frac{1}{t} \frac{\delta}{\delta b} \int R\, dt, \tag{163}$$

by which equation, combined with the following

$$x_0 = a, \tag{164}$$

we are to eliminate the parameters a & b from the expression (23), & so obtain x_t as a function of x_0, x'_0 and t, involving also the small constant α.

In the present example the elimination is easy and gives

$$x_t = x_0 + t x'_0 + \frac{\delta}{\delta x'_0} \int_0^t R\, dt - t \frac{\delta}{\delta x_0} \int_0^t R\, dt, \tag{165}$$

R being treated as that function of x_0, x'_0 and t, which is obtained by changing x_t in it to $x_0 + t x'_0$. But to illustrate the less simple process which it is in general convenient to employ, we may make

$$x'_t = b + \Delta x'_t, \quad x'_0 = b + \Delta x'_0, \tag{166}$$

& then the equations (162) become

$$\Delta x'_t - \Delta x'_0 = -\frac{\delta}{\delta a} \int_0^t R\, dt; \quad t \Delta x'_t = -\frac{\delta}{\delta b} \int_0^t R\, dt; \tag{167}$$

we may also make

$$a = a^{\backprime} + \Delta a, \quad b = b^{\backprime} + \Delta b, \quad x_t = x^{\backprime}_t + \Delta x_t, \tag{168}$$

$$a^{\backprime} = x_0, \quad b^{\backprime} = x'_0, \quad x^{\backprime}_t = a^{\backprime} + b^{\backprime} t, \tag{169}$$

$$0 = \Delta a, \quad 0 = \Delta b + \Delta x'_0, \quad \Delta x_t = \Delta a + t \Delta b; \tag{170}$$

and then we find, for what may be called the perturbation of x_t,

$$\Delta x_t = -t \Delta x'_0 = \frac{\delta}{\delta b^{\backprime}} \int_0^t R\, dt - \frac{\delta}{\delta a^{\backprime}} \int_0^t R\, dt, \tag{171}$$

R being treated here as a function of $a' + b't$; a result which entirely agrees with the expression (165).

Now it is certain that this very expression (165) can be found also by Lagrange's method of varying the parameter. For if, with (23), we had combined not (164) but the following,

$$x'_t = b, \tag{172}$$

as the second equation of definition to determine the varying quantities a & b, we should have had, rigorously,

$$\left.\begin{aligned} b' = x''_t &= -\frac{\delta R}{\delta x_t} = -\frac{\delta R}{\delta a}, \\ a' = -tb'_t &= t\frac{\delta R}{\delta x_t} = \quad \frac{\delta R}{\delta b}, \end{aligned}\right\} \tag{173}$$

and therefore approximately

$$\left.\begin{aligned} a &= a_0 + \int_0^t \frac{\delta R}{\delta b_0}\,dt = x_0 + \quad {}_0^t\frac{\delta R}{\delta x'_0}\,dt, \\ b &= b_0 - \quad {}_0^t\frac{\delta R}{\delta a_0}\,dt = x'_0 - \quad {}_0^t\frac{\delta R}{\delta x_0}\,dt, \end{aligned}\right\} \tag{174}$$

so that, by (23),

$$x_t = x_0 + tx'_0 + \int_0^t \frac{\delta R}{\delta x'_0}\,dt - t\int_0^t \frac{\delta R}{\delta x_0}\,dt, \tag{175}$$

or finally

$$x_t = x_0 + tx'_0 + \qquad R\,dt - t\frac{\delta}{\delta x_0}\int_0^t R\,d \tag{165}$$

So far, I certainly can pretend to no practical advantage of any sort for my method over that of Lagrange, or rather they both conduct to the very same system of integration & partial differentiation. And I am inclined to admit (what doubtless it would be easy to decide & what I should probably find decided already in my own published essays if I had leisure at this moment to consult them) that however far the approximation might be carried with respect to excentricities & inclinations, the equations analogous to (94) & (95) of II,* obtained by my own method, would agree entirely with those to which the method of Lagrange conducts. We see, indeed, that the results in these two methods are obtained from very different conceptions; in such a manner that if the differential equation (157) be considered as determining the varied motion of a point P along a line, while a and b represent respectively the *epoch* and *velocity* of the uniform motion of another point P' in the same line which coincides in position with P at the time t, my P' coincides also with P at the time 0, but Lagrange's at the time $t + dt$; and algebraically my a is fixed $(= x_0)$, while my b is determined (approximately) by the equation (163), or by the following,

$$b = x'_0 - \frac{\delta}{\delta x_0}\int_0^t R\,dt + \frac{1}{t}\frac{\delta}{\delta x_0}\int_0^t R\,dt, \tag{176}$$

but Lagrange's a and b both vary & are determined approximately by the equations (174) or rigorously by the equation (175). But the practical question is whether in passing to a higher order of approximation, & introducing the square & cube of the small constant α, it is easier to develope a & b from Lagrange's equations

$$b' = -\frac{\delta R}{\delta a}, \quad a' = \frac{\delta R}{\delta b}, \tag{177}$$

---

* [Second letter to Lubbock, p. 261.]

or to affect the analogous development with my different system of conceptions & of formulae.

Confining ourselves to the square of α, I suppose that Poisson's mode of applying Lagrange's method to the Theory of the Moon would correspond to the following treatment of the equations (173) or (177). Introducing the approximate values (174) for a & b into the partial differential coefficients $-\frac{\delta R}{\delta a}$ and $\frac{\delta R}{\delta b}$, we get, as an improved system of approximations,

$$\left.\begin{aligned} b &= b_0 - \int_0^t \frac{\delta R}{\delta a_0}\,dt - \int_0^t \left(\frac{\delta^2 R}{\delta a_0^2}\int_0^t \frac{\delta R}{\delta b_0}\,dt\right) dt + \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0}\int_0^t \frac{\delta R}{\delta a_0}\,dt\right) dt, \\ a &= a_0 + \int_0^t \frac{\delta R}{\delta b_0}\,dt + \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0}\int_0^t \frac{\delta R}{\delta b_0}\,dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b_0^2}\int_0^t \frac{\delta R}{\delta a_0}\,dt\right) dt; \end{aligned}\right\} \tag{178}$$

and consequently (23), being now true by definition, we have to the order of accuracy sought

$$\begin{aligned} x_t = a_0 + tb_0 &+ \int_0^t \frac{\delta R}{\delta b_0}\,dt - t\int_0^t \frac{\delta R}{\delta a_0}\,dt \\ &+ \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0}\int_0^t \frac{\delta R}{\delta b_0}\,dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b_0^2}\int_0^t \frac{\delta R}{\delta a_0}\,dt\right) dt \\ &+ t\int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0}\int_0^t \frac{\delta R}{\delta a_0}\,dt\right) dt - t\int_0^t \left(\frac{\delta^2 R}{\delta a_0^2}\int_0^t \frac{\delta R}{\delta b_0}\,dt\right) dt; \end{aligned} \tag{179}$$

R being treated as a function of a_0, b_0 and t, obtained by changing x_t in it to $a_0 + tb_0$. The lower indices 0 may be suppressed, & we may write, more simply,

$$\begin{aligned} x_t = a + bt &+ \int_0^t \frac{\delta R}{\delta b}\,dt - t \qquad dt \\ &+ \int_0^t \left(\frac{\delta^2 R}{\delta a \delta b}\int_0^t \frac{\delta R}{\delta b}\,dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b^2}\int_0^t \frac{\delta R}{\delta a}\,dt\right) dt \\ &+ t\int_0^t \left(\frac{\delta^2 R}{\delta a \delta b}\int_0^t \frac{\delta R}{\delta a}\,dt\right) dt - t\int_0^t \left(\frac{\delta^2 R}{\delta a^2}\int_0^t \frac{\delta R}{\delta b}\,dt\right) dt, \end{aligned} \tag{180}$$

if we here agree to treat a and b as denoting not the variable parameters which were introduced into the expression (23), when that expression was extended by definition so as to be true for disturbed motion, but the fixed initial constants a_0 & b_0, or x_0 & x'_0.

Consider now my method of improving the approximate expression (165), so as to take account of α^2, that is, of the square of the disturbing force. I put the expression (165) under the form

$$x_t = a + bt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt, \tag{181}$$

a and b being quantities which would be constant if α^2 or R^2 were neglected, & R being a function of them and of t, obtained by changing in it x_t to $a + bt$. This expression if it were rigorous would give by differentiation, $\left(\text{because } \frac{\delta R}{\delta b} = t\frac{\delta R}{\delta a}\right)$,

$$x'_t = b - \frac{\delta}{\delta a}\int_0^t R\,dt; \tag{182}$$

it would therefore give, for the function under the integral sign in (160), the value

$$\tfrac{1}{2}x_t'^2 - R_{x_t} = \tfrac{1}{2}b^2 - b\frac{\delta}{\delta a}\int_0^t R\,dt - R_{a+bt} - R^{(2)}, \tag{183}$$

R in the first member being a function of the disturbed variable x_t, but being in the second member the same function of the undisturbed part $a+bt$, so that R_{a+bt} is only a first approximate value of R_{x_t}; while $R^{(2)}$ has been written for abridgment instead of the following part, which is of the order of the square of R,

$$R^{(2)} = -\frac{1}{2}\left(\frac{\delta}{\delta a}\int_0^t R\,dt\right)^2 + \frac{\delta R}{\delta a}\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt. \tag{184}$$

Thus $-\delta\int_0^t R^{(2)}dt$ becomes the part of the second order in the second member of the rigorous formula (160), and is to be equated to the part of the same order in the first member of that formula. We have therefore*

$$-\delta\int_0^t R^{(2)}dt = \Delta x_t'(\delta a + t\,\delta b) - \Delta x_0'\,.\,\delta a - \frac{\delta}{\delta a}\int_0^t R\,dt\,.\,\delta\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\quad R\,dt, \tag{185}$$

if we make

$$x_t' = b - \frac{\delta}{\delta a}\int_0 R\,dt + \Delta x_t', \quad x_0' = b + \Delta x_0', \tag{186}$$

(so that $\Delta x_t'$ and $\Delta x_0'$ are here of the second order) defining a & b to be such as to satisfy exactly the equation (181) and the following

$$x_0 = a. \tag{164}$$

The formula (185) resolves itself into the 2 following

$$\left.\begin{aligned} -\frac{\delta}{\delta a}\int_0^t R^{(2)}dt &= \Delta x_t' - \Delta x_0' - \frac{\delta}{\delta a}\int_0^t R\,dt\,.\left(\frac{\delta^2}{\delta a\,\delta b} - t\frac{\delta^2}{\delta a^2}\right)\int_0^t R\,dt, \\ -\frac{\delta}{\delta b}\int_0^t R^{(2)}dt &= t\,\Delta x_t' - \frac{\delta}{\delta a}\int_0^t R\,dt\,.\left(\frac{\delta^2}{\delta b^2} - t\frac{\delta^2}{\delta a\,\delta b}\right)\int_0^t R\,dt. \end{aligned}\right\} \tag{187}$$

At the same time, the new part or part of the second order in x_t, which will result on substituting for b its value $x_0' - \Delta x_0'$, must evidently be, by (181),

$$\Delta x_t = -t\,\Delta x_0'. \tag{188}$$

We have therefore, by an easy elimination,

$$\Delta x_t = -\left(t\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\right)\int_0^t R^{(2)}dt - \frac{\delta}{\delta a}\int_0^t R\,dt\,.\left(t\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\right)^2\int^t R\,dt; \tag{189}$$

the notation being such that we have the symbolic equation

$$\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\Big)^2 = t^2\frac{\delta^2}{\delta a^2} - 2t\frac{\delta^2}{\delta a\,\delta b} + \frac{\delta^2}{\delta b^2}. \tag{190}$$

This correction (189) is therefore that which must be added to the expression (181), if we desire to treat a & b as equal to the constants x_0 and x_0'; & consequently, with this last condition, we have

$$x_t = a + bt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R^{(2)}dt - \frac{\delta}{\delta a}\int_0^t R\,dt\,.\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)^2\int_0^t R\,dt; \tag{191}$$

---

* [The last term arises from (181), which gives $\delta x_t = \delta a + t\,\delta b + \delta\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt.$]

a and b being here, as was said, the constants x_0 and x'_0, while R is the old function formed from them by changing x_t to $a+bt$, and $R^{(2)}$ again is formed from R according to the rule (184).

The perturbation of the 2nd order expressed by the second line of my formula (191) is so unlike in appearance to that expressed by the sum of the last two lines of the formula (180) deduced from Lagrange's method that I thought it useful to confirm their agreement by submitting both of them to suitable reductions,* depending on the circumstance that R in each is a function of $a+bt$ alone. And thus I found from each this common transformed expression for that part of the perturbation which is of the order of the square of the disturbing force,

$$\Delta x_t = \frac{1}{b}\int_0^t \frac{\delta R}{\delta a}\,dt \,.\int_0^t\left(\int_0^t \frac{\delta R}{\delta a}\,dt\right)dt - \frac{3}{2b}\int_0^t\left(\int_0^t \frac{\delta R}{\delta a}\,dt\right)^2 dt; \qquad (192)$$

[Four pages are missing here. From a fragment it is evident that they contained a verification of these formulae for the case $x'' = \alpha\,(a+bt)$.]

.... I believe however that I need take no pains to prove to you the existence of this inconvenience in the application of Lagrange's method, for I suspect that it was to it, & to what you thought its probable occurrence in the corresponding stage of the application of my own method also, that you attended in your answer to my IInd Letter, when you said: "But a great difficulty begins in your method when you introduce the perturbational terms in R in order to get a further approximation, because you must either develope $\frac{\delta}{\delta n}\int R\,dt$, $\frac{\delta}{\delta \epsilon}\int R\,dt$, $\frac{\delta}{\delta k}\int R\,dt$, $\frac{\delta}{\delta l}\int R\,dt$ *separately* or you must do what will be equally troublesome." I did not on first reading understand this passage, because I had reflected much less on Lagrange's method than on my own; I supposed, as I still do, that you would not reckon as a serious addition to the other difficulties of the problem the more explicit *differentiation of the development of a simple function* (such as my $\int R^{(2)}dt$) even with respect to several different elements, if that one development were once effected.

On the other hand, while I think that I escape from the important difficulty of having, at each new stage of the approximation, to *develope separately as many different and independent functions as there are varying elements*, (which is the difficulty you chiefly feared from the experience of Lagrange's method,) I will not be too sanguine that some new & equally formidable difficulty of *some other kind* may not arise in applying my method, though I see no reason why it should. And it is right to state, because in a kind of appendix to my IIId Letter (dated Oct. 18th) I may seem to have slurred over the point, that in passing to a second correction & calculating that new function which answers to the $R^{(2)}$ of the present letter, it will be necessary to develope the terms which depend upon the square of the disturbing force not only in $-R$ but also in $\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}$ which accompanies it under the integral sign in the expression of my principal function, or in the formula (57).

---

* [If R_u denotes the value of R with u inserted for t we get

$$\int_0^t R^{(2)}\,dt = \frac{1}{2}\int_0^t \frac{(R_u-R_0)^2}{b^2}\,du - \frac{R_t-R_0}{b^2}\int_0^t (R_u-R_0)\,du,$$

$$\left(\frac{\delta}{\delta b}-t\frac{\delta}{\delta a}\right)\frac{1}{2}\int_0^t \frac{(R_u-R_0)^2}{b^2}\,du = -\frac{3}{2b^2}\int_0^t (R_u-R_0)^2\,du + \frac{t}{b^2}\frac{\delta R_0}{\delta a}\int_0^t (R_u-R_0)\,du.$$

The equation (192) follows easily.]

It is now the 7th of November, this letter having been begun a week ago, but continued only at intervals since. Next week I begin to lecture (as I do every year) on Astronomy in the University of Dublin; and though I propose to introduce more of dynamics this time than I have ever done before, my lectures will still, I foresee, be, as they have always been, a very serious distraction from all private studies & researches. It is likely therefore that I may not be able to write again upon the Theory of the Moon till Christmas. But at that time perhaps I may have leisure and courage to enter on the subject again. In the mean while, I shall esteem it a great favour if you will inform me of the result of your reflexions on what I have already written; and will tell me, among other things, whether I have rightly divined the sort of process by which Poisson proposes to allow for the square of the disturbing force in applying Lagrange's method. No doubt it will be my duty to read Poisson's memoir myself, & I can procure it from the R.I. Academy—but still I wish to have your opinion too. As to foreign books in general, you can hardly conceive the difficulty I find in getting at them. For instance, our Academy, though subscribing to the *Comptes Rendus,* has not yet received a single number for 1837. Perhaps that difficulty might be a little diminished by employing for the aforesaid Academy a better London Bookseller—& you would confer a great favour on it & me, by suggesting one who would take some trouble about the matter. The present man is Boone (of the Strand, I think), very respectable I dare say and so forth, but of almost no use to us. And in the meantime, for the reason that I mentioned, I should be very glad to have a sight of what you said Jacobi has written in connexion with my speculations, if his paper could be fairly franked. I return one of the two copies of Pontécoulant's pamphlet, which is still among my *legenda*. I send also a copy of my second Essay on Dynamics & beg you to accept it. I must send another to Poisson, who in the paper that we saw at Liverpool spoke only of my first Essay & seemed, as I said, to have read only the first few pages even of that; but of course I must consider it as a compliment that he even troubled himself to read so much. Perhaps you can name some other persons, who ought to receive separate copies of my Second Essay, for the copies only reached me as I was leaving Liverpool; Pontécoulant of course ought to have one: would you take the trouble of transmitting any? And can you also send the enclosed order for a set of Italian Memoirs to the proper place in London, namely the bookseller Molini, with a request that he will send the books to me at the Royal Irish Academy, Dublin?

You will easily conceive that if I venture to attack the Theory of the Moon with the bold hope of adding anything to what Plana & you have done I shall avail myself of whatever facility may be derived from starting with a movable ellipse; & shall endeavour so to manage my formulae as to keep the time under the signs of periodicity. No doubt it will be prudent to conform in all things possible to your notation, & your choice of coordinates & of elements such as $\frac{1}{r}$ & e instead of r & k. But I dare not pledge myself to voyage out of sight of land upon so vast a sea of labour.

With best regards to Mrs Lubbock, I am, my dear Sir,

very sincerely yours

William R. Hamilton.

[The correspondence closes with a letter from J. W. Lubbock dated 12th Nov., 1837, the extant parts of which are of no mathematical interest.]

VIII.

ON THE COMPOSITION OF FORCES

Read November 8, 1841.

[*Proc. Roy. Irish Acad.* Vol. II (1840–44), pp. 166–168.]

Two rectangular forces, x and y, being supposed to be equivalent to a single resultant force p, inclined at an angle v to the force x, it is required to determine the law of the dependence of this angle on the ratio of the two component forces x and y.

Denoting by p' any other single force, intermediate between x and y and inclined to x at an angle v', which we shall suppose to be greater than v; and denoting by x' and y' the rectangular components of this new force p' in the directions of x and y, we may, by easy decompositions and recompositions, obtain a new pair of rectangular forces, x'' and y'', which are together equivalent to p' and have for components

$$x''=\frac{x}{p}x'+\frac{y}{p}y';$$

$$y''=\frac{x}{p}y'-\frac{y}{p}x';$$

the direction of x'' coinciding with that of p, but the direction of y'' being perpendicular thereto. Hence

$$\frac{y''}{x''}=\frac{xy'-yx'}{xx'+yy'};$$

that is,

$$\tan^{-1}\frac{y''}{x''}=\tan^{-1}\frac{y'}{x'}-\tan^{-1}\frac{y}{x};$$

or, finally,

$$f(v'-v)=f(v')-f(v), \tag{A}$$

at least for values of v, v' and $v'-v$, which are each greater than 0, and less than $\frac{\pi}{2}$; if f be a function so chosen that the equation

$$\frac{y}{x}=\tan f(v)$$

expresses the sought law of connexion between the ratio $\frac{y}{x}$ and the angle v. The functional equation (A) gives

$$f(mv)=mf(v)=\frac{m}{n}f(nv),$$

m and n being any whole numbers; and the case of equal components gives evidently

$$f\left(\frac{\pi}{4}\right)=\frac{\pi}{4};$$

hence

$$f\left(\frac{m}{n}\frac{\pi}{4}\right)=\frac{m}{n}\frac{\pi}{4},$$

and ultimately,

$$f(v)=v, \tag{B}$$

because it is evident, by the nature of the question, that while v increases from 0 to $\frac{\pi}{2}$, the function $f(v)$ increases therewith, and therefore could not be equal thereto for all values of v commensurable with $\frac{\pi}{4}$, unless it had the same property also for all intermediate incommensurable values. We find, therefore, that for all values of the component forces x and y, the equation

$$\frac{y}{x}=\tan v \tag{C}$$

holds good; that is, the resultant force coincides *in direction* with the diagonal of the rectangle constructed with lines representing x and y as sides.

The other part of the known law of the composition of forces, namely, that this resultant is represented also *in magnitude* by the same diagonal, may easily be proved by the process of the *Mécanique Céleste*, which, in the present notation, corresponds to making

$$x'=x, \quad y'=y, \quad x''=p,$$

and therefore gives

$$p=\frac{x^2+y^2}{p}, \quad p^2=x^2+y^2.$$

But the demonstration above assigned for the law of the *direction* of the resultant appears to Sir William Hamilton to be new.*

---

* [For an account of the history of statical proofs of the parallelogram of forces, with references to the literature of the subject, see Voss, *Encyklopädie der Mathematischen Wissenschaften*, IV. 1, pp. 43–46.]

IX.

ON THEOREMS OF CENTRAL FORCES

Read November 30, 1846.

[*Proc. Roy. Irish Acad.* Vol. III (1845–47), pp. 308, 309.]

Sir William R. Hamilton stated the following theorems of central forces, which he had proved by his calculus of quaternions, but which, as he remarked, might be also deduced from principles more elementary.

If a body be attracted to a fixed point, with a force which varies directly as the distance from that point, and inversely as the cube of the distance from a fixed plane, the body will describe a conic section, of which the plane intersects the fixed plane in a straight line, which is the polar of the fixed point with respect to the conic section.*

And in like manner, if a material point be obliged to remain upon the surface of a given sphere, and be acted on by a force, of which the tangential component is constantly directed (along the surface) towards a fixed point or pole upon that surface, and varies directly as the sine of the arcual distance from that pole, and inversely as the cube of the sine of the arcual distance from a fixed great circle; then the material point will describe a spherical conic, with respect to which the fixed great circle will be the polar of the fixed point.

Thus, a spherical conic would be described by a heavy point upon a sphere, if the vertical accelerating force were to vary inversely as the cube of the perpendicular and linear distance from a fixed plane passing through the centre.

The first theorem had been suggested to Sir W. Hamilton by a recently resumed study of a part of Sir Isaac Newton's *Principia*; and he had been encouraged to seek for the second theorem, by a recollection of a result respecting motion in a spherical conic, which was stated some years ago to the Academy by the Rev. C. Graves. In that result of Mr. Graves, the fixed pole was a focus of the conic, and the polar was therefore the director arc; consequently, the sine of the distance from the polar was proportional to the sine of the distance from the pole, and, instead of the law now mentioned to the Academy, there was the simpler law of proportionality to the inverse square of the sine of the distance from the fixed pole or focus.

---

* [Referred to as Hamilton's Theorem. Cf. Whittaker, *Analytical Dynamics* (1927), p. 79.]

X.

THE HODOGRAPH, OR A NEW METHOD OF EXPRESSING IN SYMBOLICAL LANGUAGE THE NEWTONIAN LAW OF ATTRACTION*

Read December 14, 1846.

[*Proc. Roy. Irish Acad.* Vol. III (1845–47), pp. 344–353.]

Sir William R. Hamilton made a communication respecting a new mode of geometrically conceiving, and of expressing in symbolical language, the Newtonian law of attraction, and the mathematical problem of determining the orbits and perturbations of bodies which are governed in their motions by that law.

Whatever may be the complication of the accelerating forces which act on any moving body, regarded as a moving point, and, therefore, however complex may be its *orbit*, we may always imagine a succession of straight lines, or vectors, to be drawn from some one point, as from a common origin, in such a manner as to represent, by their directions and lengths, the varying directions and degrees (or quantities) of the velocity of the moving point: and the curve which is the locus of the ends of the straight lines so drawn may be called the *hodograph*† of the body, or of its motion, by a combination of the two Greek words, ὁδός, a *way*, and γράφω, *to write* or *describe*; because the vector of this hodograph, which may also be said to be the *vector of velocity* of the body, and which is always parallel to the tangent at the corresponding point of the orbit, marks out or indicates at once the direction of the momentary path or way in which the body is moving, and the rapidity with which the body, at that moment, is moving in that path or way. This hodographic curve is even more immediately connected than the orbit with the *forces* which act upon the body, or with the varying resultant of those forces, for the tangent to the hodograph is always parallel to the direction of this resultant; and if the element of the hodograph be divided by the element of the time, the quotient of this division represents (in the usual units) the intensity of the same resultant force; so that the whole accelerating force which acts on the body at any one instant is represented, both in direction and in magnitude, by the element of the hodograph, divided by the element of the time. We have also the general proportion, that the *force is to the velocity*, in any varied motion of a point, *as the element of the hodograph is to the corresponding element of the orbit.*

These general remarks respecting varied motion, under the influence of *any* accelerating forces whatever, having been premised, let it be now supposed that the force is constantly directed towards some one *fixed point* or *centre*, which it will then be natural to choose for the origin of the vectors of the hodograph. The straight lines drawn to the moving body from the centre of force being called, in like manner, the vectors of the orbit, or the *vectors of position* of the body, we see that each such vector of position is now parallel to the tangent of the hodograph drawn at the extremity of the vector of velocity, as the latter vector was seen to be parallel to the tangent of the orbit, drawn at the extremity of the vector of position; so that the two vectors, and the two tangents drawn at their extremities, enclose at each moment a *parallelogram*, of which it is easily seen that the *plane and area are constant*, although its position and its shape are generally variable from one moment to another, in the motion thus performed under the influence of a central force. If, therefore, this constant area be given, and if either the hodograph or the orbit be known, the other of these two curves can be deduced by a simple and uniform process, on which account the two curves themselves may be called *reciprocal hodographs*.

---

* [For a treatment by quaternions, see Hamilton, *Elements of Quaternions* (2nd edition), Vol. II (1901), pp. 299 *et seq.*]

† [Hamilton afterwards discovered that Möbius had already described this curve in his *Mechanik des Himmels* (1843). See Hamilton, *Lectures on Quaternions* (1853), p. 614.]

The opposite angles of a parallelogram being equal, it is evident, that if the central force be attractive, any one vector of position is inclined to the next following element of the orbit at the same angle as that at which the corresponding vector of velocity is inclined to the next preceding element of the hodograph. Also, if from either extremity of any small element of any curve a chord of the circle which osculates to that curve along that element be drawn and bisected, the element subtends, at the middle point of this chord, an angle equal to the angle between the two tangents drawn at the two extremities of the element; that is, here, if the curve be the hodograph, to the angle between the two near vectors of position, which are parallel to the two extreme tangents of its element. We have, therefore, two small and similar triangles, from which results the following proportion, that *the half chord of curvature of the hodograph* (passing through or tending towards the fixed centre of force) *is to the radius vector of the orbit* as the element of the hodograph is to the element of the orbit, that is, by what was lately seen, *as the force is to the velocity*.*

But also, the radius of curvature of the hodograph is to the half chord of curvature of the same curve, as the radius vector of the orbit is to the perpendicular let fall from the fixed centre on the tangent to the same orbit; therefore, by compounding equal ratios, we obtain this other proportion: *the radius of curvature of the hodograph is to the radius vector of the orbit, as the rectangle under the same radius vector and the force is to the rectangle under the velocity and the perpendicular*, or to the constant parallelogram under the vectors of position and velocity. If, therefore, the law of the inverse square hold good, so that the second and third terms of this last proportion vary inversely as each other, while the fourth term remains unchanged, the first term must be also constant; that is, *with Newton's law of force* (supposed here to act towards a fixed centre) *the curvature of the hodograph is constant*: and, consequently, this curve, having been already seen to be *plane*, is now perceived to be a *circle*, of which the radius is equal to the attracting mass divided by the constant double areal velocity in the orbit. Reciprocally, we see that *no other law* of force would conduct to the same result: so that the Newtonian law may be *characterized* as being the *Law of the Circular Hodograph*.

Another mode of arriving at the same simple but important result is to observe, that because the radius of curvature of the hodograph is equal to the element of that curve, divided by the angle between the tangents at its extremities, or (in the case of a central force) by the angle between the two corresponding vectors of the orbit, which angle is equal to the double of the elementary area divided by the square of the distance (of the body from the centre of force), while the element of the hodograph has been seen to be equal to the force multiplied by the element of time, or multiplied by the same double element of orbital area, and divided by the constant of double areal velocity, therefore this radius of curvature of the hodograph must, for any central force, be equal to the force multiplied by the square of the distance and divided by the double areal velocity.

---

* By an exactly similar reasoning, the following known proportion may be proved anew, namely, that the force is to the velocity as that velocity is to the half chord of curvature of the orbit, whatever the law of central force may be.

The point on the hodograph which is the termination of any one vector of velocity may be called the *hodographic representative* of the moving body, and the foregoing principles show, that with a central force varying as the inverse square of the distance, this representative point describes, in any proposed interval of time, a *circular arc*, which contains the same number of degrees, minutes, and seconds, as the angle contemporaneously described round the centre of force by the body itself in its orbit, or by the revolving vector of position; because, whatever that angle may be, an equal angle is described in the same time by the revolving tangent to the hodograph. Thus, with the law of Newton, *the angular motion of a body in its orbit is exactly represented, with all its variations, by the circular motion on the hodograph*; and this remarkable result may be accepted, perhaps, as an additional motive for the use of the new term which it is here proposed to introduce.

Whatever the law of central force may be, if the square of the velocity in the orbit be subtracted from the double rectangle under the force and distance, which has been seen to be equal to the rectangle under the same velocity and the chord of curvature of the hodograph, the remainder is the rectangle under the segments into which that chord is cut by the centre of force, being positive when this section takes place internally, but negative when the section is external, that is, when the centre of force is outside the osculating circle of the hodograph. In the case of the law of the inverse square, this latter curve is its own osculating circle, and the rectangle under the segments of the chord is therefore constant, by an elementary theorem of geometry contained in the third book of Euclid; if, then, the square of the velocity be subtracted from the double of the attracting mass, divided by the distance of the body from the centre of force, at which that mass is conceived to be placed, the remainder is a constant quantity, which is positive if the orbit be a closed curve, that is, if the centre of force be situated in the *interior* of the circular hodograph.

In this case of a closed orbit, the positive constant, which is thus equal to the product of the segments of a hodographic chord, or *the constant product of any two opposite velocities of the body*, is easily seen, by the foregoing principles, to be equal to *the attracting mass divided by the semisum of the two corresponding distances of the body*, which semisum is therefore seen to be *constant* and may be called (as in fact it is) the *mean distance*. The *law of living force*, involving this mean distance, may therefore be deduced as an elementary consequence of this mode of hodographic representation for the case of a closed orbit; together with the corresponding forms of the law, involving a null or a negative constant, instead of the reciprocal of the mean distance, for the two cases of an orbit which is not closed, namely, when the centre of force is on or is outside the circumference of the hodographic circle.

Whichever of these situations the centre of force may have, we may call the straight *line* drawn from it to the centre of the hodograph, the hodographic *vector of eccentricity*; and the *number* which expresses the ratio of the length of this vector to the radius of the hodograph will represent, if the orbit be closed, the ratio of the semidifference to the semisum of the two extreme distances of the body from the centre of force, and may be called generally the *numerical eccentricity* of the hodograph, or of the orbit (without violating the received meaning of the term).

Whatever the value of this numerical eccentricity may be, the constant area of the parallelogram under the vectors of position and velocity may always be treated as the sum or difference of two other parallelograms, of which one is equal to the rectangle under the constant radius of the hodographic circle and the varying radius vector of the orbit, while the other is equal to the parallelogram under the vectors of position and eccentricity; and hence it is not difficult to infer that the length of the vector of position, or of the radius vector of the orbit, varies in a constant ratio, expressed by the numerical eccentricity, to the perpendicular let fall from its extremity, that is, from the position of the body, on a constant straight line or *directrix*, which is situated in the plane of the orbit and is parallel to the hodographic vector of eccentricity. The *orbit*, therefore, whether it be closed or not, is always (with the law of the inverse square) a *conic section*, having the centre of force for a *focus*—a theorem which has indeed been known since the time of Newton, but has not perhaps been proved before from principles so very elementary.*

Conceive a diameter of the hodograph to be drawn in a direction perpendicular to the vector of eccentricity; the extremities of this diameter correspond to the extremities of that chord of the orbit which is perpendicular to the shortest radius vector, and which is called the *parameter*;† from which it follows that the *semiparameter of the orbit is equal to the constant area of the parallelogram under distance and velocity, divided by the radius of the hodograph*, and, consequently, that it is equal to the *square of the constant double areal velocity, divided by the attracting mass*.

It is evident that these results agree with and illustrate those by which Newton shewed that Kepler's laws were mathematical consequences of his own great law of attraction. In applying them to the undisturbed motion of any binary system of bodies, attracting each other according to that law, we have only to substitute the sum of the two masses for the single attracting mass already considered, and to treat one of the two bodies as if it were the fixed origin of the vectors of a *relative hodograph*, which will still be circular as before. And even if we consider a ternary, or a *multiple system*, we may still regard *each* body as *tending*, by its attraction, to cause every *other* to describe an orbit of which the hodographic representative would be a perfect circle.

When there is *one predominant mass*, as in the case of the solar system, we may in general regard each other body of the system as moving in an orbit about it, which is, on the same plan, *represented* by a *varying circular hodograph*. For if, at any one moment, we know *the two heliocentric vectors* of position and velocity of a planet, we know the plane and area of the parallelogram under those two vectors, and can conceive a parallelepiped constructed, of which this momentary parallelogram shall be the base, while the volume of the solid shall represent the sum of the masses of the sun and planet; and then the height of the same solid will be equal to the radius of the momentary hodograph; so that, in order to construct this hodograph, we shall only have to describe, in the plane and with the radius determined as above, a circle which shall touch the side parallel to the heliocentric vector of position, at the extremity of the vector of velocity, and shall have its concavity, at the point of contact, turned towards the sun. The moon, or any other satellite, may also be regarded as describing, about its primary, an orbit of which the hodographic representative shall still be a varying circle.

---

* The *hodograph of the earth's annual motion* may be considered to be exhibited to *observation* in astronomy as the *curve of aberration of a star*; and it is known that this aberratic curve is a *circle*, notwithstanding the eccentricity of the earth's orbit; but the author is not aware that this circularity of the aberratic curve (for a star near the pole of the ecliptic) has ever been shown before to be a *consequence of the law of the inverse square*, except by the help of the properties of the *elliptic orbit*; whereas the spirit of the present communication is to *derive that orbit from the circle*, and to regard that circle itself as a sort of geometrical *picture of Newton's law*, instead of being only one of many corollaries from the laws of Kepler.

† [Latus-rectum.]

As formulae which may assist in symbolically tracing out the consequences of this geometrical conception, Sir William Hamilton offers the following transformations of certain general equations for the motion of a system of bodies attracting each other according to Newton's law, which he communicated to the Royal Irish Academy in July, 1845.*

The new forms of the equations are these:

$$\rho = \int \tau \qquad \frac{m'}{V.(\rho'-\rho)(\tau'-\tau)}; \quad \tau = \Sigma \int \sigma dU(\rho'-\rho);$$

in which ρ and τ are the vectors of position and velocity of the mass m at the time t; ρ' and τ' the two corresponding vectors of another mass m' at the same time; σ is another vector perpendicular to the plane and equal in length to the radius of the *momentary relative hodograph, representing the momentary relative orbit*, which the attraction of the mass m' tends to cause the body m to describe; d, $\int$, Σ are marks of differentiation, integration and summation, and V, U are the characteristics of the operations of taking respectively the vector and versor of a quaternion. Or, eliminating ρ and σ, but retaining the *hodographic vector* τ, and using Δ as the mark of differencing, the conditions of the question may be included in the following formula, which the author hopes on a future occasion to develope:

$$= \Sigma \int \frac{(m+\Delta m)\, dU(\int \Delta\tau dt)}{V(\Delta\tau . \int \Delta\tau dt)}$$

Meanwhile it is conceived that any such attempt as the foregoing, to simplify or even to transform the important and difficult problem of investigating the mathematical consequences of the Newtonian law of attraction, is likely to be received at the present time with peculiar indulgence and interest, in consequence not only of the brilliant deductive discovery lately made of the new planet† exterior to Uranus, but also of the extraordinary and exciting intelligence which has just arrived from Dorpat of the presumed discovery by Professor Mädler of a *central cluster* (the Pleiades), and of a *central sun* (Alcinoe, called also Eta Tauri): around which cluster, and which sun or star, it is believed by Mädler that our own sun and all the other stars of our sidereal system, including the milky way but exclusive of the more distant nebulae, are moving in enormous orbits, under the combined influences of their own mutual attractions, all regulated by the same great law.

---

* *Proc. Roy. Irish Acad.* Vol. III (1845–47), Appendices III and V, pp. xxxvi–lx.

† [Neptune was first observed on the night of September 23, 1846.]

Sir William Hamilton exhibited Professor Mädler's work, *Die Centralsonne*, Dorpat, 1846, in which, as a first provisional attempt to determine the orbit of our own sun, with the help of the proper motions of a great number of stars, combined with Bessel's parallax of 61 Cygni, Mädler assigns to what he regards as the Central Sun, Alcinoe, a distance amounting to thirty-four million times the distance of our sun from us; concluding, also, but still only as first approximations, that the period of our sun's revolution is about eighteen millions of years, and that its orbit has now an inclination to the ecliptic of about 84 degrees, with an ascending node of which the present longitude is nearly 237°.

A chart of observed places of Le Verrier's Planet [Neptune] was also exhibited by Sir William Hamilton; and was illustrated by comparison with Bremiker's Star-Map, which was also laid upon the table.

XI.

ON A THEOREM OF HODOGRAPHIC ISOCHRONISM*

Read March 16, 1847.

[*Proc. Roy. Irish Acad.* Vol. III (1845–47), p. 417.]

If two circular hodographs, having a common chord, which passes through or tends towards a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

ON A THEOREM OF ANTHODOGRAPHIC ISOCHRONISM*

Read May 10, 1847.

[*Proc. Roy. Irish Acad.* Vol. III (1845–47), p. 465.]

Sir William Hamilton stated and illustrated a theorem of anthodographic (or anthodic) isochronism, namely, that if two circular anthodes, having a common chord, which passes through or tends towards a common centre of force, be both cut perpendicularly by any third circle, the times of anthodically describing the intercepted arcs will be equal:—the *anthode* of a planet being the circular locus of the extremities of its vectors of slowness, or of straight lines representing in length and direction the reciprocals of its velocities, and drawn from a common origin.

This theorem is intimately connected with the analogous theorem respecting hodographic isochronism (or synchronism), which was communicated to the Academy by Sir William Hamilton, in a note read at the Meeting in last March. He had been led to perceive that former theorem by combining the principles of his first paper on a General Method in Dynamics, published in the second part of the *Philosophical Transactions* for 1834, with those of his communication of last December, since published in the Proceedings of the Academy, respecting the Law of the Circular Hodograph. This *Hodograph* was, for a planet or comet, the circular locus of the extremities of its *vectors of velocity*, as the Anthode is the locus of the extremities of the vectors of slowness; so that the rectangular coordinates of the Hodograph are x', y', z', if

$$x'=\frac{dx}{dt}, \quad y'=\frac{dy}{dt}, \quad z'=\frac{dz}{dt};$$

while those of the Anthode may be denoted as follows

$$x_{,}=-v^{-2}x', \quad y_{,}=-v^{-2}y', \quad z_{,}=-v^{-2}z',$$

where $v^2=x'^2+y'^2+z'^2$.

He had effected the passage from the theorem respecting hodographic to that respecting anthodic isochronism by the help of his calculus of quaternions; but had since been able to prove both theorems by means of certain elementary properties of the circle.†

---

* [See Appendix, Note 8, p. 630.]

† [See Hamilton, *Elements of Quaternions* (2nd edition), Vol. II, pp. 308–314, 318–320.]

For a hyperbolic comet, the Anthode is a circular arc *convex* to the sun; for a parabol comet, the Anthode is a *straight line*. And for comets of this latter class the theorem of is chronism takes this curiously simple form: "Any two diameters of any one circle (or spher in space, are anthodically described in equal times, with reference to any one point, regarded a a common centre of force." By this last theorem, the general problem of determining the tim of *orbital* description of a finite arc of a *parabola* is reduced to that of determining the time *anthodical* description of a finite *straight line* directed to the sun; and thus it is found that "th interval of time between any two positions of a parabolic comet, divided by the mass of the su is equal to the sixth part of the difference of the cubes of the sum and difference of the diagona of the parallelogram, constructed with the initial and final vectors of slowness as two adjacen sides." Another very simple expression for the time of description of a parabolic arc, to whic Sir William Hamilton is conducted by his own method, but which he sees to admit of easy proo from known principles (though he does not remember meeting the expression itself), is give by the following formula:

$$t=\tfrac{1}{2}T\tan\left(\theta-\tan^{-1}\tfrac{1}{2}\tan\tfrac{1}{2}\theta\right),$$

where θ is the true anomaly, and t is the time from perihelion, while T is the time of describin the first quadrant of true anomaly.

Part II

CALCULUS OF PRINCIPAL RELATIONS

XII.

CALCULUS OF PRINCIPAL RELATIONS*

[1836.]

INTRODUCTION.

1. It is a well-known theorem of the Integral Calculus that if y_x be a function of x, respecting which we are only told that it satisfies a given differential equation of the first order,

$$0 = f(x, y_x, y'_x), \tag{1}$$

$\left(y'_x \text{ denoting the differential coefficient or derived function } \frac{dy_x}{dx}\right)$, we are at liberty to assume an additional condition of the form

$$y_a = b, \tag{2}$$

a and b denoting any two assumed corresponding values of the two connected variables x and y_x; but that after the assumption of this new condition, the functional relation between these two variables x and y_x is in general theoretically determined, whatever practical difficulties may remain in the actual discovery of its form. So that the differential equation (1) may in general be conceived as conducting, by an easy or difficult but always possible integration, to a relation between any one pair of corresponding values a, y_a of the two connected variables of the question and any other pair of corresponding values x, y_x of the same two connected variables; or to an equation in finite differences of the first order between a, y_a, Δa and Δy_a, which may be thus denoted,

$$0 = F(a, y_a, x, y_x) = F(a, y_a, a + \Delta a, y_a + \Delta y_a), \tag{3}$$

Δa and Δy_a being here equivalent to $x - a$ and $y_x - y_a$ respectively.

2. Again, if y_x and z_x denote two functions of x, respecting which we only know that their first derived functions y'_x and z'_x are connected with them and with x by a given relation of the form

$$0 = X + Yy'_x + Zz'_x, \tag{4}$$

in which X, Y, Z denote given functions of x, y_x and z_x; then, if we denote by

$$\frac{\delta X}{\delta x}, \frac{\delta X}{\delta y_x}, \frac{\delta X}{\delta z_x}, \quad \frac{\delta Y}{\delta x}, \frac{\delta Y}{\delta y_x}, \frac{\delta Y}{\delta z_x}, \quad \frac{\delta Z}{\delta x}, \frac{\delta Z}{\delta y_x}, \frac{\delta Z}{\delta z_x},$$

the nine partial differential coefficients of the first order of these three functions, taken with respect to the three variables x, y_x and z_x, as if they were independent of each other, and if we suppose that these partial differential coefficients satisfy the condition

$$0 = X\left(\frac{\delta Y}{\delta z_x} - \frac{\delta Z}{\delta y_x}\right) + Y\left(\frac{\delta Z}{\delta x} - \frac{\delta X}{\delta z_x}\right) + Z\left(\frac{\delta X}{\delta y_x} - \frac{\delta Y}{\delta x}\right), \tag{5}$$

---

* [It was Hamilton's intention to write a book on the Calculus of Principal Relations (see Graves, *Life of Hamilton*, II, p. 177). This is all that remains of the manuscript of the proposed treatise. Papers XIII and XIV, pp. 332–390, contain material which he obviously intended to incorporate in the book. A sketch of the calculus is to be found on page 408.]

XII.

CALCULUS OF PRINCIPAL RELATIONS*

[1836.]

INTRODUCTION.

1. It is a well-known theorem of the Integral Calculus that if y_x be a function of x, respecting which we are only told that it satisfies a given differential equation of the first order,

$$0=f(x, y_x, y'_x), \tag{1}$$

$\left(y'_x\right.$ denoting the differential coefficient or derived function $\left.\frac{dy_x}{dx}\right)$, we are at liberty to assume an additional condition of the form

$$y_a=b, \tag{2}$$

a and b denoting any two assumed corresponding values of the two connected variables x and y_x; but that after the assumption of this new condition, the functional relation between these two variables x and y_x is in general theoretically determined, whatever practical difficulties may remain in the actual discovery of its form. So that the differential equation (1) may in general be conceived as conducting, by an easy or difficult but always possible integration, to a relation between any one pair of corresponding values a, y_a of the two connected variables of the question and any other pair of corresponding values x, y_x of the same two connected variables; or to an equation in finite differences of the first order between a, y_a, Δa and Δy_a, which may be thus denoted,

$$0=F(a, y_a, x, y_x)=F(a, y_a, a+\Delta a, y_a+\Delta y_a), \tag{3}$$

Δa and Δy_a being here equivalent to $x-a$ and y_x-y_a respectively.

2. Again, if y_x and z_x denote two functions of x, respecting which we only know that their first derived functions y'_x and z'_x are connected with them and with x by a given relation of the form

$$0=X+Yy'_x+Zz'_x, \tag{4}$$

in which X, Y, Z denote given functions of x, y_x and z_x; then, if we denote by

$$\frac{\delta X}{\delta x}, \frac{\delta X}{\delta y_x}, \frac{\delta X}{\delta z_x}, \quad \frac{\delta Y}{\delta x}, \frac{\delta Y}{\delta y_x}, \frac{\delta Y}{\delta z_x}, \quad \frac{\delta Z}{\delta x}, \frac{\delta Z}{\delta y_x}, \frac{\delta Z}{\delta z_x},$$

the nine partial differential coefficients of the first order of these three functions, taken with respect to the three variables x, y_x and z_x, as if they were independent of each other, and if we suppose that these partial differential coefficients satisfy the condition

$$0=X\left(\frac{\delta Y}{\delta z_x}-\frac{\delta Z}{\delta y_x}\right)+Y\left(\frac{\delta Z}{\delta x}-\frac{\delta X}{\delta z_x}\right)+Z\left(\frac{\delta X}{\delta y_x}-\frac{\delta Y}{\delta x}\right), \tag{5}$$

---

* [It was Hamilton's intention to write a book on the Calculus of Principal Relations (see Graves, *Life of Hamilton*, II, p. 177). This is all that remains of the manuscript of the proposed treatise. Papers XIII and XIV, pp. 332–390, contain material which he obviously intended to incorporate in the book. A sketch of the calculus is to be found on page 408.]

it is known that the differential equation (4) conducts by integration to a relation of the form

$$f_{\prime}(x,\ y_x,\ z_x) = \text{const.} = f_{\prime}(a,\ y_a,\ z_a), \tag{6}$$

which may also be thus written, as an equation in finite differences of the first order,

$$0 = F(a,\ y_a,\ z_a,\ a+\Delta a,\ y_a+\Delta y_a,\ z_a+\Delta z_a). \tag{7}$$

3. But in general, if the derived functions y'_x and z'_x be only connected with x, y_x and z_x by a given relation, which may be thus denoted,

$$0 = f(x,\ y_x,\ z_x,\ y'_x,\ z'_x), \tag{8}$$

then (setting aside some particular cases of exception such as that which has been last considered) we may attribute any arbitrary values to the three initial and three final values, a, y_a, z_a, $a+\Delta a$, $y_a+\Delta y_a$ and $z_a+\Delta z_a$, without obliging these initial and final values to satisfy any determinate relation, such as (7), as a necessary consequence of the given differential equation (8). Or, to express the same thing otherwise, we may in general assume any arbitrary values y_a and z_a of the functions y_x and z_x, as corresponding to an assumed value a of the variable x, without thereby establishing any relation between the values of the same functions $y_{a+\Delta a}$ and $z_{a+\Delta a}$, which correspond to any other assumed value $a+\Delta a$ of the same variable x. According to the unlimited variety of forms which we may assume for the function y_x, consistently with previously assumed values y_a and $y_{a+\Delta a}$, we shall have by the differential equation (8) an unlimited variety of corresponding forms of the function z_x, even when the initial value z_a is given or previously assumed; and in passing from one such form of z_x to another the final value $z_{a+\Delta a}$ will itself in general vary; so that it will be only in particular cases that these various forms of the function z_x will fail to give an unlimited variety of final values $z_{a+\Delta a}$.

4. To illustrate this theorem by an example and to prove that it is true in an extensive class of cases before considering the general proof of it, let us assume the particular relation (4), supposing now that the condition (5) is *not* satisfied; and let us write, for abridgement,

$$-\frac{X}{Z} = p, \qquad \cdot\frac{}{Z} = q, \tag{9}$$

so that the condition (5) shall take this simplified form

$$\frac{\delta p}{\delta y_x} + q\frac{\delta p}{\delta z_x} = \frac{\delta q}{\delta x} + p\frac{\delta q}{\delta z_x}. \tag{10}$$

We have thus the differential equation

$$z'_x = p + q y'_x, \tag{11}$$

in which p and q are any two given functions of x, y_x and z_x, *not* satisfying the condition (10). Let any function y_x of x be supposed to be assumed at pleasure, subject only to the initial and final conditions

$$y_a = b, \quad y_{a+\Delta a} = b + \Delta b, \tag{12}$$

in which the values of b and Δb are given or previously assumed; and then let the connected function z_x be conceived to be determined (as we may always at least theoretically conceive it to be, by principles already referred to, whatever practical difficulties the actual determination may present) so as to satisfy the differential equation (11) and also the initial condition

$$z_a = c, \tag{13}$$

in which c is given or assumed; and finally, from the form of the function z_x thus determined, let the final value

$$z_{a+\Delta a} = c + \Delta c \tag{14}$$

be conceived to be calculated. We propose to show that this final value $c+\Delta c$ will in general vary with a variation of the assumed form of the function y_x, even when a, b, c, Δa and Δb remain unvaried. To show this, let ϵ be any small and arbitrary constant and let $y_x + \epsilon\eta_x$ and $z_x + \epsilon\zeta_x$ denote two new connected functions of x, which differ little from the two old functions y_x, z_x and which satisfy, like them, the differential equation (11), or rather the new equation into which that transforms itself when the new functions are substituted for the old ones; let it be supposed also that these new functions $y_x + \epsilon\eta_x$, $z_x + \epsilon\zeta_x$ satisfy these three new conditions

$$y_a + \epsilon\eta_a = b, \quad y_{a+\Delta a} + \epsilon\eta_{a+\Delta a} = b + \Delta b, \quad z_a + \epsilon\zeta_a = c, \tag{15}$$

which are analogous to the three old conditions (12) and (13) and give, when combined with them,

$$\eta_a = 0, \quad \eta_{a+\Delta a} = 0, \quad \zeta_a = 0. \tag{16}$$

The new differential equation, analogous to (11), may be thus written

$$z'_x + \epsilon\zeta'_x = p + \epsilon\eta_x \frac{\delta p}{\delta y_x} + \epsilon\zeta_x \frac{\delta p}{\delta z_x} + \left(q + \epsilon\eta_x \frac{\delta q}{\delta y_x} + \epsilon\zeta_x \frac{\delta q}{\delta z_x}\right) y'_x + \epsilon q \eta'_x + \epsilon^2 E_x, \tag{17}$$

E_x denoting, for abridgement, a factor the development of which is easily obtained but is not necessary for our present purpose; and if we subtract the old differential equation (11) from the new one (17) and divide by ϵ, we find, more simply,

$$\zeta'_x = \left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right) \zeta_x + \left(\frac{\delta p}{\delta y_x} + y'_x \frac{\delta q}{\delta y_x}\right) \eta_x + q\eta'_x + \epsilon E_x : \tag{18}$$

from which equation we propose to deduce by integration an expression for the final value $\zeta_{a+\Delta a}$, which shall show that this final value does not in general vanish and therefore that the final value $z_{a+\Delta a} + \epsilon\zeta_{a+\Delta a}$ of the new function $z_x + \epsilon\zeta_x$ is in general different from the final value $z_{a+\Delta a}$ of the old function z_x.

With this view, we may multiply the differential equation (18) by a function λ_x, so chosen as to make the product

$$\lambda_x \left\{ \zeta'_x - \left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right) \zeta_a \right.$$

equal to the first differential coefficient or derived function $(\lambda_x \zeta_x)'$ of the product $\lambda_x \zeta_x$; a condition which gives

$$\lambda'_x = -\lambda_x \left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right), \tag{19}$$

and therefore, in the notation of definite integrals,

$$\lambda_x = \lambda_a e^{-\int_a^x \left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right) dx}. \tag{20}$$

When λ_x is thus chosen, it is easy to see that the first differential coefficient or derived function of the product $\lambda_x q$ is

$$\begin{aligned}(\lambda_x q)' &= \frac{d}{dx}(\lambda_x q) = \lambda_x \frac{dq}{dx} + q\lambda'_x \\ &= \lambda_x\left(\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + z'_x \frac{\delta q}{\delta z_x} + q\frac{\lambda'_x}{\lambda_x}\right) \\ &= \lambda_x\left\{\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + (p + qy'_x)\frac{\delta q}{\delta z_x} - q\left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right)\right\} \\ &= \lambda_x\left(\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + p\frac{\delta q}{\delta z_x} - q\frac{\delta p}{\delta z_x}\right),\end{aligned} \tag{21}$$

and that consequently

$$\lambda_x\left(\frac{\delta p}{\delta y_x} + y'_x \frac{\delta q}{\delta y_x}\right) = (\lambda_x q)' + \lambda_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right); \tag{22}$$

changing therefore $(\lambda_x q)'\eta_x + \lambda_x q\eta'_x$ to the equivalent expression $(q\lambda_x\eta_x)'$, we find from the equation (18) the following

$$(\lambda_x\zeta_x)' = (q\lambda_x\eta_x)' + \lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right) + \epsilon\lambda_x E_x, \tag{23}$$

which gives, in the notation of definite integrals, by the conditions (16),

$$\lambda_{a+\Delta a}\zeta_{a+\Delta a} = \int_a^{a+\Delta a} \lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right)dx + \epsilon\int_a^{a+\Delta a}\lambda_x E_x dx, \tag{24}$$

that is, finally,

$$\zeta_{a+\Delta a} = \frac{1}{\lambda_{a+\Delta a}}\int_a^{a+\Delta a}\left\{\lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right) + \epsilon\lambda_x E_x\right\}dx. \tag{25}$$

From this expression it is easy to prove what was asserted, that when the condition (10) is *not* satisfied the final value $\zeta_{a+\Delta a}$ does *not* in general vanish independently of the form of the function η_x, even when the initial value ζ_a and the initial and final values η_a and $\eta_{a+\Delta a}$ are each $=0$; for we now see that $\zeta_{a+\Delta a}$ may in general be developed according to ascending powers of the small multiplier ϵ in a series of the form

$$\zeta_{a+\Delta a} = \zeta^{(0)}_{a+\Delta a} + \epsilon\,\zeta^{(1)}_{a+\Delta a} + \text{etc.}, \tag{26}$$

and that the first term of this series does not in general vanish independently of η_x since it may be expressed as follows

$$\zeta^{(0)}_{a+\Delta a} = e^{-\int_a^{a+\Delta a} L_x dx} \cdot \int_a^{a+\Delta a} e^{+\int_c^x L_x dx} M_x\eta_x dx, \tag{27}$$

if we put, for abridgement,

$$\left.\begin{aligned} \cdot &= -\left(\frac{\delta p}{\delta z_x} + y'_x\frac{\delta q}{\delta z_x}\right), \\ M_x &= \frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}.\end{aligned}\right\} \tag{28}$$

We see, then, that except in the case expressed by the condition (10) (in which this last coefficient M_x vanishes) the differential equation (11) is not sufficient to conduct, by necessary inference, to any one determinate relation between the three initial values a, b, c and the three final values $a+\Delta a$, $b+\Delta b$, $c+\Delta c$ of the three connected variables x, y_x, z_x; and therefore that we cannot in general deduce, as a necessary consequence of the differential equation (11), any equation in finite differences of the form (7): since otherwise it would be necessary that $\zeta_{a+\Delta a}$ should in general vanish along with η_a, ζ_a and $\eta_{a+\Delta a}$ independently of the form of η_x and of the value of ϵ.

5. To make this reasoning more perfectly clear by a still more particular example, let us assume the following particular forms for the coefficients p and q,

$$p=-y_x, \quad q=x; \tag{29}$$

so that the differential equation (11) becomes now

$$z'_x=xy'_x-y_x. \tag{30}$$

We have now

$$\frac{\delta p}{\delta x}=0, \quad \frac{\delta p}{\delta y_x}=-1, \quad \frac{\delta p}{\delta z_x}=0, \quad \frac{\delta q}{\delta x}=1, \quad \frac{\delta q}{\delta y_x}=0, \quad \frac{\delta q}{\delta z_x}=0; \tag{31}$$

so that the condition (10) is not satisfied. If we suppose that y_x and z_x are two functions of x, which satisfy the differential equation (30) and also the conditions (12) and (13); and that $y_x+\epsilon\eta_x$ and $z_x+\epsilon\zeta_x$ are two other functions of x, which satisfy the analogous differential equation

$$z'_x+\epsilon\zeta'_x=x(y'_x+\epsilon\eta'_x)-(y_x+\epsilon\eta_x) \tag{32}$$

and the analogous conditions (15) or (16); we obtain this new differential equation of the form (18)

$$\zeta'_x=x\eta'_x-\eta_x, \tag{33}$$

and the general expression (20) for the multiplier λ_x becomes here

$$\lambda_x=\lambda_a=\text{const.}, \tag{34}$$

so that we may suppress this multiplier or suppose it equal to 1; and we find this new equation of the form (23)

$$\zeta'_x=(x\eta_x)'-2\eta_x, \tag{35}$$

which gives, by integration and by the conditions (16), a result of the form (25), namely,

$$\zeta_{a+\Delta a}=\int_a^{a+\Delta a}(-2\eta_x)\,dx. \tag{36}$$

And it is clear that this final value of ζ_x does not vanish independently of the form of the function η_x, even when the conditions (16) are satisfied; and therefore that, in this example, the final values of the three connected variables x, y_x, z_x are not connected with each other by any necessary relation when only the initial values and the differential equation (30) are given.

6. Resuming now the more general differential equation (8) and reasoning on it in a similar manner with a view to establish the general theorem of the third article, we find, instead of the equation (18), the following:

$$0=\frac{\delta f}{\delta z'_x}\zeta'_x+\frac{\delta f}{\delta z_x}\zeta_x+\frac{\delta f}{\delta y'_x}\eta'_x+\frac{\delta f}{\delta y_x}\eta_x+\epsilon E_x; \tag{37}$$

which is now to be multiplied by a function λ_x, chosen so as to make the product

$$\lambda_x\left(\frac{\delta f}{\delta z'_x}\zeta'_x+\frac{\delta f}{\delta z_x}\zeta_x\right)$$

equal to the first derived function or differential coefficient of the product $\lambda_x\frac{\delta f}{\delta z'_x}\zeta_x$, that is, so as to satisfy the following condition

$$\lambda'_x\frac{\delta f}{\delta z'_x}=\lambda_x\left\{\frac{\delta f}{\delta z_x}-\left(\frac{\delta f}{\delta z'_x}\right)'\right\}, \tag{38}$$

analogous to the condition (19). In this manner we find, for the multiplier λ_x, the following expression, more general than the expression (20),

$$\lambda_x=\lambda_a e^{\int_a^x L_x dx} \tag{39}$$

in which we have made, for abridgement,

$$L_x=\frac{\dfrac{\delta f}{\delta z_x}-\left(\dfrac{\delta f}{\delta z'_x}\right)'}{\dfrac{\delta f}{\delta z'_x}}. \tag{40}$$

and since we have thus the relation

$$\lambda'_x=L_x\lambda_x, \tag{41}$$

we may put the product

$$\lambda_x\left(\frac{\delta f}{\delta y'_x}\eta'_x+\frac{\delta f}{\delta y_x}\eta_x\right)$$

under the form

$$\left(\lambda_x\frac{\delta f}{\delta y'_x}\eta_x\right)'+\lambda_x\eta_x\left\{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}\right\},$$

and consequently may present the differential equation (37) under the form

$$0=\left(\lambda_x\frac{\delta f}{\delta z'_x}\zeta_x\right)'+\left(\lambda_x\frac{\delta f}{\delta y'_x}\eta_x\right)'+\lambda_x\eta_x\left\{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}\right\}+\epsilon\lambda_x E_x, \tag{42}$$

which is analogous to the equation (23) and may be similarly integrated. We are thus conducted, when we suppose that the conditions (16) are satisfied, so that η_a, $\eta_{a+\Delta a}$ and ζ_a vanish, to a development of $\zeta_{a+\Delta a}$ of the form (26), in which the first and principal term $\zeta^{(0)}_{a+\Delta a}$ will not vanish independently of the form of the function η_x unless the functions y_x, z_x satisfy the following (additional) condition:

$$\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}=0; \tag{43}$$

that is, by (40),

$$\frac{\dfrac{\delta f}{\delta y_x}-\left(\dfrac{\delta f}{\delta y'_x}\right)'}{\dfrac{\delta f}{\delta y'_x}}=\frac{\dfrac{\delta f}{\delta z_x}-\left(\dfrac{\delta f}{\delta z'_x}\right)'}{\dfrac{\delta f}{\delta z'_x}} \tag{44}$$

And since this condition (44) is not in general satisfied, either identically by all possible forms of the functions y_x, z_x or even by all forms of those functions which satisfy the given differential equation (8), the theorem of the third article is true.

7. We see, however, that when the forms of the functions y_x, z_x are such as to satisfy this new condition (44), along with the differential equation

$$0=f(x, y_x, z_x, y'_x, z'_x) \tag{8}$$

and the initial and final conditions (12) and (13), we have then the final condition

$$\zeta^{(0)}_{a+\Delta a}=0, \tag{45}$$

as a consequence of the differential equation (37) and of the initial and final conditions (16), whatever the form of the function η_x may be for intermediate values of x; supposing always that no one of the functions here considered becomes infinite or indeterminate within the extent of the question. And because the condition (45), or the differential equation (44) which expresses it, conducts in general (in combination with the other conditions above mentioned) to a final value $z_{a+\Delta a}$ of the function z_x, which has its variation $\epsilon\zeta_{a+\Delta a}$ of the form $\epsilon^2\zeta^{(1)}_{a+\Delta a}+$ etc. and therefore ultimately proportional to the square of the small multiplier ϵ, (to the first power of which multiplier the variation $\epsilon\eta_x$ of the function y_x has been supposed proportional); whereas without the condition (45) the variation $\epsilon\zeta_{a+\Delta a}$ would be of the form $\epsilon\zeta^{(0)}_{a+\Delta a}+$ etc. and therefore ultimately proportional to the first power of the same small multiplier ϵ; we see that this condition (45) conducts in general to forms of the functional relations between the variables x, y_x, z_x which are remarkably distinguished from all other forms of those relations by a peculiar law of variation of the final value of z_x. On account of this important property and of others connected with it, we shall call the equation (44), which expresses the condition (45) and which is in general a differential equation of the second order, the *principal supplementary differential equation* for determining the forms of the functions y_x, z_x in combination with the given differential equation of the first order

$$0=f(x, y_x, z_x, y'_x, z'_x), \tag{8}$$

and with the initial and final conditions (12) and (13); and when these equations and conditions conduct to a determinate final value (or class of final values) $z_{a+\Delta a}$ of the function z_x, we shall call this value (or these values) $z_{a+\Delta a}$ the *principal final value* (or values) of the function z_x, corresponding to the given initial and final values a, y_a, z_a, $a+\Delta a$, $y_{a+\Delta a}$ and to the differential equation (8).

The values which we have thus called *principal final values* of a function z_x, which is connected with another function y_x by a differential equation such as (8), are called more commonly the *maxima* or *minima* of the function z_x, though they are not always greater or less than all neighbouring values of that function; and it was the desire of discovering these particular or principal values which led to the invention of the Calculus of Variations. But although the rules of that Calculus conduct to what we have called the principal supplementary differential equations, such as (44), they offer no general method for investigating or even expressing the integrals of those differential equations. A general method for accomplishing this important object is supplied, however, by that new Calculus of Principal Relations, to the consideration of which we shall soon proceed.

8. For the particular case (4) of this general differential equation (8), that is, for the particular form

$$f=X+Yy'_x+Zz'_x \tag{46}$$

of the general function f, (X, Y and Z denoting here any given functions of x, y_x and z_x,) we have the particular expressions

$$\left.\begin{aligned} \frac{\delta f}{\delta y_x} &= \frac{\delta X}{\delta y_x} + y'_x \frac{\delta Y}{\delta y_x} + z'_x \frac{\delta Z}{\delta y_x}, \quad \frac{\delta f}{\delta y'_x} \\ \frac{\delta f}{\delta z_x} &= \frac{\delta X}{\delta z_x} + y'_x \frac{\delta Y}{\delta z_x} + z'_x \frac{\delta Z}{\delta z_x}, \quad \frac{\delta f}{\delta z'_x} = Z, \end{aligned}\right\} \tag{47}$$

$$\begin{aligned} \left(\frac{\delta f}{\delta y'_x}\right)' &= \frac{\delta Y}{\delta x} + y'_x \frac{\delta Y}{\delta y_x} + z'_x \frac{\delta Y}{\delta z_x}, \\ \left(\frac{\delta f}{\delta z'_x}\right)' &= \frac{\delta Z}{\delta x} + y'_x \frac{\delta Z}{\delta y_x} + z'_x \frac{\delta Z}{\delta z_x}, \end{aligned} \tag{48}$$

and therefore

$$\begin{aligned} \frac{\delta f}{\delta y_x} - \left(\frac{\delta f}{\delta y'_x}\right)' &= \frac{\delta X}{\delta y_x} - \frac{\delta Y}{\delta x} + z'_x \left(\frac{\delta Z}{\delta y_x} - \frac{\delta Y}{\delta z_x}\right), \\ \frac{\delta f}{\delta z_x} - \left(\frac{\delta f}{\delta z'_x}\right)' &= \frac{\delta X}{\delta z_x} - \frac{\delta Z}{\delta x} + y'_x \left(\frac{\delta Y}{\delta z_x} - \frac{\delta Z}{\delta y_x}\right); \end{aligned} \tag{49}$$

so that when we attend to the original differential equation (8), which here reduces itself to the particular form (4), we find, after reductions, that in the present case the general supplementary equation (44) transforms itself to the particular condition

$$0 = X \left(\frac{\delta Y}{\delta z_x} - \frac{\delta Z}{\delta y_x}\right) + Y \left(\frac{\delta Z}{\delta x} - \frac{\delta X}{\delta z_x}\right) + Z \left(\frac{\delta X}{\delta y_x} - \frac{\delta Y}{\delta x}\right). \tag{5}$$

In like manner, if we assume the particular form

$$f = p + q y'_x - z'_x, \tag{50}$$

so as to reduce the general differential equation (8) to the particular differential equation (11), we find

$$\begin{aligned} \frac{\dfrac{\delta f}{\delta y_x} - \left(\dfrac{\delta f}{\delta y'_x}\right)'}{\dfrac{\delta f}{\delta y'_x}} &= \frac{1}{q} \left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} - p \frac{\delta q}{\delta z_x} - q y'_x \frac{\delta q}{\delta z_x}\right), \\ \frac{\dfrac{\delta f}{\delta z_x} - \left(\dfrac{\delta f}{\delta z'_x}\right)}{\dfrac{\delta f}{\delta z'_x}} &= -\left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right); \end{aligned} \tag{51}$$

so that in this particular case the general supplementary equation (44) reduces itself to the particular condition

$$\frac{\delta p}{\delta y_x} + q \frac{\delta p}{\delta z_x} = \frac{\delta q}{\delta x} + p \frac{\delta q}{\delta z_x}. \tag{10}$$

Thus for the two particular forms (46) and (50) of the function f (of which two forms, indeed, the one includes the other) the condition (44) either cannot be satisfied at all (as in the example of the fifth article where f was equal or proportional to $-y_x + x y'_x - z'_x$); is identically satisfied (as in the case considered in the second article); or else can only be satisfied by establishing a particular relation between the variable x and the functions y_x, z_x which does not involve the differential coefficients of those functions, and therefore does not permit us to assume as we have hitherto done the arbitrary initial conditions $y_a = b$, $z_a = c$. Peculiar considerations would also be required for the case when one or other of the differential coefficients y'_x, z'_x disappears from the expression of f. But, in general, the equation (44) may evidently be put under the form

$$0 = P + Qy''_x + Rz''_x, \tag{52}$$

in which P, Q and R are known functions of x, y_x, z_x, y'_x and z'_x, namely,

$$\left.\begin{aligned} P &= \frac{\delta f}{\delta z'_x}\left(\frac{\delta f}{\delta y_x} - \frac{\delta^2 f}{\delta x\, \delta y'_x} - y'_x \frac{\delta^2 f}{\delta y_x \delta y'_x} - z'_x \frac{\delta^2 f}{\delta z_x \delta y'_x}\right) \\ &\quad \frac{\delta f}{\delta y'_x}\left(\frac{\delta f}{\delta z_x} \quad \frac{\delta^2 f}{\delta x\, \delta z'_x} - y'_x \frac{\delta^2 f}{\delta y_x \delta z'_x} - z'_x \frac{\delta^2 f}{\delta z_x \delta z'_x}\right. \\ Q &= -\frac{\delta f}{\delta z'_x}\frac{\delta^2 f}{\delta y'^2_x} + \frac{\delta f}{\delta y'_x}\frac{\delta^2 f}{\delta y'_x \delta z'_x}, \\ R &= -\frac{\delta f}{\delta z'_x}\frac{\delta^2 f}{\delta y'_x \delta z'_x} + \frac{\delta f}{\delta y'_x}\frac{\delta^2 f}{\delta z'^2_x}, \end{aligned}\right\} \tag{53}$$

or functions proportional to these; and since the functions Q and R do not in general both vanish, this supplementary equation (44) or (52) is evidently in general a differential equation of the second order, as was remarked before.

9. Another differential equation of the same order, which is in general distinct from this *principal supplementary differential equation* (44) or (52), may be obtained by differentiating the *original differential equation* of the first order (8) and may be thus denoted,

$$0 = f' = P_{,} + Q_{,} y''_x + R_{,} z''_x; \tag{54}$$

in which the coefficients $P_{,}$, $Q_{,}$, $R_{,}$ are the following functions of x, y_x, z_x, y'_x, z'_x,

$$P_{,} = \frac{\delta f}{\delta x} + y'_x \frac{\delta f}{\delta y_x} + z'_x \frac{\delta f}{\delta z_x}; \quad Q_{,} = \frac{\delta f}{\delta y'_x} \quad R_{,} = \frac{\delta f}{\delta z'_x}, \tag{55}$$

or functions proportional to these. If we had only the system of these two differential equations of the second order, (52) and (54), we could only in general deduce from them, by an easy or difficult but theoretically possible integration, two relations which might be thus denoted,

$$\left.\begin{aligned} 0 &= \chi\,(a, y_a, z_a, y'_a, z'_a, x, y_x, z_x), \\ 0 &= \chi_{,}\,(a, y_a, z_a, y'_a, z'_a, x, y_x, z_x), \end{aligned}\right\} \tag{56}$$

between the three connected variables x, y_x, z_x, their three initial values a, y_a, z_a, and the two initial differential coefficients y'_a, z'_a; or relations equivalent to these. But since the original differential equation of the first order (8) gives this other initial condition

$$0 = f_a = f\,(a, y_a, z_a, y'_a, z'_a), \tag{57}$$

we can in general conceive the two initial differential coefficients y'_a and z'_a to be eliminated between these three last equations and thus a relation obtained of the form

$$0 = F\,(a, y_a, z_a, x, y_x, z_x), \tag{58}$$

or

$$0 = F\,(a, y_a, z_a, a + \Delta a, y_{a+\Delta a}, z_{a+\Delta a}), \tag{59}$$

or, if we choose, of the form (7), or finally of this simpler form

$$0 = F\,(a, b, c, a + \Delta a, b + \Delta b, c + \Delta c) \tag{60}$$

between the three initial and three final values of the variables x, y_x, z_x, these variables being obliged to satisfy both the *original differential equation* (8) and also the *principal supplementary differential equation*, (44) or (52). And because the relation (58) or (59) or (60) thus obtained gives in general the *principal final value* $z_{a+\Delta a}$ or $c+\Delta c$ of the function z_x when the five other initial and final values a, y_a, z_a, $a+\Delta a$, $y_{a+\Delta a}$, or a, b, c, $a+\Delta a$, $b+\Delta b$, are given, we shall call this relation the *principal integral* of the original differential equation (8), or the *principal integral relation* between the final and initial values of the three connected variables x, y_x, z_x resulting from that differential equation.

10. To give an example of the application of this method of investigating a principal integral relation, let us take the following particular form of the function f,

$$f = gy_x + \tfrac{1}{2}y_x'^2 - z_x', \tag{61}$$

g being any given constant; so that the proposed differential equation, of which we are to seek the principal integral, may now be put under the form

$$z_x' = gy_x + \tfrac{1}{2}y_x'^2. \tag{62}$$

We have now the following particular expressions for the partial differential coefficients of the function f,

$$\frac{\delta f}{\delta x} = 0, \quad \frac{\delta f}{\delta y_x} = g, \quad \frac{\delta f}{\delta z_x} = 0, \quad \frac{\delta f}{\delta y_x'} = y_x', \quad \frac{\delta f}{\delta z_x'} = -1; \tag{63}$$

so that, by (55), the general coefficients $P_,$, $Q_,$, $R_,$ of the differential equation (54) become here

$$P_, = gy_x', \quad Q_, = y_x', \quad R_, = -1: \tag{64}$$

we have also for the partial differential coefficients of the second order of the same function f the expressions

$$\left.\begin{array}{lllll}
\dfrac{\delta^2 f}{\delta x^2} = 0, & \dfrac{\delta^2 f}{\delta x\,\delta y_x} = 0, & \dfrac{\delta^2 f}{\delta x\,\delta z_x} = 0, & \dfrac{\delta^2 f}{\delta x\,\delta y_x'} = 0, & \dfrac{\delta^2 f}{\delta x\,\delta z_x'} = 0, \\
\dfrac{\delta^2 f}{\delta y_x^2} = 0, & \dfrac{\delta^2 f}{\delta y_x\delta z_x} = 0, & \dfrac{\delta^2 f}{\delta y_x\delta y_x'} = 0, & \dfrac{\delta^2 f}{\delta y_x\delta z_x'} = 0, & \\
\dfrac{\delta^2 f}{\delta z_x^2} = 0, & \dfrac{\delta^2 f}{\delta z_x\delta y_x'} = 0, & \dfrac{\delta^2 f}{\delta z_x\delta z_x'} = 0, & & \\
\dfrac{\delta^2 f}{\delta y_x'^2} = 1, & \dfrac{\delta^2 f}{\delta y_x'\delta z_x'} = 0, & & & \\
\dfrac{\delta^2 f}{\delta z_x'^2} = 0; & & & &
\end{array}\right\} \tag{65}$$

and therefore, by (53), the general coefficients P, Q, R of the differential equation (52) become, in the present example,

$$P = -g, \quad Q = 1, \quad R = 0. \tag{66}$$

The general differential equations of the second order

$$0 = P + Qy_x'' + Rz_x'' \tag{52}$$

and

$$0 = P_, + Q_,y_x'' + R_,z_x'' \tag{54}$$

become therefore now

$$0 = -g + y_x'' \tag{67}$$

and

$$z_x'' = gy_x' + y_x'y_x''; \tag{68}$$

the former of these two, namely the equation (67), being the principal supplementary differential equation and the latter, namely (68), being obtained by differentiation from the original differential equation (62). If we had only the system of these two differential equations of the second order we could only deduce from them by integration the system of the two following equations of the forms (56),

$$0=y_x-y_a-y'_a(x-a)-\tfrac{1}{2}g(x-a)^2 \tag{69}$$

and

$$0=z_x-z_a-z'_a(x-a)-gy'_a(x-a)^2-\tfrac{1}{3}g^2(x-a)^3; \tag{70}$$

or relations equivalent to these. But the original differential equation (62) gives also this initial condition of the form (57),

$$0=f_a=gy_a+\tfrac{1}{2}y'^2_a-z'_a; \tag{71}$$

and, by eliminating y'_a and z'_a between the three last equations (69), (70) and (71), we find this other equation of the form (58),

$$0=F=z_x-z_a-\tfrac{1}{2}g(y_x+y_a)(x-a)-\frac{(y_x-y_a)^2}{2(x-a)}+\tfrac{1}{24}g^2(x-a)^3, \tag{72}$$

which is the *principal integral* sought of the proposed differential equation (62). This integral may also be thus written, under the form (7), as an equation in finite differences,

$$0=\Delta z_a-g(y_a+\tfrac{1}{2}\Delta y_a)\Delta a-\frac{\Delta y_a^2}{2\Delta a}+\tfrac{1}{24}g^2\Delta a^3; \tag{73}$$

or more concisely thus, under the form (60),

$$0=\Delta c-g(b+\tfrac{1}{2}\Delta b)\Delta a-\frac{\Delta b^2}{2\Delta a}+\tfrac{1}{24}g^2\Delta a^3. \tag{74}$$

11. In a very extensive class of cases the proposed differential equation, though it contains the differential coefficient z'_x, does not contain the undifferentiated function z_x; and then, as in the last example, it may be put under the form

$$z'_x=\phi(x,\, y_x,\, y'_x), \tag{75}$$

so that the function f may be regarded as being of the form

$$f=\phi(x,\, y_x,\, y'_x)-z'_x. \tag{76}$$

In all cases of this class the process of the 8th article for determining the principal integral admits of being simplified; for the principal supplementary differential equation (44) or (52) becomes simply

$$\frac{\delta\phi}{\delta y_x}=\left(\frac{\delta\phi}{\delta y'_x}\right)', \tag{77}$$

or

$$0=-\frac{\delta\phi}{\delta y_x}+\frac{\delta^2\phi}{\delta x\,\delta y'_x}+\frac{\delta^2\phi}{\delta y_x\,\delta y'_x}y'_x+\frac{\delta^2\phi}{\delta y'^2_x}y''_x, \tag{78}$$

and does not involve z_x, z'_x nor z''_x, so that its integral, like the equation (69), is of the following form, more simple than the form of either of the two general equations (56),

$$0=\chi(a,\, y_a,\, y'_a,\, x,\, y_x). \tag{79}$$

And because this integral (79) of the principal supplementary differential equation (77) or (78) enables us to express the function y_x, and therefore also y'_x, in terms of the constants a, y_a, y'_a, and of the variable x, it enables us to express the difference $z_x - z_a$ in terms of the same constants and variable, by means of the original differential equation (75), as the value of a definite integral, namely,

$$z_x - z_a = \int_a^x \phi\,(x,\ y_x,\ y'_x)\,dx: \tag{80}$$

so that instead of eliminating the *two* initial differential coefficients y'_a and z'_a between the three equations assigned for that purpose in the 8th article, it is sufficient to eliminate the *one* initial differential coefficient y'_a between the two equations (79) and (80) in order to obtain the principal integral relation connecting the initial and final values of x, y_x and z_x; which relation will evidently be of the following form,

$$0 = F = z_x - z_a - \Phi\,(a,\ y_a,\ x,\ y_x) = \Delta c - \Phi\,(a,\ b,\ a + \Delta a,\ b + \Delta b). \tag{81}$$

Thus, in the last example, in which the function $\phi\,(x,\ y_x,\ y'_x) = gy_x + \frac{1}{2}y'^2_x$, so that

$$\frac{\delta\phi}{\delta y_x} = g, \quad \frac{\delta\phi}{\delta y'_x} = y'_x, \quad \frac{\delta^2\phi}{\delta x\,\delta y'_x} = 0, \quad \frac{\delta^2\phi}{\delta y_x\,\delta y'_x} = 0, \quad \frac{\delta^2\phi}{\delta y'^2_x} = 1, \tag{82}$$

the form (78) for the principal supplementary differential relation becomes

$$0 = -g + y''_x, \tag{67}$$

as was otherwise found before; and its integral is of the form (79), namely,

$$0 = y_x - y_a - y'_a\,(x-a) - \tfrac{1}{2}g\,(x-a)^2, \tag{69}$$

which gives for the functions y_x, y'_x the expressions

$$\left.\begin{aligned} y_x &= y_a + y'_a\,(x-a) + \tfrac{1}{2}g\,(x-a)^2, \\ y'_x &= y'_a + g\,(x-a). \end{aligned}\right\} \tag{83}$$

Substituting these expressions in that of the function ϕ, we find

$$\phi\,(x,\ y_x,\ y'_x) = gy_x + \tfrac{1}{2}y'^2_x = gy_a + \tfrac{1}{2}y'^2_a + 2gy'_a\,(x-a) + g^2\,(x-a)^2, \tag{84}$$

and therefore

$$\begin{aligned} {}_x - z_a &= \int_a^x \{gy_a + \tfrac{1}{2}y'^2_a + 2gy'_a\,(x-a) + g^2\,(x-a)^2\}\,dx \\ &= (gy_a + \tfrac{1}{2}y'^2_a)\,(x-a) + gy'_a\,(x-a)^2 + \tfrac{1}{3}g^2\,(x-a)^3; \end{aligned} \tag{85}$$

so that finally, by eliminating y'_a between the two equations (69) and (85), we obtain the same principal integral relation

$$0 = z_x - z_a - \tfrac{1}{2}g\,(y_x + y_a)\,(x-a) - \frac{(y_x - y_a)^2}{2\,(x-a)} + \tfrac{1}{24}g^2\,(x-a)^3, \tag{72}$$

which was found before by a less simple process.

12. The differential equation

$$z'_x = \phi\,(x,\ y_x,\ y'_x) \tag{75}$$

conducts in general to the expression

$$z_x - z_a = \int_a^x \phi\,(x,\ y_x,\ y'_x)\,dx, \tag{80}$$

whether we do or do not employ the supplementary differential equation (77) or (78); but among all the values of the definite integral (80), corresponding to all possible forms of the function y_x and to the initial and final conditions (12), the particular value (or values) determined by this principal supplementary differential equation may be called the *principal value* (or values) of that integral for reasons already explained. We shall therefore call the value (or values) of the definite integral (80), determined in this manner, the *principal definite integral* (or integrals) of the differential expression $\phi(x, y_x, y'_x)\,dx$, taken between the limits a and x, or a and $a+\Delta a$; and we shall distinguish in writing a *principal definite integral* of this kind by drawing a stroke under the sign of integration; so that the principal integral relation (81), deduced from a proposed differential equation of the form (75), may be denoted as follows:

$$z_x - z_a = \underline{\int}_a^x \phi(x, y_x, y'_x)\,dx = \underline{\int}_a^{a+\Delta a} \phi(x, y_x, y'_x)\,dx. \tag{86}$$

For example, by what has been already shown, the *principal definite integral* of the differential expression $(gy_x + \frac{1}{2}y'^2_x)\,dx$, taken between the limits a and x, is

$$\int_a^x (gy_x + \tfrac{1}{2}y'^2_x)\,dx = \tfrac{1}{2}g(y_x + y_a)(x-a) + \frac{(y_x - y_a)^2}{2(x-a)} - \frac{g^2}{24}(x-a)^3. \tag{87}$$

13. Besides the principal supplementary differential equation, the Calculus of Variations conducts also to another important relation called usually the Equation of Limits, which may be explained and investigated as follows.

Returning to the general differential equation of the first order

$$0 = f(x, y_x, z_x, y'_x, z'_x), \tag{8}$$

let us now imagine that after finding two functions y_x, z_x which satisfy this original equation (8) and also the principal supplementary differential equation (44) or (52) and the initial and final conditions (12) and (13), and after calculating thus the principal final value $z_{a+\Delta a}$ of z_x which corresponds to a given final value $y_{a+\Delta a}$ of y_x and to given initial values y_a, z_a of the same functions y_x, z_x, we then change these two functions as before to others of the form $y_x + \epsilon\eta_x$ and $z_x + \epsilon\zeta_x$; but that at the same time we change also the variable x itself to a function of the form $x + \epsilon\xi_x$, and therefore the differential coefficients y'_x, z'_x to the quotients

$$\frac{y'_x + \epsilon\eta'_x}{1 + \epsilon\xi'_x}, \quad \frac{z'_x + \epsilon\zeta'_x}{1 + \epsilon\xi'_x};$$

and that, by making all these changes in the original differential equation (8), we form a new but analogous differential equation and then oblige the functions ξ_x, η_x, ζ_x to satisfy this new equation, namely,

$$0 = f\left(x + \epsilon\xi_x, y_x + \epsilon\eta_x, z_x + \epsilon\zeta_x, \frac{y'_x + \epsilon\eta'_x}{1 + \epsilon\xi'_x}, \frac{z'_x + \epsilon\zeta'_x}{1 + \epsilon\xi'_x}\right). \tag{88}$$

We shall then have, instead of (37), the following equation deduced from the combination of (8) and (88):

$$0 = \frac{\delta f}{\delta x}\xi_x + \frac{\delta f}{\delta y_x}\eta_x + \frac{\delta f}{\delta z_x}\zeta_x + \frac{\delta f}{\delta y'_x}(\eta'_x - y'_x\xi'_x) + \frac{\delta f}{\delta z'_x}(\zeta'_x - z'_x\xi'_x) + \epsilon E_x, \tag{89}$$

E_x being a factor which we need not at present determine; and since by the differential equation (54) we have

$$\frac{\delta f}{\delta x} = -y'_x\frac{\delta f}{\delta y_x} - z'_x\frac{\delta f}{\delta z_x} - y''_x\frac{\delta f}{\delta y'_x} - z''_x\frac{\delta f}{\delta z'_x}, \tag{90}$$

it will follow that

$$0=\frac{\delta f}{\delta z'_x}(\zeta'_x-z'_x\xi'_x-z''_x\xi_x)+\frac{\delta f}{\delta y'_x}(\eta'_x-y'_x\xi'_x-y''_x\xi_x)$$
$$+\frac{\delta f}{\delta z_x}(\zeta_x-z'_x\xi_x)+\frac{\delta f}{\delta y_x}(\eta_x-y'_x\xi_x)+\epsilon E_x. \qquad (91)$$

Multiplying this differential equation (91) by the factor λ_x of the 6th article, which satisfies the two conditions

$$\lambda_x\frac{\delta f}{\delta z_x}=\left(\lambda_x\frac{\delta f}{\delta z'_x}\right)', \quad \lambda_x\frac{\delta f}{\delta y_x}=\left(\lambda_x\frac{\delta f}{\delta y'_x}\right)', \qquad (92)$$

we find this transformed differential equation

$$0=\left(\lambda_x\left\{\frac{\delta f}{\delta z'_x}(\zeta_x-z'_x\xi_x)+\frac{\delta f}{\delta y'_x}(\eta_x-y'_x\xi_x)\right\}\right)+\epsilon\lambda_x E_x, \qquad (93)$$

which gives, by integration,

$$0=\Delta\left(\lambda_a\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}\right)+\epsilon\int_a^{a+\Delta a}\lambda_x E_x\,dx, \qquad (94)$$

f_a denoting as in (57) the initial function $f(a,y_a,z_a,y'_a,z'_a)$ and Δ still implying that we are to take the finite difference corresponding to the transition from the initial to the final values. If then, without obliging ξ_a, η_a, ζ_a, $\xi_{a+\Delta a}$ and $\eta_{a+\Delta a}$ to vanish, we suppose (as in general we may) that $\zeta_{a+\Delta a}$ is developed in a series of the form (26) according to positive and integer powers of the small constant multiplier ϵ, we find for the first and principal term $\zeta^{(0)}_{a+\Delta a}$ of this series the equation

$$\frac{\delta f_{a+\Delta a}}{\delta z'_{a+\Delta a}}(\zeta^{(0)}_{a+\Delta}-z'_{a+\Delta a}\xi_{a+\Delta a})+\frac{\delta f_{a+\Delta a}}{\delta y'_{a+\Delta a}}(\eta_{a+\Delta a}-y'_{a+\Delta a}\xi_{a+\Delta a})$$
$$=\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}e^{-\int_a^{a+\Delta a}L_x dx} \qquad (95)$$

in which L_x has the same meaning as in the 6th article, namely,

$$=\frac{\lambda'_x}{\lambda_x}=\frac{\frac{\delta f}{\delta z_x}-\left(\frac{\delta f}{\delta z'_x}\right)'}{\frac{\delta f}{\delta z'_x}}=\frac{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'}{\frac{\delta f}{\delta y'_x}}. \qquad (96)$$

This equation (95) may be simplified when the function f has the form

$$f=\phi(x,y_x,y'_x)-z'_x; \qquad (76)$$

for it then becomes

$$-\zeta^{(0)}_{a+\Delta a}+\phi_{a+\Delta a}\xi_{a+\Delta a}+\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}(\eta_{a+\Delta a}-y'_{a+\Delta a}\xi_{a+\Delta a})$$
$$=-\zeta_a+\phi_a\xi_a+\frac{\delta\phi_a}{\delta y'_a}(\eta_a-y'_a\xi_a), \qquad (97)$$

that is,

$$\zeta^{(0)}_{a+\Delta a}-\zeta_a=\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}\eta_{a+\Delta a}-\frac{\delta\phi_a}{\delta y'_a}\eta_a$$
$$+\left(\phi_{a+\Delta a}-y'_{a+\Delta a}\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}\right)\xi_{a+\Delta a}-\left(\phi_a-y'_a\frac{\delta\phi_a}{\delta y'_a}\right)\xi_a. \qquad (98)$$

For example, when the function ϕ has the form (62), so that

$$\phi=gy_x+\tfrac{1}{2}y'^2_x,\quad \phi_a=gy_a+\tfrac{1}{2}y'^2_a,\quad \phi_{a+\Delta a}=gy_{a+\Delta a}+\tfrac{1}{2}y'^2_{a+\Delta a}, \qquad (99)$$

then the formula (98) becomes

$$\zeta^{(0)}_{a+\Delta a}-\zeta_a=y_{a+\Delta a}\eta_{a+\Delta a}-y'_a\eta_a+(gy_{a+\Delta a}-\tfrac{1}{2}y'^2_{a+\Delta a})\,\xi_{a+\Delta a}-(gy_a-\tfrac{1}{2}y'^2_a)\,\xi_a. \qquad (100)$$

In general, if we write x instead of $a+\Delta a$, the formula (95) becomes

$$\frac{\delta f_x}{\delta z'_x}(\zeta^{(0)}_x-z'_x\xi_x)+\frac{\delta f_x}{\delta y'_x}(\eta_x-y'_x\xi_x)=\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}e^{-\int_a^x L_x dx}: \qquad (101)$$

and the formula (98) becomes

$$\zeta^{(0)}_x-\zeta_a=\frac{\delta\phi_x}{\delta y'_x}\eta_x-\frac{\delta\phi_a}{\delta y'_a}\eta_a+\left(\phi_x-y'_x\frac{\delta\phi_x}{\delta y'_x}\right)\xi_x-\left(\phi_a-y'_a\frac{\delta\phi_a}{\delta y'_a}\right)\xi_a. \qquad (102)$$

Thus, although the differential equation (88) is not sufficient of itself to determine the forms of the three functions ξ_x, η_x, ζ_x, nor even to determine rigorously, in general, the final value ζ_x of one of these three functions when only the final values ξ_x, η_x of the other two functions and the initial values ξ_a, η_a, ζ_a of all three are given; (because the term $\epsilon\lambda_x E_x$ in the transformed differential equation (93) is not in general immediately integrable;) yet, when the functions y_x and z_x satisfy the principal supplementary equation (44) as well as the original equation (8), we see that then the differential equation (88) is in general sufficient to determine the most important part or term $\zeta^{(0)}_x$ of the final value of the function ζ_x, or the limit to which that final value tends while the small multiplier ϵ tends to 0, whatever arbitrary forms may be assumed for the three functions ξ_x, η_x, ζ_x consistently with that one differential equation (88) and with the three given initial and the two given final values, ξ_a, η_a, ζ_a, and ξ_x, η_x.

14. Now, consistently with these five given values, initial and final, and with the differential equation of the first order (88), we may in general oblige the three functions ξ_x, η_x, ζ_x to satisfy other conditions; for example, we may in general oblige them to satisfy any two assumed supplementary differential equations of the second order; because a system of three differential equations between four variables x, ξ_x, η_x, ζ_x conducts in general to a system of three integral equations between those four variables involving five arbitrary constants, when one of the three differential equations is of the first order and the two other differential equations are both of the second order. Among all the supplementary differential equations which might be thus assumed, there is one which deserves special attention; namely that which is formed from the principal supplementary differential equation of the second order (44) by not only changing x, y_x, z_x and y'_x, z'_x to $x+\epsilon\xi_x$, $y_x+\epsilon\eta_x$, $z_x+\epsilon\zeta_x$ and $\frac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}$, $\frac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}$, but also changing, in like manner, y''_x and z''_x to

$$\frac{\left(\dfrac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x} \quad\text{and}\quad \frac{\left(\dfrac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x};$$

and which may therefore be thus written,

$$0=\psi\left(x+\epsilon\xi_x, y_x+\epsilon\eta_x, z_x+\epsilon\zeta_x, \frac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}, \frac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}, \frac{\left(\dfrac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x}, \frac{\left(\dfrac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x}\right), \tag{103}$$

if the equation (44), from which it is formed, be written thus,

$$0=\psi(x, y_x, z_x, y'_x, z'_x, y''_x, z''_x). \tag{104}$$

For, although the two equations (88) and (103) are not alone sufficient to determine completely the forms of the three functions ξ_x, η_x, ζ_x even in conjunction with the five given initial and final values ξ_a, η_a, ζ_a, ξ_x and η_x, but require another supplementary differential equation of the second order, which might, for example, be assumed as follows,

$$\xi''_x=0, \tag{105}$$

for such complete determination; yet it is easy to see that, as the two equations (8) and (44) conducted to a principal integral relation of the form

$$0=F(a, y_a, z_a, x, y_x, z_x), \tag{58}$$

so the two similar equations (88) and (103) must conduct to a similar integral relation

$$0=F(a+\epsilon\xi_a, y_a+\epsilon\eta_a, z_a+\epsilon\zeta_a, x+\epsilon\xi_x, y_x+\epsilon\eta_x, z_x+\epsilon\zeta_x), \tag{106}$$

in which the form of the function F is the same. Combining these two relations (58) and (106), we find a new equation to determine the limit $\zeta_x^{(0)}$ to which the final value of the function ζ_x tends, while the small multiplier ϵ tends to 0 and while the final values of the two other functions ξ_x, η_x and the three initial values ξ_a, η_a, ζ_a of the same three functions ξ_x, η_x, ζ_x remain unchanged but arbitrary; namely, the equation

$$0=\frac{\delta F}{\delta a}\xi_a+\frac{\delta F}{\delta y_a}\eta_a+\frac{\delta F}{\delta z_a}\zeta_a+\frac{\delta F}{\delta x}\xi_x+\frac{\delta F}{\delta y_x}\eta_x+\frac{\delta F}{\delta z_x}\zeta_x^{(0)}. \tag{107}$$

The value of the limit $\zeta_x^{(0)}$ thus obtained from a combination of the two differential equations (88) and (103) must agree with the value of the same limit obtained in the 13th article from the equation (88) alone; and this agreement must exist independently of the five arbitrary values ξ_a, η_a, ζ_a, ξ_x, η_x; comparing, therefore, the coefficients which multiply these five arbitrary values in the two expressions of $\zeta_x^{(0)}$, deduced from the two equations (107) and (101), we find these three relations:

$$\left.\begin{aligned}
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta a}&=\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\left\{y'_a\frac{\delta f_a}{\delta y'_a}+z'_a\frac{\delta f_a}{\delta z'_a}\right\}e^{-\int_a^x L_x dx}\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta y_a}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_a}{\delta y'_a}e^{-\int_a^x L_x dx}\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta z_a}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_a}{\delta z'_a}e^{-\int_a^x L_x dx},
\end{aligned}\right. \tag{108}$$

and these two others:

$$\left.\begin{aligned}
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta x}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\left\{y'_x\frac{\delta f_x}{\delta y'_x}+z'_x\frac{\delta f_x}{\delta z'_x}\right\},\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta y_x}&=\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_x}{\delta y'_x}.
\end{aligned}\right\} \tag{109}$$

In that extensive class of cases, in which the function f has the form (76) and in which therefore the function F may be put under the form (81), the five relations (108) and (109) may be reduced to the four following:

$$\frac{\delta\Phi}{\delta a}=y'_a\frac{\delta\phi_a}{\delta y'_a}-\phi_a, \quad \frac{\delta\Phi}{\delta y_a}=-\frac{\delta\phi_a}{\delta y'_a}, \tag{110}$$

and

$$\frac{\delta\Phi}{\delta x}=-y'_x\frac{\delta\phi_x}{\delta y'_x}+\phi_x, \quad \frac{\delta\Phi}{\delta y_x}=\frac{\delta\phi_x}{\delta y'_x}. \tag{111}$$

In general, the three relations (108) conduct, by elimination, to the two following:

$$\left.\begin{aligned}\left(\frac{\delta F}{\delta z_a}\right)^{-1}\frac{\delta F}{\delta a}&=-\left(\frac{\delta f_a}{\delta z'_a}\right)^{-1}\left\{y'_a\frac{\delta f_a}{\delta y'_a}+z'_a\frac{\delta f_a}{\delta z'_a}\right\},\\ \left(\frac{\delta F}{\delta z_a}\right)^{-1}\frac{\delta F}{\delta y_a}&=\left(\frac{\delta f_a}{\delta z'_a}\right)^{-1}\frac{\delta f_a}{\delta y'_a}.\end{aligned}\right\} \tag{112}$$

15. To illustrate these general relations, let us resume the example of the 10th article, in which

$$0=f_x=gy_x+\tfrac{1}{2}y'^2_x-z'_x, \tag{61}$$

$$0=f_a=gy_a+\tfrac{1}{2}y'^2_a-z'_a, \tag{71}$$

and

$$0=F=z_x-z_a-\tfrac{1}{2}g\,(y_x+y_a)\,(x-a)-\frac{(y_x-y_a)^2}{2\,(x-a)}+\tfrac{1}{24}g^2\,(x-a)^3. \tag{72}$$

In this example

$$\left.\begin{aligned}&\frac{\delta f_x}{\delta y'_x}=y'_x, \quad \frac{\delta f_x}{\delta z'_x}=-1, \quad \frac{\delta f_a}{\delta y'_a}=y'_a, \quad \frac{\delta f_a}{\delta z'_a}=-1,\\ &\frac{\delta F}{\delta x}=-\tfrac{1}{2}g\,(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2\,(x-a)^2,\\ &\frac{\delta F}{\delta y_x}=-\tfrac{1}{2}g\,(x-a)-\frac{y_x-y_a}{x-a}, \quad \frac{\delta F}{\delta z_x}=1,\\ &\frac{\delta F}{\delta a}=\tfrac{1}{2}g\,(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2\,(x-a)^2,\\ &\frac{\delta F}{\delta y_a}=-\tfrac{1}{2}g\,(x-a)+\frac{y_x-y_a}{x-a}, \quad \frac{\delta F}{\delta z_a}=-1;\end{aligned}\right. \tag{113}$$

the general relations (112) become, therefore,

$$\left.\begin{aligned}&-\tfrac{1}{2}g\,(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2\,(x-a)^2=y'^2_a-z'_a,\\ &\tfrac{1}{2}g\,(x-a)-\frac{y_x-y_a}{x-a}=-y'_a;\end{aligned}\right\} \tag{114}$$

to which forms the two first of the three relations (108) also reduce themselves, while the third of those relations (108) becomes identical, because, in this example, the function L_x vanishes by (40); and the two general relations (109) become

$$\begin{aligned} -\tfrac{1}{2}g\,(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2\,(x-a)^2&=y_x'^2-z_x'\,; \\ -\tfrac{1}{2}g\,(x-a)-\frac{y_x-y_a}{x-a}&=-y_x'. \end{aligned} \tag{115}$$

And accordingly we may verify the existence of these four relations (114) and (115) in the present example by substituting for y_x, y_x' their expressions (83), deduced by integration from the principal supplementary differential equation of the second order (67), and at the same time substituting for z_x', z_a' their values deduced from the original differential equation of the first order (62) and its initial form (71).

The same example enables us to illustrate the four simpler but less general relations (110) and (111); namely, by making, according to the 11th article,

$$\begin{aligned} &\phi_x=gy_x+\tfrac{1}{2}y_x'^2, \quad \phi_a=gy_a+\tfrac{1}{2}y_a'^2, \\ &\Phi=\tfrac{1}{2}g\,(y_x+y_a)\,(x-a)+\frac{(y_x-y_a)^2}{2\,(x-a)}-\tfrac{1}{24}g^2\,(x-a)^3; \end{aligned} \tag{116}$$

and therefore

$$\begin{aligned} &\frac{\delta\phi_x}{\delta y_x'}=y_x', \quad \frac{\delta\phi_a}{\delta y_a'}=y_a' \\ &\frac{\delta\Phi}{\delta x}=\tfrac{1}{2}g\,(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2\,(x-a)^2, \\ &\frac{\delta\Phi}{\delta y_x}=\tfrac{1}{2}g\,(x-a)+\frac{y_x-y_a}{x-a}, \\ &\frac{\delta\Phi}{\delta a}=-\tfrac{1}{2}g\,(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2\,(x-a)^2, \\ &\frac{\delta\Phi}{\delta y_a}=\tfrac{1}{2}g\,(x-a)-\frac{y_x-y_a}{x-a}; \end{aligned} \tag{117}$$

for the two relations (110) thus become

$$\left.\begin{aligned} -\tfrac{1}{2}g\,(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2\,(x-a)^2&=\tfrac{1}{2}y_a'^2-gy_a, \\ \tfrac{1}{2}g\,(x-a)-\frac{y_x-y_a}{x-a}&=-y_a'; \end{aligned}\right\} \tag{118}$$

and the two relations (111) become

$$\begin{aligned} \tfrac{1}{2}g\,(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2\,(x-a)^2&=-\tfrac{1}{2}y_x'^2+gy_x, \\ \tfrac{1}{2}g\,(x-a)+\frac{y_x-y_a}{x-a}&=y_x'; \end{aligned} \tag{119}$$

and these relations (118) and (119), which agree with those marked (114) and (115), are satisfied, like them, by the expressions (83) for y_x and y_x'.

Reciprocally, it is important to observe that those former expressions are included in these recent relations; in such a manner that they might have been obtained from the equations (110) and (111), if the form of the function Φ had been known as well as the forms of ϕ_x and ϕ_a or from the more general equations (112) and (109), if we had known the form of the function F as well as the forms of f_x and f_a. For, having thus obtained the relations

$$\left.\begin{aligned} y'_a &= \frac{y_x - y_a}{x-a} - \tfrac{1}{2} g\,(x-a), \\ y'_x &= \frac{y_x - y_a}{x-a} + \tfrac{1}{2} g\,(x-a), \end{aligned}\right\} \tag{120}$$

we might thence have easily deduced the expressions

$$\left.\begin{aligned} y_x &= y_a + y'_a(x-a) + \tfrac{1}{2} g\,(x-a)^2, \\ y'_x &= y'_a + g\,(x-a), \end{aligned}\right\} \tag{83}$$

which we have otherwise deduced before, by integration, from the principal supplementary differential equation

$$0 = -g + y''_x. \tag{67}$$

CHAPTER I.

General Theory of the Principal Integral of any Total Differential Equation of the First Order, but not of the First Degree, between three or more Variables.

1. Let $x_1, x_2, \dots x_n$ denote any n sought functions of any one variable x, the number n being supposed to be greater than unity; let $x'_1, x'_2, \dots x'_n$ denote, according to the well-known notation of Lagrange, the n derived functions corresponding, of the first order, or the first differential coefficients

$$\frac{dx_1}{dx}, \quad \frac{dx_2}{dx}, \quad \dots \quad \frac{dx_n}{dx}$$

of the n sought functions respectively; let

$$f(x, x_1, x_2, \dots x_n, x'_1, x'_2, \dots x'_n)$$

denote any known function of $x, x_1, x_2, \dots x_n, x'_1, x'_2, \dots x'_n$; and therefore let the equation

$$0 = f(x, x_1, x_2, \dots x_n, x'_1, x'_2, \dots x'_n) \tag{A}$$

represent any proposed total differential equation of the first order, restricting the forms of the n sought functions $x_1, \dots x_n$ and assisting to determine those forms by establishing a known relation between those n functions themselves, their n derived functions of the first order and the independent variable x. Since the number n of the sought functions $x_1, \dots x_n$ has been supposed to be greater than unity, the one equation (A) is not in general sufficient to determine the forms of all and we may on the contrary assume any $n-1$ supplementary equations, differential or not differential, to connect these n sought functions, in combination with that given equation of the first order (A).

Among all the supplementary equations which might be thus assumed, we shall select as deserving of special attention, for reasons that will hereafter appear, the following $n-1$ equations, which are in general of the second order, and we shall call them the *principal supplementaries* of the original equation (A):

$$\frac{f'(x_1) - \{f'(x'_1)\}'}{f'(x'_1)} = \frac{f'(x_2) - \{f'(x'_2)\}'}{f'(x'_2)} = \dots = \frac{f'(x_n) - \{f'(x'_n)\}'}{f'(x'_n)}. \tag{B}$$

In this notation, which is borrowed from Lagrange, the $2n$ symbols

$$f'(x_1),\ f'(x_2),\ \dots\ f'(x_n),\quad f'(x'_1),\ f'(x'_2),\ \dots\ f'(x'_n)$$

denote respectively the $2n$ partial derivatives, or partial differential coefficients, of the first order of the known function f, taken with respect to $x_1, x_2, \dots x_n, x'_1, x'_2, \dots x'_n$; and the n symbols

$$\{f'(x'_1)\}',\ \{f'(x'_2)\}',\ \dots\ \{f'(x'_n)\}'$$

denote the total derivatives, or total differential coefficients, of the first order of the functions $f'(x'_1), f'(x'_2), \dots f'(x'_n)$ respectively, considered as depending on the independent variable x, not only so far as that variable enters into them explicitly by entering into the known composition of the function f, but also so far as it enters into them implicitly by entering into the unknown composition of the n sought functions $x_1, x_2, \ldots x_n$ and of their derivatives $x_1', x_2', \ldots x_n'$; so that, according to Lagrange's analogous notations for partial and total derivatives of the second order,* the symbols $\{f'(x_1')\}', \ldots \{f'(x_n')\}'$ are equivalent, respectively, to the following more developed expressions:

$$f^{\prime,\prime}(x, x_1') + x_1' f^{\prime,\prime}(x_1, x_1') + x_2' f^{\prime,\prime}(x_2, x_1') + \ldots + x_n' f^{\prime,\prime}(x_n, x_1')$$
$$+ x_1'' f''(x_1') + x_2'' f^{\prime,\prime}(x_1', x_2') + \ldots + x_n'' f^{\prime,\prime}(x_1', x_n'),$$

$$f^{\prime,\prime}(x, x_n') + x_1' f^{\prime,\prime}(x_1, x_n') + x_2' f^{\prime,\prime}(x_2, x_n') + \ldots + x_n' f^{\prime,\prime}(x_n, x_n')$$
$$+ x_1'' f^{\prime,\prime}(x_1', x_n') + x_2'' f^{\prime,\prime}(x_2', x_n') + \ldots + x_n'' f''(x_n').$$

The object of the present Chapter is to establish a general method for rigorously expressing, and for at least approximately calculating, the integrals of all such systems of original and principal supplementary equations as the system (A) and (B) for any number of variables $x, x_1, \ldots x_n$ and for any form of the given function f, (some particular exceptions being set aside); but, before proceeding to the establishment of such a method, it may be proper to mention some of the chief reasons, (connected with the Calculus of Variations,) for the selection of the system of equations (B) as the supplementary system to be combined with the original equation (A).

2. Such reasons may be drawn from the comparison of that original equation (A) with another equation of like form, obtained by slightly altering as follows the original values of the variables $x, x_1, \ldots x_n$. If we represent by $\xi, \xi_1, \xi_2, \ldots \xi_n$ any functions of x and by ϵ any small multiplier independent of x, we may then consider

$$x_1 + \epsilon\xi_1, \quad x_2 + \epsilon\xi_2, \ \ldots\ x_n + \epsilon\xi_n$$

as n functions of $x + \epsilon\xi$, which do not much differ in their forms, or in their laws of functional dependence on $x + \epsilon\xi$, from the n former functions $x_1, x_2, \ldots x_n$, considered as depending on x; and if we wish that these n new functions $x_1 + \epsilon\xi_1$, &c. of $x + \epsilon\xi$ should be connected with each other by a differential relation of exactly the same form as that original differential relation, which was previously given to connect the n old functions x_1, &c. of x, we must then establish this new differential equation of the first order, analogous to and formed from the original equation (A):

$$0 = f\left(x + \epsilon\xi, x_1 + \epsilon\xi_1, \ldots x_n + \epsilon\xi_n, \frac{x_1' + \epsilon\xi_1'}{1 + \epsilon\xi'}, \ldots \frac{x_n' + \epsilon\xi_n'}{1 + \epsilon\xi'}\right); \qquad \text{(C)}$$

in which $\xi', \xi_1', \ldots \xi_n'$ are the first derivatives or differential coefficients of $\xi, \xi_1, \ldots \xi_n$, considered as functions of x; and consequently

$$\frac{x_1' + \epsilon\xi_1'}{1 + \epsilon\xi'}, \ldots \frac{x_n' + \epsilon\xi_n'}{1 + \epsilon\xi'}$$

are the first derivatives or differential coefficients of $x_1 + \epsilon\xi_1, \ldots x_n + \epsilon\xi_n$, considered as functions of $x + \epsilon\xi$. Developing this new differential equation (C) according to the ascending powers of the small multiplier ϵ, and suppressing that part of the development which vanishes on account of the original differential equation (A), and finally dividing by ϵ, we find

$$0 = \xi f'(x) + \xi_1 f'(x_1) + \ldots + \xi_n f'(x_n)$$
$$+ (\xi_1' - x_1'\xi') f'(x_1') + \ldots + (\xi_n' - x_n'\xi') f'(x_n') + \epsilon E; \qquad \text{(D)}$$

---

* [Lagrange, *Théorie des fonctions analytiques* (1797).]

the coefficient E denoting, for abridgement, a development of which the actual calculation is not necessary for our present purpose, because it disappears when we pass (as we shall shortly do) to the limit at which ϵ vanishes. The part independent of ϵ in the equation (D) may be transformed, by observing that the total derivative of the original equation (A) gives this relation:

$$0 = f'(x) + x_1' f'(x_1) + \dots + x_n' f'(x_n) + x_1'' f'(x_1') + \dots + x_n'' f'(x_n'); \tag{E}$$

for thus we find, by eliminating $f'(x)$ and by observing that $\xi_1' - x_1'\xi' - x_1''\xi = (\xi_1 - x_1'\xi)'$, &c.,

$$\begin{aligned} 0 = (\xi_1 - x_1'\xi) f'(x_1) + \dots + (\xi_n - x_n'\xi) f'(x_n) \\ + (\xi_1 - x_1'\xi)' f'(x_1') + \dots + (\xi_n - x_n'\xi)' f'(x_n') + \epsilon E. \end{aligned} \tag{F}$$

Let λ denote a function of x, so chosen that when we multiply by it the sum of the terms $(\xi_n - x_n'\xi) f'(x_n)$ and $(\xi_n - x_n'\xi)' f'(x_n')$, the product shall be an exact derivative, independently of the forms of the functions ξ and ξ_n; that is, let λ satisfy the following differential equation of the first order,

$$\lambda f'(x_n) = \{\lambda f'(x_n')\}'. \tag{G}$$

The differential equation (F) may then be thus transformed,

$$\begin{aligned} 0 = \{(\xi_1 - x_1'\xi)\lambda f'(x_1') + \dots + (\xi_n - x_n'\xi)\lambda f'(x_n')\}' \\ + (\xi_1 - x_1'\xi)(\lambda f'(x_1) - \{\lambda f'(x_1')\}') + \dots \\ + (\xi_{n-1} - x_{n-1}'\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x_{n-1}')\}') + \epsilon\lambda E; \end{aligned} \tag{H}$$

and it gives, in the notation of finite differences and of definite integrals,

$$\begin{aligned} 0 = \Delta\{(\xi_1 - x_1'\xi)\lambda f'(x_1') + \dots + (\xi_n - x_n'\xi)\lambda f'(x_n')\} \\ \int_a^x (\xi_1 - x_1'\xi)(\lambda f'(x_1) - \{\lambda f'(x_1')\}')\,dx + \dots \\ \int_a^x (\xi_{n-1} - x_{n-1}'\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x_{n-1}')\}')\,dx + \epsilon\int_a^x \lambda E\,dx; \end{aligned} \tag{I}$$

the sign of a finite difference Δ implying here that we are to subtract the initial from the final value of the function to which it is prefixed, so that, the initial value of x being supposed to be a, the symbol $\Delta F(x)$ is equivalent in this notation to $F(x) - F(a)$. Now if we take successively smaller and smaller values of the multiplier ϵ, that is, values nearer and nearer to 0, so as to make ϵ tend to 0 as its limit, the product $\epsilon\int_a^x \lambda E\,dx$ will also tend to 0 as its limit, and thus the definite integral $\int_a^x \lambda E\,dx$ will disappear from the limiting or ultimate form of the equation (I), as being multiplied by an ultimately evanescent factor ϵ. But when the term $\epsilon\int_a^x \lambda E\,dx$ is suppressed in the equation (I), the function ξ_n enters into that equation by its initial and final values only and no trace of the intermediate values (or form) of this function ξ_n remains; to accomplish which removal of the effects of all but the extreme values of ξ_n from the equation (I) was (as will easily be perceived) the thing aimed at in establishing the foregoing relation (G) for the determination of the multiplier λ. As yet, however, the equation (I) involves not merely the extreme values, but also the intermediate forms, of the other n new functions $\xi, \xi_1, \dots \xi_{n-1}$, because it involves the definite integrals

$$\begin{gathered}\int_a^x (\xi_1 - x_1'\xi)(\lambda f'(x_1) - \{\lambda f'(x_1')\}')\,dx,\\ \\ \int_a^x (\xi_{n-1} - x_{n-1}'\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x_{n-1}')\}')\,dx;\end{gathered} \tag{J}$$

it is therefore evidently an object of particular interest to make these $n-1$ definite integrals vanish, without restricting the forms of the n functions $\xi, \xi_1, \dots \xi_{n-1}$, and for this purpose to make the coefficients of

$$\xi_1 - x_1'\xi, \dots \xi_{n-1} - x_{n-1}'\xi$$

under the signs of integration vanish, by establishing the following $n-1$ relations between the n original functions $x_1, \dots x_n$, the multiplier λ and the independent variable x:

$$\begin{gathered}\lambda f'(x_1) = \{\lambda f'(x_1')\}',\\ \\ \lambda f'(x_{n-1}) = \{\lambda f'(x_{n-1}')\}';\end{gathered} \tag{K}$$

which can in general be done, consistently with the original given relation (A) and with the assumed equation (C). And if we eliminate λ and its derivative λ', or rather the ratio of the latter to the former, between the n equations (G) and (K), we obtain the $n-1$ supplementary equations (B); which were accordingly selected as possessing this remarkable character, among others, of causing the effects of the forms of the $n+1$ new functions $\xi, \xi_1, \dots \xi_n$ to disappear from the ultimate state of the integral of the varied equation (C), and of reducing this ultimate state of that integral (I) to the form of a linear relation between the final and initial values of these new functions, namely,

$$0 = \Delta\{(\xi_1 - x_1'\xi)\lambda f'(x_1') + \dots + (\xi_n - x_n'\xi)\lambda f'(x_n')\}; \tag{L}$$

a relation which may also be thus written,

$$\begin{aligned}0 = {} & (\xi_1 - x_1'\xi)\lambda f'(x_1') + \dots + (\xi_n - x_n'\xi)\lambda f'(x_n')\\ & - (\alpha_1 - a_1'\alpha)\gamma f'(a_1') - \dots - (\alpha_n - a_n'\alpha)\gamma f'(a_n'),\end{aligned} \tag{M}$$

if we employ the symbols

$$a_1, \dots a_n, \quad a_1', \dots a_n', \quad \alpha, \alpha_1, \dots \alpha_n, \quad \alpha', \alpha_1', \dots \alpha_n', \quad \gamma$$

to denote the initial values of the functions

$$x_1, \dots x_n, \quad x_1', \dots x_n', \quad \xi, \xi_1, \dots \xi_n, \quad \xi', \xi_1', \dots \xi_n', \quad \lambda,$$

(corresponding to the initial value a of x,) and employ also the symbols $f'(a_1'), \dots f'(a_n')$ to denote the corresponding initial values of $f'(x_1'), \dots f'(x_n')$.

3. In the important but particular case, in which the original differential equation (A), or the given function f, is of the first order with respect to the differential coefficients $x_1', \dots x_n'$, so that the expressions of the partial derivatives $f'(x_1'), \dots f'(x_n')$ do not contain those coefficients $x_1', \dots x_n'$, the supplementary equations (B) reduce themselves to the first order; and a reduction of the same sort takes place, with respect to all or some of these equations, in some other cases of exception, which are of less importance. But, in general, the $n-1$ supplementary differential equations (B) are of the 2nd order and compose, when combined with the total derivative (E) of the original differential equation of the 1st order (A), a system of n total differential equations of the 2nd order, of which the complete integrals must involve $2n$ arbitrary constants, (besides the arbitrary initial value a of the independent variable x); and since we may choose for these $2n$ constants the $2n$ initial values $a_1, \dots a_n, a'_1, \dots a'_n$, we may represent the n integrals of the equations (B) and (E) as follows:

$$\left.\begin{array}{l} x_1 = \phi_1(x, a, a_1, \dots a_n, a'_1, \dots a'_n), \\ \\ x_n = \phi_n(x, a, a_1, \dots a_n, a'_1, \dots a'_n); \end{array}\right\} \qquad (\mathrm{N})$$

the forms of the n functions $\phi_1, \dots \phi_n$ remaining, as yet, unknown. Besides, the original differential equation (A) gives this initial condition:

$$0 = f(a, a_1, \dots a_n, a'_1, \dots a'_n); \qquad (\mathrm{O})$$

so that the constants $a_1, \dots a_n, a'_1, \dots a'_n$ are not any longer all arbitrary (when a is considered as known), but must be so assumed as to satisfy this condition (O), if we wish to represent, by the equations (N), no longer the integrals of the derived system (B) and (E) but the integrals of the original system (A) and (B). And if we knew these integrals of that original system, that is, if we knew the forms of the n functions $\phi_1, \dots \phi_n$ in the n equations (N), as well as the form of the function f in the initial condition (O), we could in general eliminate the n constants $a'_1, \dots a'_n$ between these $n+1$ equations (N) and (O) and so arrive at a relation which would involve only $x, x_1, \dots x_n$ and $a, a_1, \dots a_n$ and which would be thus denoted:

$$0 = \psi(x, x_1, \dots x_n, a, a_1, \dots a_n); \qquad (\mathrm{P})$$

but not in general at two (or more) distinct relations of this kind; since, after the assumption of the n initial values $a_1, \dots a_n$ of the n functions $x_1, \dots x_n$ (corresponding to any assumed initial value of the independent variable x), there would still remain $n-1$ arbitrary constants to dispose of, and consequently the final values of any $n-1$ of the same n functions (corresponding to any final value of the same independent variable x) would still admit, in general, of being assumed at pleasure. It is sufficient, for the present, to concede the possibility of such an elimination, and to perceive the general existence of a determinate relation such as (P), (that is, a relation between the final and initial values of the independent variable x and of the n functions $x_1, \dots x_n$ as a consequence of the differential equations (A) and (B)); for it will soon be shown that, instead of our being obliged to integrate first those differential equations and then to deduce the relation (P) by elimination from the integrals thus found, we may, on the contrary, with advantage, seek first by independent processes to discover the relation (P), and, when it has once in any way been found, may then deduce from it the whole system of integrals (N).

4. The equation (M) between the extreme values of the $n+1$ functions $\xi, \xi_1, \dots \xi_n$, introduced in the 2nd article, was obtained as a limiting form of the integral of the single differential equation (C), combined with the relations (A) and (B) between the n functions $x_1, \dots x_n$; and therefore this equation (M) will still be true, after the introduction of any new equations between the same $n+1$ functions $\xi, \xi_1, \dots \xi_n$, if we combine these new equations with the equation (C) and with the system (A) and (B). If, then, by the introduction of any such new or supplementary

relations between the functions $\xi, \xi_1, \dots \xi_n$ we can obtain, in any new way, a new limiting linear relation between the extreme values of those functions, not visibly coincident with the relation (M) but of the form

$$\left.\begin{aligned} 0 = X\xi + X_1\xi_1 + \dots + X_n\xi_n \\ + A\alpha + A_1\alpha_1 + \dots + A_n\alpha_n, \end{aligned}\right\} \qquad (Q)$$

in which the coefficients $X, X_1, \dots X_n, A, A_1, \dots A_n$ are independent of $\xi, \xi_1, \dots \xi_n$; if also, by the nature of this new process, it is permitted to assume at pleasure any $2n+1$ of the $2n+2$ extreme values $\alpha, \alpha_1, \dots \alpha_n, \xi, \xi_1, \dots \xi_n$, provided that the limiting state of the remaining extreme value (corresponding to the limit $\epsilon = 0$) is then determined so as to satisfy the new relation (Q); (as, by the nature of the process which conducted to the former relation (M), it was permitted to assume at pleasure any $2n+1$ of the same $2n+2$ extreme values, provided that the limiting state of the remaining value was determined so as to satisfy that former relation); we shall be able to conclude that these two limiting relations (M) and (Q), though differing in appearance, must in reality coincide with each other, in such a manner that the $2n+2$ coefficients of the one must be proportional to those of the other, and that thus, by the introduction of a new multiplier L, we shall have the $2n+2$ equations following:

$$\left.\begin{aligned} X &= -L\lambda\{x'_1 f'(x'_1) + \dots + x'_n f'(x'_n)\}, \\ X_1 &= L\lambda f'(x'_1), \ \dots \ X_n = L\lambda f'(x'_n), \\ A &= L\gamma\{a'_1 f'(a'_1) + \dots + a'_n f'(a'_n)\}, \\ A_1 &= -L\gamma f'(a'_1), \dots A_n = -L\gamma f'(a'_n). \end{aligned}\right\} \qquad (R)$$

5. Now among the various supplementary relations which might be assumed to connect the $n+1$ functions $\xi, \xi_1, \dots \xi_n$, those relations which are derived from the principal supplementary differential equations of the second order (B) by changing

$$x, x_1, \dots x_n, \quad x'_1, \dots x'_n, \quad x''_1, \dots x''_n$$

to

$$x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \dots\ x_n+\epsilon\xi_n,\ \frac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'},\ \dots\ \frac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \dots\ \frac{\left(\dfrac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}$$

are deserving of special mention, namely the $n-1$ new supplementary equations, which may be thus denoted,

$$\left.\begin{aligned} 0 &= \overset{1}{f}\left(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \dots\ x_n+\epsilon\xi_n,\ \frac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'},\ \dots\ \frac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \dots\ \frac{\left(\dfrac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}\right) \\ 0 &= \overset{n-1}{f}\left(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \dots\ x_n+\epsilon\xi_n,\ \frac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'},\ \dots\ \frac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x'_1+\epsilon\xi'_1}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \dots\ \frac{\left(\dfrac{x'_n+\epsilon\xi'_n}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}\right) \end{aligned}\right\} \qquad (S)$$

if the $n-1$ former principal supplementary equations (B) be denoted as follows:

$$\left.\begin{aligned} 0 &= \overset{1}{f}(x, x_1, \dots x_n, x'_1, \dots x'_n, x''_1, \dots x''_n), \\ 0 &= \overset{n-1}{f}(x, x_1, \dots x_n, x'_1, \dots x'_n, x''_1, \dots x''_n); \end{aligned}\right\} \qquad (T)$$

and these $n-1$ equations (S), when combined with the equation (C), conduct to a limiting linear relation of the desired form (Q) between the $2n+2$ extreme values $\alpha, \alpha_1, \ldots \alpha_n, \xi, \xi_1, \ldots \xi_n$, leaving $2n+1$ of them arbitrary. For, although the n differential equations (C) and (S), even along with any finite relation of arbitrary constants or initial data, are not in general sufficient to determine completely the forms of the $n+1$ functions $\xi, \xi_1, \ldots \xi_n$ considered as depending on x, yet these n differential equations are in general sufficient to determine completely the forms of the n functions $x_1+\epsilon\xi_1, \ldots x_n+\epsilon\xi_n$ considered as depending on $x+\epsilon\xi$, if the $2n$ initial constants $a_1+\epsilon\alpha_1, \ldots a_n+\epsilon\alpha_n, \dfrac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'}, \ldots \dfrac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}$, (as well as $a+\epsilon\alpha$,) be known or even any $2n-1$ of these $2n$ constants, because the equation (C) conducts to an initial condition connecting them; and thus we find n integral equations of the forms (N), namely,

$$\left.\begin{aligned} x_1+\epsilon\xi_1 &= \phi_1\left(x+\epsilon\xi,\ a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \ldots\ a_n+\epsilon\alpha_n,\ \frac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'}\qquad \frac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}\right) \\ x_n+\epsilon\xi_n &= \phi_n\,(x+\epsilon\xi, \ldots\ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad), \end{aligned}\right\} \tag{U}$$

together with this initial condition of the form (O)

$$0=f\left(a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \ldots\ a_n+\epsilon\alpha_n\ \ \frac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'},\ \ldots \frac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}\right); \tag{V}$$

which $n+1$ equations (U) and (V) conduct, by elimination of the n constants

$$\frac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'}, \ldots \frac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'},$$

to this following relation

$$0=\psi\,(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \ldots\ x_n+\epsilon\xi_n,\ a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \ldots\ a_n+\epsilon\alpha_n), \tag{W}$$

the form of the function ψ in this relation being the same as in the final relation (P); so that, developing this new relation (W) according to the ascending powers of ϵ, suppressing the terms which vanish on account of the old relation (P), dividing across by ϵ and finally letting ϵ tend to 0, we have the equation

$$\begin{aligned} 0=\xi\psi'\,(x)+\xi_1\psi'\,(x_1)+\xi_2\psi'\,(x_2)+\ldots+\xi_n\psi'\,(x_n \\ +\alpha\psi'\,(a)+\alpha_1\psi'\,(a_1)+\alpha_2\psi'\,(a_2)+\ldots+\alpha_n\psi'\,(a_n).\rfloor \end{aligned} \tag{X}$$

[Here there are 19 pages of manuscript missing.]

11. The reduced equation (G^2) may be put under the form

$$z_{2n-1}'=V_1z_1'+V_2z_2'+\ldots+V_{2n-2}z_{2n-2}', \tag{A^3}$$

in which the $2n-2$ coefficients $V_1, V_2, \ldots V_{2n-2}$ are to be considered as known functions of the $2n-1$ variables $z_1, z_2, \ldots z_{2n-1}$ and of these alone. If we denote by the symbols $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-3}$ any $2n-3$ assumed functions of these $2n-1$ variables and treat these functions as auxiliary variables, we may in general conceive that the $2n-3$ former variables $z_1, z_2, \ldots z_{2n-3}$ are expressed, reciprocally, as $2n-3$ functions of these $2n-3$ new variables $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-3}$ and of the two other old variables z_{2n-2}, z_{2n-1}; the partial derivatives, or differential coefficients, of the first order of these $2n-3$ new functions may be denoted by the symbols

$$z'_1(\overset{1}{z}_1),\ z'_1(\overset{1}{z}_2),\ \dots\ z'_1(\overset{1}{z}_{2n-3}),\ z'_1(z_{2n-2}),\ z'_1(z_{2n-1}),$$

$$z'_{2n-3}(\overset{1}{z}_1),\ z'_{2n-3}(\overset{1}{z}_2),\ \dots\ z'_{2n-3}(\overset{1}{z}_{2n-3}),\ z'_{2n-3}(z_{2n-2}),\ z'_{2n-3}(z_{2n-1});$$

and their total derivatives of the same order may in like manner be denoted as follows:

$$\left.\begin{aligned} z'_1 &= z'_1(\overset{1}{z}_1)\overset{1}{z}{}'_1 + \dots + z'_1(\overset{1}{z}_{2n-3})\overset{1}{z}{}'_{2n-3} + z'_1(z_{2n-2})z'_{2n-2} + z'_1(z_{2n-1})z'_{2n-1}, \\ z'_{2n-3} &= z'_{2n-3}(\overset{1}{z}_1)\overset{1}{z}{}'_1 + \dots + z'_{2n-3}(\overset{1}{z}_{2n-3})\overset{1}{z}{}'_{2n-3} + z'_{2n-3}(z_{2n-2})z'_{2n-2} + z'_{2n-3}(z_{2n-1})z'_{2n-1}. \end{aligned}\right\} \qquad (\mathrm{B}^3)$$

By the substitution of these expressions, the equation (A³) takes the form

$$0 = \overset{1}{Z}_1\overset{1}{z}{}'_1 + \overset{1}{Z}_2\overset{1}{z}{}'_2 + \dots + \overset{1}{Z}_{2n-3}\overset{1}{z}{}'_{2n-3} + \overset{1}{Z}_{2n-2}z'_{2n-2} + \overset{1}{Z}_{2n-1}z'_{2n-1}; \qquad (\mathrm{C}^3)$$

in which the coefficients $\overset{1}{Z}_1, \overset{1}{Z}_2, \dots \overset{1}{Z}_{2n-1}$ have the following values:

$$\left.\begin{aligned} \overset{1}{Z}_1 &= V_1 z'_1(\overset{1}{z}_1) + \dots + V_{2n-3}z'_{2n-3}(\overset{1}{z}_1), \\ \overset{1}{Z}_{2n-3} &= V_1 z'_1(\overset{1}{z}_{2n-3}) + \dots + V_{2n-3}z'_{2n-3}(\overset{1}{z}_{2n-3}), \end{aligned}\right\} \qquad (\mathrm{D}^3)$$

$$\left.\begin{aligned} \overset{1}{Z}_{2n-2} &= V_1 z'_1(z_{2n-2}) + \dots + V_{2n-3}z'_{2n-3}(z_{2n-2}) + V_{2n-2}; \\ \overset{1}{Z}_{2n-1} &= V_1 z'_1(z_{2n-1}) + \dots + V_{2n-3}z'_{2n-3}(z_{2n-1}) - 1. \end{aligned}\right\} \qquad (\mathrm{E}^3)$$

This equation (C³) will admit of being put under the simpler form

$$\overset{1}{z}{}'_{2n-3} = \overset{1}{V}_1\overset{1}{z}{}'_1 + \overset{1}{V}_2\overset{1}{z}{}'_2 + \dots + \overset{1}{V}_{2n-4}\overset{1}{z}{}'_{2n-4} + \overset{1}{V}_{2n-1}z'_{2n-1}, \qquad (\mathrm{F}^3)$$

in which the $2n-4$ coefficients $\overset{1}{V}_1, \overset{1}{V}_2, \dots \overset{1}{V}_{2n-4}$ will be known functions of the $2n-3$ new variables $\overset{1}{z}_1, \overset{1}{z}_2, \dots \overset{1}{z}_{2n-3}$ and of z_{2n-1}, not involving z_{2n-2}, if we can so choose the new variables as to satisfy the $2n-3$ conditions following:

$$\overset{1}{Z}_{2n-2} = 0; \qquad (\mathrm{G}^3)$$

$$\frac{\overset{1}{Z}{}'_1(z_{2n-2})}{\overset{1}{Z}_1} = \frac{\overset{1}{Z}{}'_2(z_{2n-2})}{\overset{1}{Z}_2} = \dots = \frac{\overset{1}{Z}{}'_{2n-3}(z_{2n-2})}{\overset{1}{Z}_{2n-3}}. \qquad (\mathrm{H}^3)$$

The expressions (D³) give

$$\begin{aligned} \overset{1}{Z}{}'_i(z_{2n-2}) = {} & V_1 z''_1(\overset{1}{z}_i, z_{2n-2}) + \dots + V_{2n-3}z''_{2n-3}(\overset{1}{z}_i, z_{2n-2}) \\ & + \{V'_1(z_1) \,.\, z'_1(z_{2n-2}) + \dots + V'_1(z_{2n-3}) \,.\, z'_{2n-3}(z_{2n-2}) + V'_1(z_{2n-2})\} z'_1(z_i) \\ & + . \\ & + \{V'_{2n-3}(z_1) \,.\, z'_1(z_{2n-2}) + \dots + V'_{2n-3}(z_{2n-3}) \,.\, z'_{2n-3}(z_{2n-2}) + V'_{2n-3}(z_{2n-2})\} z'_{2n-3}(z_i); \end{aligned} \qquad (\mathrm{I}^3)$$

the index i being here any one of the integer numbers 1, 2, ... $2n-3$; and the first expression (E³) gives

$$\begin{aligned}\overset{1}{Z}{}'_{2n-2}(\overset{1}{z}_i) = {} & V_1 z_1''(\overset{1}{z}_i, z_{2n-2}) + \ldots + V_{2n-3} z''_{2n-3}(\overset{1}{z}_i, z_{2n-2}) \\ & + \{V_1'(z_1) . z_1'(z_{2n-2}) + \ldots + V'_{2n-3}(z_1) . z'_{2n-3}(z_{2n-2}) + V'_{2n-2}(z_1)\} z_1'(\overset{1}{z}_i) \\ & + \ldots\ldots \\ & + \{V_1'(z_{2n-3}) . z_1'(z_{2n-2}) + \ldots + V'_{2n-3}(z_{2n-3}) . z'_{2n-3}(z_{2n-2}) + V'_{2n-2}(z_{2n-3})\} z'_{2n-3}(\overset{1}{z}_i); \end{aligned} \qquad (\mathrm{K}^3)$$

so that we have the equation

$$\overset{1}{Z}{}_i'(z_{2n-2}) - \overset{1}{Z}{}'_{2n-2}(\overset{1}{z}_i) = W_1 z_1'(\overset{1}{z}_i) + \ldots + W_{2n-3} z'_{2n-3}(\overset{1}{z}_i), \qquad (\mathrm{L}^3)$$

in which

$$\begin{aligned} W_k = {} & \{V_k'(z_1) - V_1'(z_k)\} z_1'(z_{2n-2}) \\ & + \{V_k'(z_2) - V_2'(z_k)\} z_2'(z_{2n-2}) \\ & + \ldots\ldots \\ & + \{V_k'(z_{2n-3}) - V'_{2n-3}(z_k)\} z'_{2n-3}(z_{2n-2}) \\ & + V_k'(z_{2n-2}) - V'_{2n-2}(z_k), \end{aligned} \qquad (\mathrm{M}^3)$$

the index k (like i) denoting any integer from 1 to $2n-3$ inclusive. And since, by (D³), we have

$$\overset{1}{Z}_i = V_1 z_1'(\overset{1}{z}_i) + \ldots + V_{2n-3} z'_{2n-3}(\overset{1}{z}_i), \qquad (\mathrm{N}^3)$$

we see that we shall satisfy the $2n-3$ conditions (G³) and (H³) if we can so select the $2n-3$ auxiliary variables $\overset{1}{z}_1, \ldots \overset{1}{z}_{2n-3}$ as to satisfy all the following conditions:

$$0 = V_1 z_1'(z_{2n-2}) + \ldots + V_{2n-3} z'_{2n-3}(z_{2n-2}) + V_{2n-2}, \qquad (\mathrm{O}^3)$$

and

$$\frac{W_1}{V_1} = \frac{W_2}{V_2} = \ldots = \frac{W_{2n-3}}{V_{2n-3}}. \qquad (\mathrm{P}^3)$$

In this manner we are led to endeavour to integrate a new auxiliary system of total differential equations of the 1st order, namely, the following system of $2n-3$ such equations between the $2n-2$ variables $z_1, z_2, \ldots z_{2n-2}$:

$$0 = V_1 z_1' + V_2 z_2' + \ldots + V_{2n-3} z'_{2n-3} + V_{2n-2}; \qquad (\mathrm{Q}^3)$$

$$\left.\begin{aligned} & \frac{1}{V_1}\{V_1'(z_1) . z_1' + V_2'(z_1) . z_2' + \ldots + V'_{2n-3}(z_1) . z'_{2n-3} + V'_{2n-2}(z_1) - V_1'\} \\ & = \frac{1}{V_2}\{V_1'(z_2) . z_1' + V_2'(z_2) . z_2' + \ldots + V'_{2n-3}(z_2) . z'_{2n-3} + V'_{2n-2}(z_2) - V_2'\} \\ & \\ & = \frac{1}{V_{2n-3}}\{V_1'(z_{2n-3}) . z_1' + V_2'(z_{2n-3}) . z_2' + \ldots + V'_{2n-3}(z_{2n-3}) . z'_{2n-3} + V'_{2n-2}(z_{2n-3}) - V'_{2n-3}\}; \end{aligned}\right\} \qquad (\mathrm{R}^3)$$

in which z_{2n-1} is treated as constant and z_{2n-2} is taken for the independent variable, so that the total derivatives $V_1', \ldots V'_{2n-3}$ are equivalent to the following more developed expressions:

$$\left.\begin{aligned} V_1' &= V_1'(z_1) . z_1' + \ldots + V_1'(z_{2n-3}) . z'_{2n-3} + V_1'(z_{2n-2}), \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V'_{2n-3} &= V'_{2n-3}(z_1) . z_1' + \ldots + V'_{2n-3}(z_{2n-3}) . z'_{2n-3} + V'_{2n-3}(z_{2n-2}). \end{aligned}\right\} \qquad (\mathrm{S}^3)$$

For, if we can integrate these $2n-3$ equations and thereby discover expressions for $z_1, z_2, \ldots z_{2n-3}$ as functions of the independent variable z_{2n-2}, involving also the variable z_{2n-1}, which has been treated as constant in the integration, and involving $2n-3$ arbitrary constants, we have only to treat these $2n-3$ new constants as variable and substitute them for the $2n-3$ auxiliary variables $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-3}$, which have hitherto been left undetermined; since thus we shall accomplish the desired reduction of the differential equation (A^3) to the form (F^3), in which the coefficient of z'_{2n-2} is null, the coefficient of $\overset{1}{z}'_{2n-3}$ is unity and the coefficients of $\overset{1}{z}'_1, \overset{1}{z}'_2, \ldots \overset{1}{z}'_{2n-4}$ are independent of z_{2n-2}. The same reasoning shows that if we can integrate the following new system of $2n-5$ total differential equations of the 1st order:

$$0 = \overset{1}{V}_1 \overset{1}{z}'_1 + \overset{1}{V}_2 \overset{1}{z}'_2 + \ldots + \overset{1}{V}_{2n-5} \overset{1}{z}'_{2n-5} + \overset{1}{V}_{2n-4}; \tag{T^3}$$

$$\frac{1}{\overset{1}{V}_1}\{\overset{1}{V}'_1(\overset{1}{z}_1) \,.\, \overset{1}{z}'_1 + \ldots + \overset{1}{V}'_{2n-5}(\overset{1}{z}_1) \,.\, \overset{1}{z}'_{2n-5} + \overset{1}{V}'_{2n-4}(\overset{1}{z}_1) - \overset{1}{V}'_1\}$$

$$\frac{1}{\overset{1}{V}_{2n-5}}\{\overset{1}{V}'_1(\overset{1}{z}_{2n-5}) \,.\, \overset{1}{z}'_1 + \ldots + \overset{1}{V}'_{2n-5}(\overset{1}{z}_{2n-5}) \,.\, \overset{1}{z}'_{2n-5} + \overset{1}{V}'_{2n-4}(\overset{1}{z}_{2n-5}) - \overset{1}{V}'_{2n-5}\}; \tag{U^3}$$

in which $\overset{1}{z}_{2n-3}$ and z_{2n-1} are treated as constant and $\overset{1}{z}_{2n-4}$ is taken for the independent variable; and if we can thus discover $2n-5$ expressions for $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-5}$ as functions of $\overset{1}{z}_{2n-4}$, involving also in general $\overset{1}{z}_{2n-3}$ and z_{2n-1} and $2n-5$ arbitrary constants; then, by treating these latter constants as $2n-5$ new auxiliary variables $\overset{2}{z}_1, \overset{2}{z}_2, \ldots \overset{2}{z}_{2n-5}$, we can reduce the equation (F^3) to this new and simpler form:

$$\overset{2}{z}'_{2n-5} = \overset{2}{V}_1 \overset{2}{z}'_1 + \overset{2}{V}_2 \overset{2}{z}'_2 + \ldots + \overset{2}{V}_{2n-6} \overset{2}{z}'_{2n-6} + \overset{2}{V}_{2n-3} \overset{1}{z}'_{2n-3} + \overset{2}{V}_{2n-1} z'_{2n-1}; \tag{V^3}$$

in which the $2n-6$ coefficients $\overset{2}{V}_1, \overset{2}{V}_2, \ldots \overset{2}{V}_{2n-6}$ will be known functions of $\overset{2}{z}_1, \overset{2}{z}_2, \ldots \overset{2}{z}_{2n-5}, \overset{1}{z}_{2n-3}$ and z_{2n-1}, $\overset{2}{V}_{2n-3}$ will be a known function of the same variables and of $\overset{1}{z}_{2n-4}$, and $\overset{2}{V}_{2n-1}$ will involve in general the variable z_{2n-2} along with all the former.

By successive reductions of this sort, depending on the integrations of several successive systems of fewer and fewer total differential equations of the first order between fewer and fewer variables; namely, on the integration of a system of $2n-7$ equations between $2n-6$ variables, a system of $2n-9$ equations between $2n-8$ variables, and so on till we come to the integration of a single total differential equation between only two variables; it is in general possible, at least in theory, to reduce the differential equation (A^3) to the form:

$$\overset{n-1}{z}'_1 = \overset{n-1}{V}_3 \overset{n-2}{z}'_3 + \overset{n-1}{V}_5 \overset{n-3}{z}'_5 + \ldots + \overset{n-1}{V}_{2n-3} \overset{1}{z}'_{2n-3} + \overset{n-1}{V}_{2n-1} z'_{2n-1}; \tag{W^3}$$

in which the coefficient $\overset{n-1}{V}_3$ is, in general, a known function of the n variables $\overset{n-1}{z}_1$, $\overset{n-2}{z}_3$, $\overset{n-3}{z}_5$, ${}_{n-5}$, $\overset{1}{z}_{2n-3}$, z_{2n-1}, involving also another variable which, according to the same analogy of notation, is to be denoted by the symbol $\overset{n-2}{z}_2$; $\overset{n-1}{V}_5$ involves in general the same variables as $\overset{n-1}{V}_3$ but involves also an additional variable $\overset{n-3}{z}_4$; and thus new variables $\overset{n-4}{z}_6$, $\overset{n-5}{z}_8$, &c. are introduced successively as we pass to new coefficients $\overset{n-1}{V}_7$, $\overset{n-1}{V}_9$, &c. till we come to the last coefficient $\overset{n-1}{V}_{2n-1}$ which is a known function of all the $2n-1$ variables of the new system, namely,

$$\overset{n-1}{z}_1,\ \overset{n-2}{z}_2,\ \overset{n-2}{z}_3,\ \overset{n-3}{z}_4,\ \overset{n-3}{z}_5,\ \dots \overset{2}{z}_{2n-6},\ \overset{2}{z}_{2n-5},\ \overset{1}{z}_{2n-4},\ \overset{1}{z}_{2n-3},\ z_{2n-2},\ z_{2n-1}.$$

And now, at last, we see that the sought integral of the total differential equation (A^3) of the 1st order and 1st degree between $2n-1$ variables may in general be represented by the following system of n equations, involving one arbitrary function of $n-1$ variables:

$$\overset{n-1}{z}_1 = \chi(\overset{n-2}{z}_3,\ \overset{n-3}{z}_5,\ \dots \overset{1}{z}_{2n-3},\ z_{2n-1}); \tag{X^3}$$

$$\overset{n-1}{V}_3 = \chi'(\overset{n-2}{z}_3),\ \overset{n-1}{V}_5 = \chi'(\overset{n-3}{z}_5),\ \dots \overset{n-1}{V}_{2n-1} = \chi'(z_{2n-1}); \tag{Y^3}$$

in which the coefficients $\overset{n-1}{V}_3$, $\overset{n-1}{V}_5$, ... $\overset{n-1}{V}_{2n-1}$ are the known functions just now described, and in which the $2n-3$ new variables $\overset{n-1}{z}_1, \overset{n-2}{z}_2, \overset{n-2}{z}_3, \overset{n-3}{z}_4, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-4}, \overset{1}{z}_{2n-3}$ are themselves known functions of the $2n-1$ old variables $z_1, z_2, \dots z_{2n-1}$, which entered into the equation (A^3); while the function χ of the $n-1$ variables $\overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ remains entirely arbitrary. We see also that the equation (A^3) may be integrated by another important (though only particular) system of n equations, namely, the following:

$$\overset{n-1}{z}_1 = e_1,\ \overset{n-2}{z}_3 = e_2,\ \overset{n-3}{z}_5 = e_3,\ \dots \overset{1}{z}_{2n-3} = e_{n-1},\ z_{2n-1} = e_n; \tag{Z^3}$$

in which the n expressions $\overset{n-1}{z}_1, \overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ are equated to n arbitrary constants $e_1, e_2, e_3, \dots e_{n-1}, e_n$, instead of the first being treated as an arbitrary function of the rest. And to apply all this research respecting the integration of the equation (A^3) [or (G^2)] to the problem of integrating any partial differential equation of the 1st order

$$0 = F(\phi, x_1, \dots x_n, y_1, \dots y_n); \tag{X^1}$$

we have only to consider $\overset{n-1}{z}_1, \overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ and $\overset{n-1}{V}_3, \overset{n-1}{V}_5, \dots \overset{n-1}{V}_{2n-1}$ as equal to $2n-1$ known functions of the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$; the forms of these functions being supposed to be discovered by combining the processes of the present and of the preceding article. For thus we shall either have, at once, the general integral of the proposed partial differential equation, with an arbitrary function χ of $n-1$ variables, by eliminating, or by conceiving eliminated, the n partial differential coefficients $y_1, \dots y_n$ of the sought function ϕ between the proposed equation (X^1) itself and the general integral system (X^3), (Y^3), which is properly the method of Pfaff; or we shall have, if we prefer it, a simpler but only particular integral of the same partial differential equation (X^1), involving no arbitrary function but involving n arbitrary constants, by eliminating the same n partial differential coefficients $y_1, \dots y_n$ between the equations (X^1) and (Z^3); and then from this particular integral the general one can be easily deduced.

12. In order to apply the general method, explained in the two last articles, to the case where the relation

$$0 = F(\phi, x_1, \dots x_n, y_1, \dots y_n), \tag{X^1}$$

between the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$, is supposed to result by elimination of n other variables, such as $u_1, \dots u_n$, from a system of $n+1$ given equations of the following forms

$$0=f(\phi, x_1, \dots x_n, u_1, \dots u_n), \tag{A^4}$$

and

$$\left.\begin{aligned} y_1 &= \frac{f'(u_1)}{u_1 f'(u_1)+\dots+u_n f'(u_n)}, \\ y_n &= \frac{f'(u_n)}{u_1 f'(u_1)+\dots+u_n f'(u_n)}, \end{aligned}\right\} \tag{B^4}$$

we must endeavour to deduce from these equations expressions for the $2n+1$ partial derivatives, or partial differential coefficients,

$$F'(\phi),\ F'(x_1),\ \dots F'(x_n),\ F'(y_1),\ \dots F'(y_n);$$

or at least for their $2n$ ratios, because these ratios enter into the formulae (V^2), (W^2). By the nature of these $2n+1$ partial derivatives, they are equal or proportional to the coefficients of that limiting and linear relation which connects the otherwise arbitrary increments that may be attributed to the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$, consistently with the relation (X^1), when these increments are made smaller and smaller; in such a manner, that if we represent these increments respectively by $\epsilon\xi, \epsilon\xi_1, \dots \epsilon\xi_n, \epsilon\eta_1, \dots \epsilon\eta_n$, ϵ being any small multiplier, and if, after developing the equation

$$0=F(\phi+\epsilon\xi, x_1+\epsilon\xi_1, \dots x_n+\epsilon\xi_n, y_1+\epsilon\eta_1, \dots y_n+\epsilon\eta_n) \tag{C^4}$$

according to the ascending powers of ϵ and suppressing the part of the development which vanishes by (X^1), we then divide by ϵ and finally make ϵ tend to zero, we shall thus be conducted to the limiting and linear relation following, between $\xi, \xi_1, \dots \xi_n, \eta_1, \dots \eta_n$:

$$0=\xi F'(\phi)+\xi_1 F'(x_1)+\dots+\xi_n F'(x_n)+\eta_1 F'(y_1)+\dots+\eta_n F'(y_n). \tag{D^4}$$

The equation (A^4) conducts in like manner to this other limiting and linear relation,

$$0=\xi f'(\phi)+\xi_1 f'(x_1)+\dots+\xi_n f'(x_n)+v_1 f'(u_1)+\dots+v_n f'(u_n), \tag{E^4}$$

if, besides changing $\phi, x_1, \dots x_n$ to $\phi+\epsilon\xi, x_1+\epsilon\xi_1, \dots x_n+\epsilon\xi_n$, we change also $u_1, \dots u_n$ to $u_1+\epsilon v_1, \dots u_n+\epsilon v_n$ and make ϵ tend to 0 as before; and, by making the same changes in the n equations (B^4), we should obtain n other limiting and linear relations between $\xi, \xi_1, \dots \xi_n$, $v_1, \dots v_n$ and $\eta_1, \dots \eta_n$, which, when combined with the relation (E^4), would conduct by elimination to a new limiting and linear relation between $\xi, \xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, not involving $v_1, \dots v_n$; but we may simplify this elimination by observing that the equations (B^4) give

$$1=y_1 u_1+\dots+y_n u_n, \tag{F^4}$$

and therefore at the limit (corresponding to the limiting value 0 of ϵ)

$$0=\eta_1 u_1+\dots+\eta_n u_n+v_1 y_1+\dots+v_n y_n, \tag{G^4}$$

while the limiting relation (E^4) gives, on account of the equations (B^4),

$$0=\xi f'(\phi)+\xi_1 f'(x_1)+\dots+\xi_n f'(x_n)+(v_1 y_1+\dots+v_n y_n)\{u_1 f'(u_1)+\dots+u_n f'(u_n)\}. \tag{H^4}$$

We may therefore eliminate all the n variables $v_1, \dots v_n$ together by eliminating the one combination $v_1 y_1+\dots+v_n y_n$ between the two relations (G^4) and (H^4); and thus we obtain

$$\left.\begin{aligned} 0= &-\{\xi f'(\phi)+\xi_1 f'(x_1)+\dots+\xi_n f'(x_n)\} \\ &+(\eta_1 u_1+\dots+\eta_n u_n)\{u_1 f'(u_1)+\dots+u_n f'(u_n)\}. \end{aligned}\right\} \tag{I^4}$$

Comparing this with the relation (D^4), we find, in general, the following expressions for the sought partial differential coefficients of the function F:

$$F'(\phi) = -Mf'(\phi), \quad F'(x_1) = -Mf'(x_1), \ldots F'(x_n) = -Mf'(x_n), \tag{K^4}$$

and

$$\left.\begin{aligned} F'(y_1) &= Mu_1\{u_1f'(u_1)+\ldots+u_nf'(u_n)\}, \\ F'(y_n) &= Mu_n\{u_1f'(u_1)+\ldots+u_nf'(u_n)\}; \end{aligned}\right\} \tag{L^4}$$

the multiplier M remaining indeed still undetermined but disappearing from the expressions of the ratios of the coefficients $F'(\phi)$, &c., which alone are required for our purpose. For, since the expressions (L^4) give by (F^4)

$$M = \frac{y_1F'(y_1)+\ldots+y_nF'(y_n)}{u_1f'(u_1)+\ldots+u_nf'(u_n)}, \tag{M^4}$$

and therefore

$$\left.\begin{aligned} &\frac{F'(y_1)}{y_1F'(y_1)+\ldots+y_nF'(y_n)} = u_1, \\ &\cdots\cdots\cdots \\ &\frac{F'(y_n)}{y_1F'(y_1)+\ldots+y_nF'(y_n)} = u_n \end{aligned}\right\} \tag{N^4}$$

and also, by (K^4),

$$\left.\begin{aligned} \frac{F'(\phi)}{y_1F'(y_1)+\ldots+y_nF'(y_n)} &= -\frac{f'(\phi)}{u_1f'(u_1)+\ldots+u_nf'(u_n)}, \\ \frac{F'(x_1)}{y_1F'(y_1)+\ldots+y_nF'(y_n)} &= -\frac{f'(x_1)}{u_1f'(u_1)+\ldots+u_nf'(u_n)}, \\ \frac{F'(x_n)}{y_1F'(y_1)+\ldots+y_nF'(y_n)} &= -\frac{f'(x_n)}{u_1f'(u_1)+\ldots+u_nf'(u_n)}; \end{aligned}\right\} \tag{O^4}$$

we can now easily eliminate all the partial differential coefficients of the function F from the auxiliary system of total differential equations (V^2) and (W^2), and may write these equations as follows:

$$x_1' = u_1\phi', \ldots x_n' = u_n\phi'; \tag{P^4}$$

and

$$\left.\begin{aligned} y_1' &= \frac{\{f'(x_1)+y_1f'(\phi)\}\phi'}{u_1f'(u_1)+\ldots+u_nf'(u_n)}, \\ y_n' &= \frac{\{f'(x_n)+y_nf'(\phi)\}\phi'}{u_1f'(u_1)+\ldots+u_nf'(u_n)}. \end{aligned}\right\} \tag{Q^4}$$

It results from the general theory explained in the 10th article that the equations of this system (P^4) and (Q^4) must be capable of being reduced to $2n-1$ distinct differential equations between $2n$ variables; and, accordingly, we may consider any one of the $2n$ equations of this system as being a consequence of the $2n-1$ other equations of the same system, because the equations (A^4) and (B^4) give this differential relation, analogous to (I^4) and deduced by a similar process:

$$\begin{aligned} 0 = f'(\phi)\,.\,\phi' + f'(x_1)\,.\,x_1' + \ldots + f'(x_n)\,.\,x_n' \\ -(u_1y_1'+\ldots+u_ny_n')\{u_1f'(u_1)+\ldots+u_nf'(u_n)\}; \end{aligned} \tag{R^4}$$

while the same $n+1$ equations, (A^4) and (B^4), enable us to consider any $n+1$ of the $3n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n, u_1, \dots u_n$, for example, the $n+1$ variables $u_1, \dots u_n$ and y_n, as known functions of the $2n$ other variables $\phi, x_1, \dots x_n, y_1, \dots y_{n-1}$. If, then, omitting any one of the $2n$ equations (P^4), (Q^4), for example, the last of these equations, we can integrate the rest as a system of $2n-1$ total differential equations of the first order between $2n$ variables, such as $\phi, x_1, \dots x_n, y_1, \dots y_{n-1}$; and if we then change the $2n-1$ arbitrary constants of this integration, $c_1, \dots c_{2n-1}$, to so many auxiliary variables $z_1, \dots z_{2n-1}$, and differentiate them as such; we shall be able, by what was proved in the tenth article, to transform the partial differential equation of the 1st order (X^1), in which $y_1, \dots y_n$ are the partial differential coefficients of ϕ taken with respect to $x_1, \dots x_n$ and which results by elimination from the system (A^4) and (B^4), into a total differential equation, such as (G^2), of the 1st order and 1st degree between the $2n-1$ auxiliary variables $z_1, \dots z_{2n-1}$. But when, to simplify this preliminary reduction, we select, as we are at liberty to do, the variable ϕ itself for that hitherto undetermined and independent variable x, on which all the rest are conceived to depend in these total differential equations and with respect to which the total derivatives or total differential coefficients, $\phi', x_1', \dots x_n'$, $y_1', \dots y_n'$, have all been imagined to be taken; and when, in consequence, we change ϕ to x and ϕ' to 1, as at the end of the 10th article; the equations (P^4) then become:

$$x_1' = u_1, \ \dots x_n' = u_n; \qquad (S^4)$$

the equation (A^4) reduces itself to the original equation (A) of the 1st article; the equations (B^4) transform themselves to (D^1); and the equations (Q^4) to (O^1); which former equations, (A), (D^1) and (O^1), had been found in the 7th article to compose a system equivalent to the system of the original equation (A) and its principal supplementaries (B); whereas it was for the very purpose of accomplishing or dispensing with the integration of that system of many total differential equations that we were led to desire to integrate the one partial differential equation. Since, then, the preliminary reduction required by the general method of Pfaff (for the transformation of any partial differential equation of the first order between $n+1$ variables to a total differential equation of the 1st order and 1st degree between $2n-1$ other variables) conducts us back, in the research of a principal integral, to that very system of total differential equations with which we originally set out, and requires that those equations should previously be integrated as an auxiliary system, it appears impossible to derive any aid from this method of that eminent mathematician towards completing the solution of the special problem of the present Chapter.

13. Yet some interesting consequences result from the foregoing discussion; and especially the existence of a very intimate connexion between the *general integral* of any proposed partial differential equation of the first order with any number of variables, (a few particular forms being excepted,) and the *principal integral* of a certain total differential equation of the same order, involving the same number of variables; which connexion, if it fails to enable us to discover the latter integral through the former, yet at least allows us to deduce conversely the former from the latter; and gives thereby a new importance to the theory of principal integrals. For it is easy now to perceive (from the investigations of the last article) that in order to integrate any proposed partial differential equation of the 1st order

$$0 = F(\phi, x_1, \dots x_n, y_1, \dots y_n); \qquad (X^1)$$

in which F denotes a known function of $\phi, x_1, \dots x_n, y_1, \dots y_n$, while ϕ denotes a sought function of the n independent variables $x_1, \dots x_n$ and $y_1, \dots y_n$ denote its partial differential coefficients of the 1st order

$$y_1 = \phi'(x_1), \dots y_n = \phi'(x_n); \tag{Y^1}$$

we may in general proceed as follows. We may change the previously dependent variable or function ϕ to a new and independent variable x, on which the previously independent variables $x_1, \dots x_n$ are now to be imagined to depend; so that $x_1, \dots x_n$ are now to be considered as functions of x, of which the 1st derived functions or differential coefficients may be denoted by $x_1', \dots x_n'$. These n derived functions are next to be conceived as being connected with the functions $x_1, \dots x_n$ themselves and with the independent variable x by a differential relation of the form

$$0 = f(x, x_1, \dots x_n, x_1', \dots x_n'), \tag{A}$$

obtained by eliminating $y_1, \dots y_n$ between the $n+1$ equations following,

$$0 = F(x, x_1, \dots x_n, y_1, \dots y_n), \tag{P^1}$$

and

$$\left.\begin{aligned} x_1' &= \frac{F'(y_1)}{y_1 F'(y_1) + \dots + y_n F'(y_n)}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' &= \frac{F'(y_n)}{y_1 F'(y_1) + \dots + y_n F'(y_n)}; \end{aligned}\right\} \tag{Y^2}$$

and then we are to find, if we can, the *principal integral* of the total differential equation (A), which will (by the theory of such principal integrals, explained in the early articles of the present chapter) be of the form

$$0 = \psi(x, x_1, \dots x_n, a, a_1, \dots a_n), \tag{P}$$

$a_1, \dots a_n$ being the initial values of the variables $x_1, \dots x_n$ corresponding to the initial value a of x. This *principal integral* (P) of the total differential equation (A), when it is changed by restoring ϕ for x to the form

$$0 = \psi(\phi, x_1, \dots x_n, a, a_1, \dots a_n), \tag{U^1}$$

is itself (by what has been already proved) a *particular integral* of the proposed partial differential equation (X^1); and since it contains n arbitrary constants $a_1, \dots a_n$ (besides the initial value a of x, which may be treated as $=0$ or any other assumed and absolute number) we may immediately deduce from it the *general integral* of the same proposed equation (X^1) by the process mentioned at the end of the 9th article; namely, by treating these n arbitrary constants as so many connected variables, of which any one, for example a_1, may be considered as an arbitrary function of the rest, and by then eliminating, or conceiving eliminated, the n constants $a_1, \dots a_n$, thus rendered variable, between the particular integral (U^1) and the n equations following:

$$\left.\begin{aligned} &a_1 = \chi(a_2, a_3, \dots a_n), \\ &\chi'(a_2) = -\frac{\psi'(a_2)}{\psi'(a_1)}, \dots \chi'(a_n) = -\frac{\psi'(a_n)}{\psi'(a_1)}. \end{aligned}\right\} \tag{T^4}$$

It results also from what has been shown in former articles that by eliminating $x_1', \dots x_n'$ between the following equations

$$\left.\begin{aligned} y_1 &= \frac{f'(x_1')}{x_1' f'(x_1') + \dots + x_n' f'(x_n')}, \\ & \\ y_n &= \frac{f'(x_n')}{x_1' f'(x_1') + \dots + x_n' f'(x_n')}, \end{aligned}\right\} \qquad (\mathrm{D}^1)$$

with the help of the relation (A), we should (in general) return to the equation (P^1); or, putting ϕ for x, to the proposed partial differential equation (X^1): so that the supposition made respecting that equation at the beginning of the 12th article, namely, that it results by elimination from relations of the forms (A^4) and (B^4), is generally permitted and ought not to be considered as restricting to any particular class the form of the function F, or of the partial differential equation. However, like most (and perhaps all) other general methods, the method of integration proposed in the present article is subject to some particular exceptions, among which the most important is the case where the proposed partial differential equation (X^1), or the proposed function F, is *linear* with respect to all the n partial differential coefficients $y_1, \dots y_n$ of the sought function ϕ; so that the n derivatives $F'(y_1), \dots F'(y_n)$ and also the combination $y_1 F'(y_1) + \dots + y_n F'(y_n)$ are all, either immediately or at least in virtue of the proposed equation $F = 0$, expressed as known functions of the sought function ϕ (or x) and of the n independent variables $x_1, \dots x_n$, not involving $y_1, \dots y_n$. For then the variables $y_1, \dots y_n$ disappear from each of the n equations (V^2), and therefore we cannot deduce from the system (X^1), (Y^2), in this particular case, (although we can in general,) any one determinate relation, such as (A), between $x, x_1, \dots x_n, x_1', \dots x_n'$, to the exclusion of all other such relations; since in the present particular case we have many different relations of that form: whereas the determinateness or uniqueness of the relation thus deduced is essential to the success of our method. On the other hand, in this particular and simple case, when the proposed partial differential equation (X^1) is *linear*, we know from the researches of Lagrange that a particular and comparatively simple method may be applied, in which the equations (V^2) are still useful as auxiliary relations; and which consists in integrating those relations (V^2) as an auxiliary system of n total differential equations between the $n+1$ variables $x, x_1, \dots x_n$, and in then treating any one of the n arbitrary constants of this particular and auxiliary integration as an arbitrary function of the rest.

14. It may serve to illustrate still more fully the intimate connexion which exists between the theory of the general integral of a partial differential equation and that of the principal integral of a total differential system, and thus to exhibit more plainly the meaning and utility of the latter, if we suppose for a moment that this latter theory is complete and avail ourselves of its assistance to accomplish the several processes required for the completion of the former theory, as set forth in the method of Pfaff.

[Here the manuscript ends.]

XIII.

CALCULUS OF PRINCIPAL RELATIONS

[1836.]

[Note Book 42.]

[CONTENTS

PAGE
1. The principal integral or principal function. 332
2. The partial differential equation satisfied by the principal function. . . 334
3. Solution of the partial differential equation by successive approximation. . 335
4. An alternative method of approximation. 339
5. The first method may be used without forming the partial differential equation. 340
6. The case of two variables. 344
7. Examples.] 348

[The principal integral or principal function.]

(Jan. 20th, 1836.)

[1.] In general let

$$dS = \Phi(x_1, x_2, \ldots, x_i, dx_1, dx_2, \ldots, dx_i), \tag{1}$$

this function Φ being homogeneous of the first dimension with respect to $dx_1, dx_2, \ldots, dx_i$, so that*

$$dS = dx_1\frac{\delta dS}{\delta dx_1} + dx_2\frac{\delta dS}{\delta dx_2} + \ldots + dx_i\frac{\delta dS}{\delta dx_i}. \tag{2}$$

Then, by the first expression for dS,

$$\delta dS = \frac{\delta dS}{\delta x_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta x_i}\delta x_i + \frac{\delta dS}{\delta dx_1}\delta dx_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta dx_i; \tag{3}$$

and, by the second expression for dS,

$$\delta dS = dx_1\delta\frac{\delta dS}{\delta dx_1} + \ldots + dx_i\delta\frac{\delta dS}{\delta dx_i} + \frac{\delta dS}{\delta dx_1}\delta dx_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta dx_i; \tag{4}$$

and therefore, by comparing these equations, we find

$$0 = \frac{\delta dS}{\delta x_1}\delta x_1 - \delta\frac{\delta dS}{\delta dx_1}dx_1 + \ldots + \frac{\delta dS}{\delta x_i}\delta x_i - \delta\frac{\delta dS}{\delta dx_i}dx_i. \tag{5}$$

Also, by (3),

$$\begin{aligned}\delta S = \int\delta dS = \Delta\left(\frac{\delta dS}{\delta dx_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta x_i\right)\\ -\int\left\{\left(\frac{\delta dS}{\delta x_1} - d\frac{\delta dS}{\delta dx_1}\right)\delta x_1 + \ldots + \left(\frac{\delta dS}{\delta x_i} - d\frac{\delta dS}{\delta dx_i}\right)\delta x_i\right\};\end{aligned} \tag{6}$$

---

* [$\dfrac{\delta dS}{\delta dx_i}$ stands for the partial derivative of dS with respect to dx_i.]

and if we establish the i equations

$$\frac{\delta dS}{\delta x_1} = d\frac{\delta dS}{\delta dx_1}, \ \ldots, \ \frac{\delta dS}{\delta x_i} = d\frac{\delta dS}{\delta dx_i}, \tag{7}$$

(which are, by (5), equivalent only to $i-1$ distinct equations, because the general relation (5) gives, in particular,

$$0 = \left(\frac{\delta dS}{\delta x_1} - d\frac{\delta dS}{\delta dx_1}\right) dx_1 + \ldots + \left(\frac{\delta dS}{\delta x_i} - d\frac{\delta dS}{\delta dx_i}\right) dx_i),$$

the variation δS of the integral $\int dS$ will take the *simplest possible form*, (as being that form which is most independent of the variations $\delta x_1, \delta x_2, \ldots, \delta x_i$, since it depends only on their extreme and not on their intermediate values,) namely the form

$$\delta S = \Delta\left(\frac{\delta dS}{\delta dx_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta x_i\right). \tag{8}$$

We shall call the integral $S = \int dS$, determined in this way, the *principal integral** of the given element dS, or of the function Φ, in equation (1) and shall denote it, for distinction, by the symbolic expression

$$S = \underline{\int} dS = \underline{\int} \Phi\,(x_1, \ldots, x_i, dx_1, \ldots, dx_i), \tag{9}$$

drawing a stroke under the sign $\int$ of integration.

If we denote by $a_1, a_2, \ldots, a_i$ the initial values (or values at the first limit of the integral) of the i variables $x_1, x_2, \ldots, x_i$, if also we put for abridgement

$$\frac{\delta dS}{\delta dx_1} = y_1, \ \frac{\delta dS}{\delta dx_2} = y_2, \ \ldots, \ \frac{\delta dS}{\delta dx_i} = y_i \tag{10}$$

and denote the initial values of $y_1, \ldots, y_i$ by $b_1, \ldots, b_i$, we may consider the *principal integral*, $S = \underline{\int} dS$, as a function of $x_1, x_2, \ldots, x_i, a_1, a_2, \ldots, a_i$, of which the variation is

$$\begin{aligned}\delta S &= \frac{\delta S}{\delta x_1}\delta x_1 + \ldots + \frac{\delta S}{\delta x_i}\delta x_i + \frac{\delta S}{\delta a_1}\delta a_1 + \ldots + \frac{\delta S}{\delta a_i}\delta a_i \\ &= y_1\delta x_1 + \ldots + y_i\delta x_i - b_1\delta a_1 - \ldots - b_i\delta a_i;\end{aligned} \tag{11}$$

so that we have the $2i$ following equations:

$$y_1 = \frac{\delta S}{\delta x_1}, \quad y_i = \frac{\delta S}{\delta x_i}, \tag{12}$$

$$b_1 = -\frac{\delta S}{\delta a_1}, \ \ldots, \ b_i = -\frac{\delta S}{\delta a_i}. \tag{13}$$

If the form of the function S, as depending on $x_1, \ldots, x_i, a_1, \ldots, a_i$, were known, we could substitute it in the i equations (13) and thus transform them into i relations between the i varying or final quantities $x_1, \ldots, x_i$, and the $2i$ initial data $a_1, \ldots, a_i, b_1, \ldots, b_i$, which i relations, with $2i$ arbitrary constants, would be forms for the i integrals of the i ordinary differential equations of the second order (7). And therefore the i relations between the $3i$ quantities $x_1, \ldots, x_i, a_1, \ldots, a_i, b_1, \ldots, b_i$, which might be found in one way by integrating the i ordinary differential equations of the second order (7), may also be deduced in another way from the one *principal relation* between the *principal function* S and the $2i$ quantities $x_1, \ldots, x_i, a_1, \ldots, a_i$ by taking the partial differential coefficients (of the first order) of that one principal function with respect to the initial variables $a_1, \ldots, a_i$ and then equating these coefficients to $-b_1, \ldots, -b_i$ respectively; which is my chief result respecting the properties of this principal integral S, considered as depending on its limits, and my chief reason for calling that integral a *principal function*; and for giving to that new branch of Algebra, which proposes by new methods to find and to use the form of this principal function, the name of the CALCULUS OF PRINCIPAL RELATIONS.

---

* [The definition of S is, of course, exactly analogous to that of the principal function in dynamics, to which, in fact, it would reduce if $\Phi = L\,dt$, where L is the Lagrangian of the dynamical system.]

[*The partial differential equation satisfied by the principal function.*]

(Jan. 21st, 1836.)

[2.] Among the chief methods for finding the form of the Principal Function S is the following, by a partial differential equation of the first order or by a pair of such equations. Since $y_1, y_2, \ldots, y_i$ are functions only of the ratios of $dx_1, \ldots, dx_i$, we can in general eliminate these $i-1$ ratios and obtain one relation between $y_1, \ldots, y_i$, involving also in general $x_1, \ldots, x_i$ and depending for its form upon the form of dS or of the function Φ in (1); and we may represent this relation as follows:

$$0 = \Psi(y_1, \ldots, y_i, x_1, \ldots, x_i). \tag{14}$$

In like manner we have by considering initial values

$$0 = \Psi(b_1, \ldots, b_i, a_1, \ldots, a_i), \tag{15}$$

the form of the function Ψ being the same as in (14). And if in these relations we substitute for $y_1, \ldots, y_i$ and $b_1, \ldots, b_i$ their values (12) and (13), we obtain the two partial differential equations

$$0 = \Psi\left(\frac{\delta S}{\delta x_1}, \ldots, \frac{\delta S}{\delta x_i}, x_1, \ldots, x_i\right) \tag{16}$$

and

$$0 = \Psi\left(-\frac{\delta S}{\delta a_1}, \ldots, -\frac{\delta S}{\delta a_i}, a_1, \ldots, a_i\right). \tag{17}$$

In integrating these equations we are to determine the arbitrary functions which may present themselves by the following conditions.

First, S must vanish when $x_1 - a_1, x_2 - a_2, \ldots, x_i - a_i$ all vanish—at least that form of S which corresponds to moderate values of those increments, and indeed every form of S excepting those cases of periodicity in which $x_1, x_2, \ldots, x_i$, being considered as functions of some one indefinitely and continuously increasing variable t, acquire all together the same values $a_1, \ldots, a_i$ for some new value $t = t_2$ which they had for the old or original value $t = t_1$. For, generally, if $x_1, x_2, \ldots, x_i$ be considered as so many functions of t while $a_1, a_2, \ldots a_i$ are considered as the values to which those functions reduce when t is made equal to 0, and if therefore the principal integral S be put under the form

$$S = \int_{t_1}^{t_2} \Phi(x_1, \ldots\ x_i, x'_1, \ldots, x'_i)\, dt, \tag{18}$$

in which

$$= \frac{dx_1}{dt}, \qquad x'_i = \frac{dx_i}{dt} \tag{19}$$

then the function S by its integral nature must vanish when $t = t_1$. It is important to observe that the value of the integral S is not affected by the arbitrary form of x_i as a function of t, if the forms of $x_1, \ldots, x_{i-1}$ be deduced from this by the differential equations (7) and if the conditions at the limits be satisfied.

Secondly, at the origin of the progression, that is, when $t=t_1$, the general values of the partial differential coefficients $\frac{\delta S}{\delta x_1}, \dots, \frac{\delta S}{\delta x_i}$ and at the same time those of $-\frac{\delta S}{\delta a_1}, \dots, -\frac{\delta S}{\delta a_i}$ must reduce to those functions of $a_1, \dots, a_i$ and of the ratios of $x_1-a_1, \dots, x_i-a_i$ which may be otherwise deduced from the general values of $b_1, \dots, b_i$ by changing the ratios of $da_1, \dots, da_i$ to the ratios of $x_1-a_1, \dots, x_i-a_i$, or from the general values of $y_1, \dots, y_i$ by changing the ratios of $dx_1, \dots, dx_i$ to those of $x_1-a_1, \dots, x_i-a_i$ and at the same time changing $x_1, \dots, x_i$ to $a_1, \dots, a_i$.

Thirdly—and this condition includes the two former ones—at the origin of the progression or first limit of the integration $(t=t_1)$ the principal function or integral S must bear the (nascent) ratio of unity or equality to the function formed from dS by changing the differentials $dx_1, \dots, dx_i$ to the increments $x_1-a_1, \dots, x_i-a_i$ and by changing $x_1, \dots, x_i$ themselves to $a_1, \dots, a_i$; that is,

$$\lim_{t=t_1} \frac{S}{t} = \lim_{t=t_1} \Phi \left(a_1, a_2, \dots \qquad\qquad \frac{x_i-a_i}{t} \right. \tag{20}$$

or, in other symbols,

$$1 = \lim_{t=t_1} \frac{S}{\Phi(a_1, \dots, a_i, x_1-a_1, \dots, x_i-a_i)}. \tag{21}$$

[*Solution of the partial differential equation by successive approximation.**]

[**3.**] We may in general consider S as a function of $a_1, a_2, \dots, a_{i-1}, a_i, \frac{x_1-a_1}{x_i-a_i}, \dots, \frac{x_{i-1}-a_{i-1}}{x_i-a_i}$, x_i-a_i, and for small or moderate values of $t-t_1$ and of $x_1-a_1, \dots, x_i-a_i$ we may in general develope this function according to ascending integer powers of the small or moderate increment x_i-a_i (setting aside singular exceptions) in a series of the form

$$S = A(x_i-a_i) + B(x_i-a_i)^2 + C(x_i-a_i)^3 + \&c., \tag{22}$$

which may also be thus written, more simply and symmetrically,

$$S = S_1 + S_2 + S_3 + \&c., \tag{23}$$

S_n being a homogeneous function of the nth dimension of the i increments $x_1-a_1, \dots, x_i-a_i$, involving also in general $a_1, \dots, a_i$. We may now conceive this expression substituted in the partial differential equation (16) so as to give an equation of the following form:

$$0 = \Psi \left\{ \frac{\delta S_1}{\delta x_1} + \frac{\delta S_2}{\delta x_1} + \&c., \dots, \frac{\delta S_1}{\delta x_2} + \frac{\delta S_2}{\delta x_2} + \&c., \dots, \frac{\delta S_1}{\delta x_i} + \frac{\delta S_2}{\delta x_i} + \&c., \right. \\ \left. a_1 + (x_1-a_1), \dots, a_i + (x_i-a_i) \right\}, \tag{24}$$

in which $\frac{\delta S_n}{\delta x_k}$ is a homogeneous function of dimension $n-1$ of the increments $x_1-a_1, \dots, x_i-a_i$. And we may in general develope this equation (24) by Taylor's theorem as follows:

$$0 = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \&c., \tag{25}$$

in which Ψ_n is homogeneous of dimension n with respect to the increments $x_1-a_1, \dots, x_i-a_i$; and then may deduce from it the following indefinite series of separate equations in partial differential coefficients of the first order,

$$0 = \Psi_0, \quad 0 = \Psi_1, \quad 0 = \Psi_2, \quad 0 = \Psi_3, \quad \&c. \tag{26}$$

---

* [See Appendix, Note 9, p. 631.]

To develope these equations, let us write generally

$$\begin{aligned}
&\Psi\,(b_1+\beta_1,\, b_2+\beta_2,\, \ldots,\, b_i+\beta_i,\, a_1+\alpha_1,\, a_2+\alpha_2,\, \ldots,\, a_i+\alpha_i)\\
&\quad=\Psi\,(b_1,\, b_2,\, \ldots,\, b_i,\, a_1,\, a_2,\, \ldots,\, a_i)+\beta_1\Psi'\,(b_1)+\beta_2\Psi'\,(b_2)+\ldots+\beta_i\Psi'\,(b_i)\\
&\qquad+\alpha_1\Psi'\,(a_1)+\alpha_2\Psi'\,(a_2)+\ldots+\alpha_i\Psi'\,(a_i)+\tfrac{1}{2}\beta_1^2\Psi''\,(b_1)+\beta_1\beta_2\Psi^{\prime,\prime}\,(b_1,b_2)+\tfrac{1}{2}\beta_2^2\Psi''\,(b_2)+\ldots\\
&\qquad+\tfrac{1}{6}\beta_1^3\Psi'''\,(b_1)+\tfrac{1}{2}\beta_1^2\beta_2\Psi^{\prime\prime,\prime}\,(b_1,b_2)+\tfrac{1}{2}\beta_1\beta_2^2\Psi^{\prime,\prime\prime}\,(b_1,b_2)+\&c. \qquad (27)
\end{aligned}$$

Adopting this notation which has been already used for similar purposes by Lagrange and other mathematicians, this second side of equation (24) may be thus developed:

$$\begin{aligned}
&\Psi\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_2}{\delta x_1}+\&c.,\, \ldots,\, \frac{\delta S_1}{\delta x_i}+\frac{\delta S_2}{\delta x_i}+\&c.,\, a_1+x_1-a_1,\, \ldots,\, a_i+x_i-a_i\right)\\
&\quad=\Psi\left(\frac{\delta S_1}{\delta x_1},\, \frac{\delta S_1}{\delta x_2},\, \ldots,\, \frac{\delta S_1}{\delta x_i},\, a_1,\, a_2,\, \ldots,\, a_i\right)\\
&\qquad\qquad+\Psi'\,(a_1)\,(x_1-a_1)+\Psi'\,(a_2)\,(x_2-a_2)+\ldots+\Psi'\,(a_i)\,(x_i-a_i)\\
&\quad+\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\left(\frac{\delta S_2}{\delta x_1}+\frac{\delta S_3}{\delta x_1}+\&c.\right)+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\left(\frac{\delta S_2}{\delta x_2}+\frac{\delta S_3}{\delta x_2}+\&c.\right)+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}+\frac{\delta S_3}{\delta x_i}+\&c.\right)\\
&\quad+\tfrac{1}{2}\Psi''\,(a_1)\,(x_1-a_1)^2+\Psi^{\prime,\prime}\,(a_1,a_2)\,(x_1-a_1)\,(x_2-a_2)+\ldots+\tfrac{1}{2}\Psi''\,(a_i)\,(x_i-a_i)^2\\
&\quad+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\, a_1\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)(x_1-a_1)+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\, a_2\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)(x_2-a_2)\\
&\qquad\qquad+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_2},\, a_2\right)\left(\frac{\delta S_2}{\delta x_2}+\&c.\right)(x_2-a_2)+\&c.\\
&\quad+\tfrac{1}{2}\Psi''\;\frac{\delta S_1}{\delta x_1}\Bigr)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)^2+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\, \frac{\delta S_1}{\delta x_2}\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)\Bigl(\frac{\delta S_2}{\delta x_2}+\&c.\\
&\qquad\qquad+\ldots+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}+\&c.\right)^2+\&c.; \qquad (28)
\end{aligned}$$

and thus the three first partial differential equations of the series (28) may be developed as follows:

$$0=(\Psi_0=)\,\Psi\left(\frac{\delta S_1}{\delta x_1},\frac{\delta S_1}{\delta x_2},\, \ldots,\, \frac{\delta S_1}{\delta x_i},\, a_1,\, a_2,\, \ldots,\, a_i\right); \qquad (29)$$

$$\begin{aligned}
0=(\Psi_1=)\,&\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_2}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_2}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_2}{\delta x_i}\\
&+\Psi'\,(a_1)\,(x_1-a_1)+\Psi'\,(a_2)\,(x_2-a_2)+\ldots+\Psi'\,(a_i)\,(x_i-a_i); \qquad (30)
\end{aligned}$$

$$\begin{aligned}
0=(\Psi_2=)\,&\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_3}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_3}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_3}{\delta x_i}\\
&+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_1}\right)\left(\frac{\delta S_2}{\delta x_1}\right)^2+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_2}{\delta x_1}\frac{\delta S_2}{\delta x_2}+\ldots+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}\right)^2\\
&+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\, a_1\right)\frac{\delta S_2}{\delta x_1}(x_1-a_1)+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_1},\, a_2\right)\frac{\delta S_2}{\delta x_2}(x_2-a_2)+\ldots+\Psi^{\prime,\prime}\left(\frac{\delta S_1}{\delta x_i},\, a_i\right)\frac{\delta S_2}{\delta x_i}(x_i-a_i)\\
&+\tfrac{1}{2}\Psi''\,(a_1)\,(x_1-a_1)^2+\Psi^{\prime,\prime}\,(a_1,\, a_2)\,(x_1-a_1)\,(x_2-a_2)+\ldots+\tfrac{1}{2}\Psi''\,(a_i)\,(x_i-a_i)^2; \qquad (31)
\end{aligned}$$

and the others may be similarly developed.

We have next to integrate these equations; at least to discover functions S_1, S_2, S_3, &c. which shall satisfy them. It might seem that this integration would introduce in general an arbitrary function for every differential equation; and thus an infinite number of arbitrary functions into the general expression of the sum $S_1+S_2+S_3+\&\text{c.}=S$; but the conditions already mentioned enable us to foresee that the form of S_1 required for our present purpose must be

$$S_1=\Phi(a_1, a_2, \ldots, a_i, x_1-a_1, x_2-a_2, \ldots, x_i-a_i), \tag{32}$$

which form accordingly may be easily shown to satisfy the partial differential equation (29) (see below); and then the remaining functions S_2, S_3, &c. may be determined, as we are about to prove, by the remaining equations (30), (31) ... without any new integrations being required —a result of great importance in the Calculus of Principal Relations as enabling us to develope the Principal Function without ambiguity for the case of moderate increments of the variables $x_1, \ldots, x_i$.

To show first of all that the form (32) for S_1 satisfies equation (29), we may observe that this form gives by partial differentiation for $\frac{\delta S_1}{\delta x_1}, \ldots, \frac{\delta S_1}{\delta x_i}$ the same functions of $a_1, \ldots, a_i$ and of the ratios of $x_1-a_1, \ldots, x_i-a_i$, which might be otherwise deduced from the expressions for $b_1, \ldots, b_i$ by changing the ratios of $da_1, \ldots, da_i$ to the ratios of $x_1-a_1, \ldots, x_i-a_i$; since then we had, independently of the ratios of $da_1, \ldots, da_i$, the relation (15) between $b_1, \ldots, b_i, a_1, \ldots, a_i$ we must also have, independently of the ratios of $x_1-a_1, \ldots, x_i-a_i$, the relation (29) between

$$\frac{\delta S_1}{\delta x_1}, \ldots, \frac{\delta S_1}{\delta x_i}, a_1,$$

(Again, the equation (5) shows that the variations $\delta x_1, \ldots, \delta x_i, \delta y_1,$, δy_i are connected by the relation

$$0=\frac{\delta dS}{\delta x_1}\delta x_1-dx_1\delta y_1+\ldots+\frac{\delta dS}{\delta x_i}\delta x_i-dx_i\delta y_i, \tag{33}$$

which may by (7) be put in the form

$$0=dy_1\delta x_1-dx_1\delta y_1+\ldots+dy_i\delta x_i-dx_i\delta y_i; \tag{34}$$

since then, by (14), we have

$$0=\Psi'(x_1)\delta x_1+\Psi'(y_1)\delta y_1+\ldots+\Psi'(x_i)\delta x_i+\Psi'(y_i)\delta y_i, \tag{35}$$

and since these two last expressions must both be satisfied independently of any other relation between $\delta x_1, \ldots, \delta x_i$ and $\delta y_1, \ldots, \delta y_i$, we see that we must have, separately,

$$\Psi'(y_1)=-Lx_1', \quad \Psi'(y_2)=-Lx_2', \quad \ldots, \Psi'(y_i)=-Lx_i', \tag{36}$$

x_1', etc. having the meanings (19) and L being some common multiplier; and in like manner, L being still the same common multiplier, we have

$$\Psi'(x_1)=+Ly_1', \quad \Psi'(x_2)=+Ly_2', \quad \ldots, \quad \Psi'(x_i)=+Ly_i', \tag{37}$$

in which

$$y_1'=\frac{dy_1}{dt}, \ \ldots, \ y_i'=\frac{dy_i}{dt}. \tag{38}$$

We might proceed in this way to determine the ratios of $\Psi'\left(\frac{\delta S_1}{\delta x_1}\right), \ldots, \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)$ and to show that they are the same as the ratios of $x_1-a_1, \ldots, x_i-a_i$, but the following method is more simple.)

Since S_1 is a homogeneous function of the first dimension of x_1-a_1 x_i-a_i, it must satisfy the condition

$$S_1=(x_1-a_1)\frac{\delta S_1}{\delta x_1}+(x_2-a_2)\frac{\delta S_1}{\delta x_2}+\ldots+(x_i-a_i)\frac{\delta S_1}{\delta x_i}; \tag{39}$$

which gives, by being varied with respect to $x_1, \ldots, x_i$,

$$0=(x_1-a_1)\,\delta\frac{\delta S_1}{\delta x_1}+(x_2-a_2)\,\delta\frac{\delta S_1}{\delta x_2}+\ldots+(x_i-a_i)\,\delta\frac{\delta S_1}{\delta x_i}, \tag{40}$$

the quantities $a_1, \ldots, a_i$ being treated as constants. But on this last supposition, the equation (29) gives

$$0=\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\delta\frac{\delta S_1}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\delta\frac{\delta S_1}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\delta\frac{\delta S_1}{\delta x_i}; \tag{41}$$

and since these two linear relations (40) and (41) between the variations $\delta\frac{\delta S_1}{\delta x_1}, \ldots, \delta\frac{\delta S_1}{\delta x_i}$ must in general hold together and be equivalent only to one relation, the coefficients in the one must be proportional to those in the other; so that, in general,

$$\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)=\lambda(x_1-a_1),\ \Psi'\left(\frac{\delta S_1}{\delta x_2}\right)=\lambda(x_2-a_2),\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)=\lambda(x_i-a_i), \tag{42}$$

λ being some common multiplier of which the form can be found when those of S_1 and Ψ are known.

Whatever this form of λ may be, we see now that

$$\begin{aligned}\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_n}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_n}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_n}{\delta x_i}&\\ =\lambda\left\{(x_1-a_1)\frac{\delta S_n}{\delta x_1}+(x_2-a_2)\frac{\delta S_n}{\delta x_2}+\ldots+(x_i-a_i)\frac{\delta S_n}{\delta x_i}\right\}&=\lambda n S_n,\end{aligned} \tag{43}$$

on account of the homogeneous form of S_n. Hence the equations (30), (31) and the other similar equations for S_4, S_5, &c. will determine (in general) the several functions S_2, S_3, S_4, S_5, &c. without any integration being required after the form of S_1 has been found by the equation (32): which is one of the most useful theorems in this Calculus.

In particular, equation (30) gives

$$S_2=-\frac{1}{2\lambda}\{\Psi'(a_1)(x_1-a_1)+\Psi'(a_2)(x_2-a_2)+\ldots+\Psi'(a_i)(x_i-a_i)\}. \tag{44}$$

To transform this expression for the first correction S_2 of the first approximate value S_1 of S, we may observe that the equation (39) gives, when varied with respect to all the quantities x_1, &c. and a_1, &c.,

$$0=(x_1-a_1)\,\delta\frac{\delta S_1}{\delta x_1}+\ldots+(x_i-a_i)\,\delta\frac{\delta S_1}{\delta x_i}-\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_1}{\delta a_1}\right)\delta a_1-\ldots-\left(\frac{\delta S_1}{\delta x_i}+\frac{\delta S_1}{\delta a_i}\right)\delta a_i; \tag{45}$$

while the equation (32) gives in like manner

$$0=\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\delta\frac{\delta S_1}{\delta x_1}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\delta\frac{\delta S_1}{\delta x_i}+\Psi'(a_1)\,\delta a_1+\ldots+\Psi'(a_i)\,\delta a_i; \tag{46}$$

and since these two last equations must coincide, we have in general along with the relations (42) the following other relations:

$$\Psi'(a_1) = -\lambda\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_1}{\delta a_1}\right),\ \Psi'(a_2) = -\lambda\left(\frac{\delta S_1}{\delta x_2}+\frac{\delta S_1}{\delta a_2}\right),\ \ldots,\ \Psi'(a_i) = -\lambda\left(\frac{\delta S_1}{\delta x_i}+\frac{\delta S_1}{\delta a_i}\right). \quad (47)$$

And thus the expression (44) transforms itself into the following:

$$S_2 = \tfrac{1}{2}(x_1-a_1)\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_1}{\delta a_1}\right)+\tfrac{1}{2}(x_2-a_2)\left(\frac{\delta S_1}{\delta x_2}+\frac{\delta S_1}{\delta a_2}\right)+\ldots+\tfrac{1}{2}(x_i-a_i)\left(\frac{\delta S_1}{\delta x_i}+\frac{\delta S_1}{\delta a_i}\right). \quad (48)$$

If then we neglect only terms which are of the third dimension with respect to the small increments $x_1-a_1, \ldots, x_i-a_i$, the principal function S may be thus expressed:

$$\begin{aligned} S &= \int \Phi(x_1, x_2, \ldots, x_i, dx_1, dx_2, \ldots, dx_i) \\ &= S_1 + \tfrac{1}{2}(x_1-a_1)\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_1}{\delta a_1}\right)+\ldots+\tfrac{1}{2}(x_i-a_i)\left(\frac{\delta S_1}{\delta x_i}+\frac{\delta S_1}{\delta a_i}\right), \end{aligned} \quad (49)$$

in which, by (32),

$$S_1 = \Phi(a_1, a_2, \ldots, a_i, x_1-a_1, x_2-a_2, \ldots, x_i-a_i).$$

And it is remarkable that in the same order of approximation this expression (49) for the principal function S may be transformed as follows:

$$S = \Phi \quad x_1+a_1 \quad x_2+a_2 \qquad x_i+a_i, \ x_1-a_1,\ x_2-a_2,\ \ldots,\ x_i-a_i \quad (50)$$

[*An alternative method of approximation.*]

[**4.**] Before proceeding further in this integration of the partial differential equation (16), let us observe that if we consider z as an independent and continuously flowing variable on which all the rest depend, and which is $=0$ at the beginning and $=x$ at the end of the progression, we may in general denote the principal function or integral S as follows:

$$S = \int_a^x \Phi\left(z_1, z_2, \ldots, z_i, \frac{dz_1}{dz}, \frac{dz_2}{dz}, \ldots, \frac{dz_i}{dz}\right) dz, \quad (51)$$

$z_1, z_2, \ldots, z_i$ being functions of z which may be thus denoted

$$z_1 = f_1(z),\ z_2 = f_2(z),\ \ldots,\ z_i = f_i(z), \quad (52)$$

and which satisfy the i initial conditions

$$f_1(a) = a_1,\ f_2(a) = a_2,\ \ldots,\ f_i(a) = a_i, \quad (53)$$

and the i final conditions

$$f_1(x) = x_1,\ f_2(x) = x_2,\ \ldots,\ f_i(x) = x_i. \quad (54)$$

And if, as a first approximation, we make the supposition of uniformly flowing values, or linear forms, of the functions $z_1, z_2, \ldots, z_i$ so as to suppose

$$z_1 = a_1 + (z-a)\frac{x_1-a_1}{x-a},\ z_2 = a_2 + (z-a)\frac{x_2-a_2}{x-a},\ \ldots,\ z_i = a_i + (z-a)\frac{x_i-a_i}{x-a} \quad (55)$$

and

$$\frac{dz_1}{dz} = f_1'(z) = \frac{x_1-a_1}{x-a},\ \frac{dz_2}{dz} = f_2'(z) = \frac{x_2-a_2}{x-a},\ \ldots,\ \frac{dz_i}{dz} = f_i'(z) = \frac{x_i-a_i}{x-a} \quad (56)$$

and therefore by (51)

$$S=\int_a^x \Phi\left(a_1+\overline{z-a}\,\frac{x_1-a_1}{x-a},\ \ldots,\ a_i+\overline{z-a}\,\frac{x_i-a_i}{x-a},\ \frac{x_1-a_1}{x-a},\ \ldots,\ \frac{x_i-a_i}{x-a}\right)dz; \tag{57}$$

we find, by developing the coefficient under the integral sign as far as the first power inclusive of $z-a$,

$$\begin{aligned}\frac{dS}{dz}&=\Phi\left(a_1+\overline{z-a}\,\frac{x_1-a_1}{x-a},\ \ldots,\ a_i+\overline{z-a}\,\frac{x_i-a_i}{x-a},\ \frac{x_1-a_1}{x-a},\ \ldots,\ \frac{x_i-a_i}{x-a}\right)\\ &=\Phi\left(a_1,\ \ldots,\ a_i,\ \frac{x_1-a_1}{x-a},\ \ldots,\ \frac{x_i-a_i}{x-a}\right)+\frac{z-a}{x-a}\{\Phi'(a_1)(x_1-a_1)+\ldots+\Phi'(a_i)(x_i-a_i)\},\end{aligned} \tag{58}$$

$\Phi'(a_1),\ \ldots,\ \Phi'(a_i)$ being here formed by varying $\Phi\left(a_1, a_2, \ldots, a_i, \frac{x_1-a_1}{x-a}, \frac{x_2-a_2}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right)$ as if $\frac{x_1-a_1}{x-a}$, etc. were constants; and therefore, by integration,

$$\begin{aligned}S=(x-a)\,\Phi\left(a_1,\ \ldots,\ a_i,\ \frac{x_1-a_1}{x-a},\ \ldots,\ \frac{x_i-a_i}{x-a}\right)&\\ +\tfrac{1}{2}(x-a)\{\Phi'(a_1)(x_1-a_1)+\ldots+\Phi'(a_i)(x_i-a_i)\},&\end{aligned} \tag{59}$$

that is,

$$\begin{aligned}S=\ &\Phi(a_1,\ a_2,\ \ldots,\ a_i,\ x_1-a_1,\ x_2-a_2,\ \ldots,\ x_i-a_i)\\ &+\frac{x_1-a_1}{2}\left(\frac{\delta}{\delta a_1}+\frac{\delta}{\delta x_1}\right)\Phi(a_1,\ a_2,\ \ldots,\ a_i,\ x_1-a_1,\ x_2-a_2,\ \ldots,\ x_i-a_i)\\ &+\&\text{c.}\\ &+\frac{x_i-a_i}{2}\left(\frac{\delta}{\delta a_i}+\frac{\delta}{\delta x_i}\right)\Phi(a_1,\ a_2,\ \ldots,\ a_i,\ x_1-a_1,\ x_2-a_2,\ \ldots,\ x_i-a_i);\end{aligned} \tag{60}$$

which agrees with the expression (49) and is therefore accurate as far as the second dimension inclusive, although $\frac{dz_1}{dz},\ \ldots,\ \frac{dz_i}{dz}$ are not accurate as far as the first dimension inclusive with respect to the small quantities $x_1-a_1,\ x_2-a_2,\ \ldots,\ x_i-a_i$. The theory of this fact will soon be fully explained.*

[*The first method may be used without forming the partial differential equation.*]

(Jan. 22nd, 1836.)

[5.] Proceeding now to equation (31) and seeking to transform the expression which it gives for S_3 into one more commodious and especially into one more closely connected with the form of the original function Φ in the expression for the element dS in (1), we may suppose in general that the equation (29) has been so prepared, by resolving it with respect to $\frac{\delta S_1}{\delta x_i}$, as to be of the form

$$0=-\frac{\delta S_1}{\delta x_i}+\text{funct}^{\text{n}}\left(\frac{\delta S_1}{\delta x_1},\ \frac{\delta S_1}{\delta x_2},\ \ldots,\ \frac{\delta S_1}{\delta x_{i-1}},\ a_1,\ a_2,\ \ldots,\ a_{i-1},\ a_i\right); \tag{61}$$

---

* [This alternative method of finding an approximate expression for S is very similar to that adopted by various writers on Rayleigh's Principle (Rayleigh, *Phil. Trans.* (1870), A, CLXI, p. 77; Ritz, *Crelle* (1908), CXXXV, p. 1). Approximate values which satisfy the end conditions (53) and (54) are substituted for $z_1,\ \ldots,\ z_n$ in the integral (57) and an approximate value thus found for the principal value, which can then be used to solve the original set of differential equations (7).]

and then we shall have

$$\Psi'\left(\frac{\delta S_1}{\delta x_i}\right) = -1; \tag{62}$$

$$\begin{aligned} &\Psi''\left(\frac{\delta S_1}{\delta x_1}\right) = 0,\ \Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_i}\right) = 0,\ \Psi'{,}'\left(\frac{\delta S_1}{\delta x_2}, \frac{\delta S_1}{\delta x_i}\right) = 0,\ \ldots, \\ &\Psi'{,}'\left(\frac{\delta S_1}{\delta x_i}, a_1\right) = 0,\ \ldots,\ \Psi'{,}'\left(\frac{\delta S_1}{\delta x_i}, a_i\right) = 0; \end{aligned} \tag{63}$$

and $\Psi''\left(\frac{\delta S_1}{\delta x_1}\right), \ldots, \Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_1\right), \ldots, \Psi''(a_1), \ldots$ will be the partial differential coefficients of the second order of $\frac{\delta S_1}{\delta x_i}$ considered as a function of $\frac{\delta S_1}{\delta x_1}$, &c. If then we put for abbreviation

$$\frac{\delta S_1}{\delta x_1} = v_1,\ \frac{\delta S_1}{\delta x_2} = v_2,\ \ldots,\ \frac{\delta S_1}{\delta x_i} = v_i, \tag{64}$$

we shall have besides (62) and (63) the expressions

$$\begin{aligned} &\Psi'\left(\frac{\delta S_1}{\delta x_1}\right) = \frac{\delta v_i}{\delta v_1},\ \Psi'\left(\frac{\delta S_1}{\delta x_2}\right) = \frac{\delta v_i}{\delta v_2},\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta x_{i-1}}\right) = \frac{\delta v_i}{\delta v_{i-1}}; \\ &\Psi'\left(\frac{\delta S_1}{\delta a_1}\right) = \frac{\delta v_i}{\delta a_1},\ \Psi'\left(\frac{\delta S_1}{\delta a_2}\right) = \frac{\delta v_i}{\delta a_2},\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta a_i}\right) = \frac{\delta v_i}{\delta a_i}; \end{aligned} \tag{65}$$

$\frac{\delta v_i}{\delta v_1}, \frac{\delta v_i}{\delta a_1}$, &c. denoting here the partial differential coefficients of the function v_i, taken with respect to v_1, a_1, &c.: we have, too,

$$\begin{aligned} &\Psi''\left(\frac{\delta S_1}{\delta x_1}\right) = \frac{\delta^2 v_i}{\delta v_1^2},\ \Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}\right) = \frac{\delta^2 v_i}{\delta v_1 \delta v_2},\ \ldots,\ \Psi''\left(\frac{\delta S_1}{\delta x_{i-1}}\right) = \frac{\delta^2 v_i}{\delta v_{i-1}^2} \\ &\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_1\right) = \frac{\delta^2 v_i}{\delta v_1 \delta a_1},\ \ldots,\ \Psi'{,}'\left(\frac{\delta S_1}{\delta x_{i-1}}, a_i\right) = \frac{\delta^2 v_i}{\delta v_{i-1} \delta a_i}, \\ &\Psi''(a_1) = \frac{\delta^2 v_i}{\delta a_1^2},\ \Psi'{,}'(a_1, a_2) = \frac{\delta^2 v_i}{\delta a_1 \delta a_2},\ \ldots,\ \Psi''(a_i) = \frac{\delta^2 v_i}{\delta a_i^2}; \end{aligned} \tag{66}$$

and it remains to calculate these differential coefficients of the function v_i from those of the function Φ, or S_1, in the expression (1), or (32).

It may somewhat simplify the proceeding if we put for abridgement

$$x_1 - a_1 = u_1,\ x_2 - a_2 = u_2,\ \ldots,\ x_i - a_i = u_i, \tag{67}$$

and therefore by (32)

$$S_1 = \Phi(a_1, a_2, \ldots, a_i, u_1, u_2, \ldots, u_i). \tag{68}$$

This function is homogeneous of the first dimension (as we have seen) with respect to $u_1, u_2, \ldots u_i$; we have therefore the relation

$$S_1 = u_1 v_1 + u_2 v_2 + \ldots + u_i v_i, \tag{69}$$

because by (64) we have

$$v_1 = \frac{\delta S_1}{\delta u_1} = \Phi'(u_1),\ \ldots,\ v_i = \frac{\delta S_1}{\delta u_i} = \Phi'(u_i). \tag{70}$$

Eliminating the ratios of $u_1, u_2, \ldots, u_i$ between these last expressions, we might deduce as before a relation of the form

$$0 = \Psi(\Phi'(u_1), \Phi'(u_2), \ldots, \Phi'(u_i), a_1, a_2, \ldots, a_i) = \Psi(v_1, v_2, \ldots, v_i, a_1, a_2, \ldots, a_i); \tag{71}$$

and might then deduce from this the sought partial differential coefficients $\frac{\delta v_i}{\delta v_1}, \frac{\delta v_i}{\delta v_2}, \ldots$. Without actually performing this elimination (which we cannot perform while we leave the form of Φ undetermined) we may still deduce these differential coefficients as follows:

The complete variation of S_1 is, by (68) and (70),

$$\delta S_1 = v_1 \delta u_1 + v_2 \delta u_2 + \ldots + v_i \delta u_i + \Phi'(a_1)\,\delta a_1 + \Phi'(a_2)\,\delta a_2 + \ldots + \Phi'(a_i)\,\delta a_i; \tag{72}$$

and comparing this with the variation of the expression (69) we find

$$0 = u_1 \delta v_1 + u_2 \delta v_2 + \ldots + u_i \delta v_i - \Phi'(a_1)\,\delta a_1 - \Phi'(a_2)\,\delta a_2 - \ldots - \Phi'(a_i)\,\delta a_i, \tag{73}$$

which gives

$$\frac{\delta v_i}{\delta v_1} = -\frac{u_1}{u_i}, \quad \frac{\delta v_i}{\delta v_2} = -\frac{u_2}{u_i}, \quad \ldots, \quad \frac{\delta v_i}{\delta v_{i-1}} = -\frac{u_{i-1}}{u_i} \tag{74}$$

and

$$\frac{\delta v_i}{\delta a_1} = \frac{\Phi'(a_1)}{u_i}, \quad \frac{\delta v_i}{\delta a_2} = \frac{\Phi'(a_2)}{u_i}, \quad \ldots, \quad \frac{\delta v_i}{\delta a_i} = \frac{\Phi'(a_i)}{u_i}. \tag{75}$$

In this manner then the partial differential coefficients of v_i of the first order are determined.

Proceeding to the second order, how is $\frac{\delta^2 v_i}{\delta v_1^2}$ to be calculated? By supposing $\delta a_1 = 0, \ldots, \delta a_i = 0$, $\delta v_2 = 0, \ldots, \delta v_{i-1} = 0$, then taking the variation of $\frac{\delta v_i}{\delta v_1} = -\frac{u_1}{u_i}$ and dividing it by δv_1. We are therefore to put, by (70),

$$0 = \Phi'{,}'(u_1, u_2)\,\delta u_1 + \Phi''(u_2)\,\delta u_2 + \Phi'{,}'(u_2, u_3)\,\delta u_3 + \ldots + \Phi'{,}'(u_2, u_i)\,\delta u_i,$$
$$0 = \Phi'{,}'(u_1, u_3)\,\delta u_1 + \Phi'{,}'(u_2, u_3)\,\delta u_2 + \Phi''(u_3)\,\delta u_3 + \ldots + \Phi'{,}'(u_3, u_i)\,\delta u_i,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$0 = \Phi'{,}'(u_1, u_{i-1})\,\delta u_1 + \Phi'{,}'(u_2, u_{i-1})\,\delta u_2 + \ldots + \Phi'{,}'(u_{i-1}, u_i)\,\delta u_i,$$

establishing thus $i-2$ relations between $\delta u_1, \delta u_2, \ldots, \delta u_i$ or rather between their $i-1$ ratios, which leave one of these ratios undetermined. We have also

$$\delta v_1 = \Phi''(u_1)\,\delta u_1 + \Phi'{,}'(u_1, u_2)\,\delta u_2 + \ldots + \Phi'{,}'(u_1, u_i)\,\delta u_i,$$

and

$$\delta v_i = \Phi'{,}'(u_1, u_i)\,\delta u_1 + \Phi'{,}'(u_2, u_i)\,\delta u_2 + \ldots + \Phi''(u_i)\,\delta u_i;$$

and hence, by elimination, we can in general express $\delta u_1 - \frac{u_1}{u_i}\delta u_i$ as a linear function of $\delta v_1, \delta v_i$, also $\delta \frac{\delta v_i}{\delta v_1} = -\delta \frac{u_1}{u_i}$ and therefore finally calculate $\frac{\delta^2 v_i}{\delta v_1^2}$.

In general the i equations

$$\delta v_1 = \delta \Phi'(u_1), \quad \delta v_2 = \delta \Phi'(u_2), \quad \ldots, \quad \delta v_i = \delta \Phi'(u_i), \tag{76}$$

or any $i-1$ of them, enable us by elimination to express $\delta \frac{u_1}{u_i}, \ldots, \delta \frac{u_{i-1}}{u_i}$ and consequently

$\delta u_1 - \lambda u_1, \ldots, \delta u_i - \lambda u_i$ (where λ is any arbitrary multiplier) as linear functions of $\delta v_1, \ldots, \delta v_i$, $\delta a_1, \ldots, \delta a_i$; and then the values thus found for $\delta u_1, \ldots, \delta u_i$ are to be substituted in the following expression, which is deduced from (73) and which may be shown (by (73) and by the homogeneous forms of $\Phi'(a_1), \ldots, \Phi'(a_i)$) not to contain the arbitrary multiplier λ:

$$\delta^2 v_i = -\frac{1}{u_i}\{\delta u_1 \delta v_1 + \delta u_2 \delta v_2 + \ldots + \delta u_i \delta v_i - \delta\Phi'(a_1)\,\delta a_1 - \ldots - \delta\Phi'(a_i)\,\delta a_i\}. \tag{77}$$

It only remains therefore to simplify and perform the elimination between the equations (76), which may be thus expanded:

$$\left.\begin{aligned}
\delta v_1 &= \Phi''(u_1)\,\delta u_1 + \Phi^{\prime,\prime}(u_1, u_2)\,\delta u_2 + \ldots + \Phi^{\prime,\prime}(u_1, u_i)\,\delta u_i + \Phi^{\prime,\prime}(u_1, a_1)\,\delta a_1 \\
&\qquad + \ldots + \Phi^{\prime,\prime}(u_1, a_i)\,\delta a_i, \\
\delta v_2 &= \Phi^{\prime,\prime}(u_1, u_2)\,\delta u_1 + \Phi''(u_2)\,\delta u_2 + \ldots + \Phi^{\prime,\prime}(u_2, u_i)\,\delta u_i + \Phi^{\prime,\prime}(u_2, a_1)\,\delta a_1 \\
&\qquad + \ldots + \Phi^{\prime,\prime}(u_2, a_i)\,\delta a_i, \\
\delta v_i &= \Phi^{\prime,\prime}(u_1, u_i)\,\delta u_1 + \Phi^{\prime,\prime}(u_2, u_i)\,\delta u_2 + \ldots + \Phi''(u_i)\,\delta u_i + \Phi^{\prime,\prime}(u_i, a_1)\,\delta a_1 \\
&\qquad + \ldots + \Phi^{\prime,\prime}(u_i, a_i)\,\delta a_i.
\end{aligned}\right\} \tag{78}$$

For this purpose we may employ the relations which result from the homogeneous form of Φ, namely,

$$\Phi = u_1\Phi'(u_1) + u_2\Phi'(u_2) + \ldots + u_i\Phi'(u_i); \tag{79}$$

$$\left.\begin{aligned}
\Phi'(a_1) &= u_1\Phi^{\prime,\prime}(a_1, u_1) + u_2\Phi^{\prime,\prime}(a_1, u_2) + \ldots + u_i\Phi^{\prime,\prime}(a_1, u_i), \\
\Phi'(a_i) &= u_1\Phi^{\prime,\prime}(a_i, u_1) + u_2\Phi^{\prime,\prime}(a_i, u_2) + \ldots + u_i\Phi^{\prime,\prime}(a_i, u_i);
\end{aligned}\right\} \tag{80}$$

and

$$\left.\begin{aligned}
0 &= u_1\Phi''(u_1) + u_2\Phi^{\prime,\prime}(u_1, u_2) + \ldots + u_i\Phi^{\prime,\prime}(u_1, u_i), \\
&\ldots\ldots\ldots\ldots\ldots \\
0 &= u_1\Phi^{\prime,\prime}(u_1, u_i) + u_2\Phi^{\prime,\prime}(u_2, u_i) + \ldots + u_i\Phi''(u_i).
\end{aligned}\right\} \tag{81}$$

Besides, if we take as the arbitrary multiplier λ in the expressions $\delta u_1 - \lambda u_1$, &c. the following (see (72)):

$$\lambda = \frac{1}{S_1}\{\delta S_1 - \Phi'(a_1)\,\delta a_1 - \ldots - \Phi'(a_i)\,\delta a_i\} = \frac{1}{S_1}(v_1\,\delta u_1 + \ldots + v_i\,\delta u_i), \tag{82}$$

we shall have, by (69), the relation

$$0 = v_1(\delta u_1 - \lambda u_1) + v_2(\delta u_2 - \lambda u_2) + \ldots + v_i(\delta u_i - \lambda u_i). \tag{83}$$

We are therefore to determine by elimination, if we can, the i expressions $\delta u_1 - \lambda u_1, \ldots, \delta u_i - \lambda u_i$ as linear functions of $\delta v_1, \delta v_2, \ldots, \delta v_i, \delta a_1, \delta a_2, \ldots, \delta a_i$ by means of this last relation and any $i-1$, or all, of the i equations following:

$$\left.\begin{aligned}
\delta v_1 &= \Phi''(u_1)(\delta u_1 - \lambda u_1) + \ldots + \Phi^{\prime,\prime}(u_1, u_i)(\delta u_i - \lambda u_i) + \Phi^{\prime,\prime}(a_1, u_1)\,\delta a_1 \\
&\qquad + \ldots + \Phi^{\prime,\prime}(a_i, u_1)\,\delta a_i, \\
\delta v_i &= \Phi^{\prime,\prime}(u_1, u_i)(\delta u_1 - \lambda u_1) + \ldots + \Phi''(u_i)(\delta u_i - \lambda u_i) + \Phi^{\prime,\prime}(a_1, u_i)\,\delta a_1 \\
&\qquad + \ldots + \Phi^{\prime,\prime}(a_i, u_i)\,\delta a_i.
\end{aligned}\right\} \tag{84}$$

If we put for abridgement

$$\delta u_1-\lambda u_1=\delta' u_1,\ \ldots,\ \delta u_i-\lambda u_i=\delta' u_i, \tag{85}$$

and

$$\left.\begin{aligned}\delta v_1-\Phi'{,}'(a_1,\ u_1)\,\delta a_1-\ldots-\Phi'{,}'(a_i,\ u_1)\,\delta a_i=\delta' v_1,\\ \delta v_i-\Phi'{,}'(a_1,\ u_i)\,\delta a_1-\ldots-\Phi'{,}'(a_i,\ u_i)\,\delta a_i=\delta' v_i,\end{aligned}\right\} \tag{86}$$

The $i+1$ relations (83) and (84), equivalent only to i distinct ones, will take these simpler forms:

$$0=v_1\delta' u_1+v_2\delta' u_2+\ldots+v_i\delta' u_i, \tag{87}$$

and

$$\left.\begin{aligned}\delta' v_1&=\Phi''(u_1)\,\delta' u_1+\Phi'{,}'(u_1,\ u_2)\,\delta' u_2+\ldots+\Phi'{,}'(u_1,\ u_i)\,\delta' u_i,\\ &\ldots\ldots\ldots\ldots\ldots\\ \delta' v_i&=\Phi'{,}'(u_1,\ u_i)\,\delta' u_1+\Phi'{,}'(u_2,\ u_i)\,\delta' u_2+\ldots+\Phi''(u_i)\,\delta' u_i,\end{aligned}\right\} \tag{88}$$

in which the coefficients are connected by the conditions of homogeneity (81).

[*The case of two variables.*]

[6.] Consider first the case of only *two* variables x_1, x_2 $(i=2)$ with two corresponding variables u_1, u_2, &c. We have now to deduce $\delta' u_1$, $\delta' u_2$, from the three relations following, or from any two of them:

$$0=v_1\delta' u_1+v_2\delta' u_2, \tag{89}$$

and

$$\left.\begin{aligned}\delta' v_1=\Phi''(u_1)\,\delta' u_1+\Phi'{,}'(u_1,\ u_2)\,\delta' u_2,\\ \delta' v_2=\Phi'{,}'(u_1,\ u_2)\,\delta' u_1+\Phi''(u_2)\,\delta' u_2;\end{aligned}\right\} \tag{90}$$

and the coefficients are connected by the 2 relations

$$0=u_1\Phi''(u_1)+u_2\Phi'{,}'(u_1,\ u_2),\quad 0=u_1\Phi'{,}'(u_1,\ u_2)+u_2\Phi''(u_2); \tag{91}$$

we have also

$$u_1v_1+u_2v_2=\Phi=S_1. \tag{92}$$

By (90), we have

$$u_2\delta' v_1-u_1\delta' v_2=\{u_2\Phi''(u_1)-u_1\Phi'{,}'(u_1,\ u_2)\}\,\delta' u_1+\{u_2\Phi'{,}'(u_1,\ u_2)-u_1\Phi''(u_2)\}\,\delta' u_2; \tag{93}$$

therefore, by (89), we have the following expressions for $\delta' u_1$, $\delta' u_2$:

$$\left.\begin{aligned}\delta' u_1&=\frac{v_2(u_2\delta' v_1-u_1\delta' v_2)}{v_2\{u_2\Phi''(u_1)-u_1\Phi'{,}'(u_1,\ u_2)\}-v_1\{u_2\Phi'{,}'(u_1,\ u_2)-u_1\Phi''(u_2)\}},\\ \delta' u_2&=\frac{-v_1(u_2\delta' v_1-u_1\delta' v_2)}{v_2\{u_2\Phi''(u_1)-u_1\Phi'{,}'(u_1,\ u_2)\}-v_1\{u_2\Phi'{,}'(u_1,\ u_2)-u_1\Phi''(u_2)\}}.\end{aligned}\right\} \tag{94}$$

In the common denominator, we have by (91)

$$\frac{\Phi''(u_1)}{u_2^2}=-\frac{\Phi'{,}'(u_1,\ u_2)}{u_1u_2}=\frac{\Phi''(u_2)}{u_1^2}=\frac{\Phi''(u_1)+\Phi''(u_2)}{u_1^2+u_2^2}; \tag{95}$$

and therefore

$$\left.\begin{aligned}u_2\Phi''(u_1)-u_1\Phi'{,}'(u_1,\ u_2)=u_2\{\Phi''(u_1)+\Phi''(u_2)\},\\ -u_2\Phi'{,}'(u_1,\ u_2)+u_1\Phi''(u_2)=u_1\{\Phi''(u_1)+\Phi''(u_2)\},\end{aligned}\right\} \tag{96}$$

so that the common denominator is $(u_2v_2+u_1v_1)\{\Phi''(u_1)+\Phi''(u_2)\}$, and the expressions (94) become (attending to (92))

$$\delta' u_1=\frac{v_2}{\Phi}\cdot\frac{u_2\delta' v_1-u_1\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)},\quad \delta' u_2=-\frac{v_1}{\Phi}\cdot\frac{u_2\delta' v_1-u_1\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)}. \tag{97}$$

Hence

$$\left.\begin{aligned} \delta' u_1 - \frac{\delta' v_1}{\Phi''(u_1) + \Phi''(u_2)} &= -\frac{u_1}{\Phi} \cdot \frac{v_1 \delta' v_1 + v_2 \delta' v_2}{\Phi''(u_1) + \Phi''(u_2)}, \\ \delta' u_2 - \frac{\delta' v_2}{\Phi''(u_1) + \Phi''(u_2)} &= -\frac{u_2}{\Phi} \cdot \frac{v_1 \delta' v_1 + v_2 \delta' v_2}{\Phi''(u_1) + \Phi''(u_2)}; \end{aligned}\right\} \tag{98}$$

and therefore by the meanings (85) of $\delta' u_1$, $\delta' u_2$, ...

$$u_2 \delta u_1 - u_1 \delta u_2 = u_2 \delta' u_1 - u_1 \delta' u_2 = \frac{u_2 \delta' v_1 - u_1 \delta' v_2}{\Phi''(u_1) + \Phi''(u_2)}; \tag{99}$$

an expression which might also have been deduced more immediately from (94). Hence, by the meanings (86) of $\delta' v_1$, $\delta' v_2$,

$$\begin{aligned} u_2 \delta u_1 - u_1 \delta u_2 = \frac{u_2 \delta v_1 - u_1 \delta v_2}{\Phi''(u_1) + \Phi''(u_2)} &- \frac{u_2 \Phi^{\prime,\prime}(a_1, u_1) - u_1 \Phi^{\prime,\prime}(a_1, u_2)}{\Phi''(u_1) + \Phi''(u_2)} \delta a_1 \\ &- \frac{u_2 \Phi^{\prime,\prime}(a_2, u_1) - u_1 \Phi^{\prime,\prime}(a_2, u_2)}{\Phi''(u_1) + \Phi''(u_2)} \delta a_2. \end{aligned} \tag{100}$$

In general the equation (77) may be put under the form

$$\delta^2 v_i = -\frac{1}{u_i} (\delta u_1 \delta' v_1 + \delta u_2 \delta' v_2 + \dots + \delta u_i \delta' v_i) + \frac{1}{u_i} \delta'^2 \Phi, \tag{101}$$

δ' referring only to the variations of a_1, a_2, ..., a_i; so that, since

$$0 = u_1 \delta' v_1 + u_2 \delta' v_2 + \dots + u_i \delta' v_i, \tag{102}$$

we have

$$\delta^2 v_i = -\frac{1}{u_i} (\delta' u_1 \delta' v_1 + \delta' u_2 \delta' v_2 + \dots + \delta' u_i \delta' v_i - \delta'^2 \Phi), \tag{103}$$

in which we may, by (102), introduce or suppress any set of terms in $\delta' u_1$, $\delta' u_2$, ..., $\delta' u_i$, which are proportional to u_1, u_2, ..., u_i.

In the particular case $i = 2$, we have therefore by (98)

$$\delta^2 v_2 = -\frac{1}{u_2} \frac{\delta' v_1^2 + \delta' v_2^2}{\Phi''(u_1) + \Phi''(u_2)} + \frac{1}{u_2} \delta'^2 \Phi, \tag{104}$$

in which

$$\left.\begin{aligned} \delta' v_1 &= \delta v_1 - \Phi^{\prime,\prime}(a_1, u_1) \delta a_1 - \Phi^{\prime,\prime}(a_2, u_1) \delta a_2, \\ \delta' v_2 &= \delta v_2 - \Phi^{\prime,\prime}(a_1, u_2) \delta a_1 - \Phi^{\prime,\prime}(a_2, u_2) \delta a_2; \end{aligned}\right. \tag{105}$$

also

$$u_1 \delta' v_1 + u_2 \delta' v_2 = 0. \tag{106}$$

If we do not choose to suppose $\delta^2 v_1 = 0$, then instead of (104) we have the more symmetrical relation

$$0 = u_1 \delta^2 v_1 + u_2 \delta^2 v_2 + \frac{\delta' v_1^2 + \delta' v_2^2}{\Phi''(u_1) + \Phi''(u_2)} - \delta'^2 \Phi. \tag{107}$$

Comparing these two last equations (106), (107), of which the former may be thus written

$$0 = u_1 \delta v_1 + u_2 \delta v_2 - \Phi'(a_1) \delta a_1 - \Phi'(a_2) \delta a_2, \tag{108}$$

with the two following:

$$0 = \delta \Psi(v_1, v_2, a_1, a_2) = \Psi'(v_1) \delta v_1 + \Psi'(v_2) \delta v_2 + \Psi'(a_1) \delta a_1 + \Psi'(a_2) \delta a_2, \tag{109}$$

and

$$\begin{aligned}0=\delta^2\Psi=\Psi'(v_1)\,\delta^2v_1+\Psi'(v_2)\,\delta^2v_2+\Psi''(v_1)\,\delta v_1^2+2\Psi^{\prime,\prime}(v_1,v_2)\,\delta v_1\,\delta v_2+\Psi''(v_2)\,\delta v_2^2\\ +2\Psi^{\prime,\prime}(v_1,a_1)\,\delta v_1\,\delta a_1+2\Psi^{\prime,\prime}(v_1,a_2)\,\delta v_1\,\delta a_2+2\Psi^{\prime,\prime}(v_2,a_1)\,\delta v_2\,\delta a_1+2\Psi^{\prime,\prime}(v_2,a_2)\,\delta v_2\,\delta a_2\\ +\Psi''(a_1)\,\delta a_1^2+2\Psi^{\prime,\prime}(a_1,a_2)\,\delta a_1\,\delta a_2+\Psi''(a_2)\,\delta a_2^2,\end{aligned}\tag{110}$$

we find that

$$\frac{\Psi'(v_1)}{u_1}=\frac{\Psi'(v_2)}{u_2}=-\frac{\Psi'(a_1)}{\Phi'(a_1)}=-\frac{\Psi'(a_2)}{\Phi'(a_2)}=\lambda,\tag{111}$$

$$\begin{aligned}\delta^2\Psi=\lambda\,(u_1\delta^2v_1+u_2\delta^2v_2)+\frac{\delta'v_1^2+\delta'v_2^2}{\Phi''(u_1)+\Phi''(u_2)}+2\,(V_1\delta v_1+V_2\delta v_2+A_1\delta a_1+A_2\delta a_2)\\ \times\{u_1\delta v_1+u_2\delta v_2-\Phi'(a_1)\,\delta a_1-\Phi'(a_2)\,\delta a_2\},\end{aligned}\tag{112}$$

λ having the same meaning as in (111), and V_1, V_2, A_1, A_2 being multipliers to be determined by the condition that this last equation shall hold good independently of the variations δv_1, δv_2, δa_1, δa_2, $\delta^2 v_1$, $\delta^2 v_2$. Taking therefore the four partial differential coefficients of the equation (112) with respect to δv_1, δv_2, δa_1, δa_2, we find

$$\begin{aligned}\delta\frac{\delta\Psi}{\delta v_1}=\frac{\lambda\,\delta'v_1}{\Phi''(u_1)+\Phi''(u_2)}+u_1\,(V_1\delta v_1+V_2\delta v_2+A_1\delta a_1+A_2\delta a_2)\\ +V_1\{u_1\delta v_1+u_2\delta v_2-\Phi'(a_1)\,\delta a_1-\Phi'(a_2)\,\delta a_2\},\end{aligned}\tag{113}$$

$$\delta\frac{\delta\Psi}{\delta v_2}=\frac{\lambda\,\delta'v_2}{\Phi''(u_1)+\Phi''(u_2)}+u_2\,(\ldots\ldots\ldots)+V_2\{\ldots\ldots\ldots\},\tag{114}$$

$$\begin{aligned}\delta\frac{\delta\Psi}{\delta a_1}=-\frac{\lambda}{\Phi''(u_1)+\Phi''(u_2)}\{\Phi^{\prime,\prime}(a_1,u_1)\,\delta'v_1+\Phi^{\prime,\prime}(a_1,u_2)\,\delta'v_2\}-\delta'\frac{\delta\Phi}{\delta a_1}-\Phi'(a_1)\,(V_1\delta v_1+\ldots)\\ +A_1\{u_1\delta v_1+\ldots\},\end{aligned}\tag{115}$$

$$\begin{aligned}\delta\frac{\delta\Psi}{\delta a_2}=-\frac{\lambda}{\Phi''(u_1)+\Phi''(u_2)}\{\Phi^{\prime,\prime}(a_2,u_1)\,\delta'v_1+\Phi^{\prime,\prime}(a_2,u_2)\,\delta'v_2\}-\delta'\frac{\delta\Phi}{\delta a_2}-\Phi'(a_2)\,(\ldots\ldots\ldots)\\ +A_2\{\ldots\ldots\ldots\}.\end{aligned}\tag{116}$$

We could thus express the partial differential coefficients of the first and second orders of Ψ by means of those of Φ, the expressions of these differential coefficients of Ψ involving also the 5 arbitrary multipliers λ, V_1, V_2, A_1, A_2, which cannot be determined without assuming some new condition, such as that contained in the form (61). But without making such assumption we can transform the two equations of the form (30) and (31), namely,

$$0=\Psi'(v_1)\frac{\delta S_2}{\delta u_1}+\Psi'(v_2)\frac{\delta S_2}{\delta u_2}+\Psi'(a_1)\,u_1+\Psi'(a_2)\,u_2,\tag{117}$$

$$\begin{aligned}0=\Psi'(v_1)\frac{\delta S_3}{\delta u_1}+\Psi'(v_2)\frac{\delta S_3}{\delta u_2}+\tfrac{1}{2}\Psi''(v_1)\left(\frac{\delta S_2}{\delta u_1}\right)^2+\Psi^{\prime,\prime}(v_1,v_2)\frac{\delta S_2}{\delta u_1}\frac{\delta S_2}{\delta u_2}+\tfrac{1}{2}\Psi''(v_2)\left(\frac{\delta S_2}{\delta u_2}\right)^2\\ +\Psi^{\prime,\prime}(v_1,a_1)\frac{\delta S_2}{\delta u_1}u_1+\Psi^{\prime,\prime}(v_1,a_2)\frac{\delta S_2}{\delta u_1}u_2+\Psi^{\prime,\prime}(v_2,a_1)\frac{\delta S_2}{\delta u_2}u_1+\Psi^{\prime,\prime}(v_2,a_2)\frac{\delta S_2}{\delta u_2}u_2\\ +\tfrac{1}{2}\Psi''(a_1)\,u_1^2+\Psi^{\prime,\prime}(a_1,a_2)\,u_1u_2+\tfrac{1}{2}\Psi''(a_2)\,u_2^2,\end{aligned}\tag{118}$$

so as to eliminate the differential coefficients of Ψ and introduce those of Φ in their stead.

For it is evident that the equation (117) may be formed from the equation

$$\delta\Psi = 0 \tag{119}$$

by merely changing δv_1, δv_2, δa_1, δa_2 to $\frac{\delta S_2}{\delta u_1}$, $\frac{\delta S_2}{\delta u_2}$, u_1, u_2 respectively, and that the equation (118) may be formed from

$$\delta^2\Psi = 0 \tag{120}$$

by making the changes just mentioned and changing also $\delta^2 v_1$, $\delta^2 v_2$ to $2\frac{\delta S_3}{\delta u_1}$, $2\frac{\delta S_3}{\delta u_2}$; since then, by (111), we have

$$\delta\Psi = \lambda\{u_1 \delta v_1 + u_2 \delta v_2 - \Phi'(a_1)\,\delta a_1 - \Phi'(a_2)\,\delta a_2\}, \tag{121}$$

the equation (117) gives independently of λ

$$0 = u_1 \frac{\delta S_2}{\delta u_1} + u_2 \frac{\delta S_2}{\delta u_2} - u_1 \Phi'(a_1) - u_2 \Phi'(a_2), \tag{122}$$

that is, on account of the homogeneous form of S_2,

$$S_2 = \tfrac{1}{2}\{u_1 \Phi'(a_1) + u_2 \Phi'(a_2)\}, \tag{123}$$

a result agreeing with (48); and, in like manner, (118) gives, by (112),

$$\begin{aligned} 0 = 2\left(u_1 \frac{\delta S_3}{\delta u_1} + u_2 \frac{\delta S_3}{\delta u_2}\right) - \Delta'^2\Phi + \frac{\Delta' v_1^2 + \Delta' v_2^2}{\Phi''(u_1) + \Phi''(u_2)} + \frac{2}{\lambda}\left(V_1 \frac{\delta S_2}{\delta u_1} + V_2 \frac{\delta S_2}{\delta u_2} + A_1 u_1 + A_2 u_2\right) \\ \times \left\{u_1 \frac{\delta S_2}{\delta u_1} + u_2 \frac{\delta S_2}{\delta u_2} - u_1 \Phi'(a_1) - u_2 \Phi'(a_2)\right\}, \end{aligned} \tag{124}$$

in which the part involving the arbitrary multiplier vanishes by (122), and in which

$$\Delta' v_1 = \frac{\delta S_2}{\delta u_1} - u_1 \Phi^{\prime,\prime}(a_1, u_1) - u_2 \Phi^{\prime,\prime}(a_2, u_1), \quad \Delta' v_2 = \frac{\delta S_2}{\delta u_2} - u_1 \Phi^{\prime,\prime}(a_1, u_2) - u_2 \Phi^{\prime,\prime}(a_2, u_2). \quad (125$$

The expression (123) gives

$$\left.\begin{aligned} \frac{\delta S_2}{\delta u_1} = \tfrac{1}{2}\Phi'(a_1) + \tfrac{1}{2}u_1 \Phi^{\prime,\prime}(a_1, u_1) + \tfrac{1}{2}u_2 \Phi^{\prime,\prime}(a_2, u_1), \\ \frac{\delta S_2}{\delta u_2} = \tfrac{1}{2}\Phi'(a_2) + \tfrac{1}{2}u_1 \Phi^{\prime,\prime}(a_1, u_2) + \tfrac{1}{2}u_2 \Phi^{\prime,\prime}(a_2, u_2). \end{aligned}\right| \tag{126}$$

Therefore

$$\left.\begin{aligned} \Delta' v_1 = \tfrac{1}{2}\{\Phi'(a_1) - u_1 \Phi^{\prime,\prime}(a_1, u_1) - u_2 \Phi^{\prime,\prime}(a_2, u_1)\}, \\ \Delta' v_2 = \tfrac{1}{2}\{\Phi'(a_2) - u_1 \Phi^{\prime,\prime}(a_1, u_2) - u_2 \Phi^{\prime,\prime}(a_2, u_2)\}, \end{aligned}\right\} \tag{127}$$

in which, by (80),

$$\Phi'(a_1) = u_1 \Phi^{\prime,\prime}(a_1, u_1) + u_2 \Phi^{\prime,\prime}(a_1, u_2), \quad \Phi'(a_2) = u_1 \Phi^{\prime,\prime}(a_2, u_1) + u_2 \Phi^{\prime,\prime}(a_2, u_2); \tag{128}$$

therefore

$$\Delta' v_1 = \tfrac{1}{2}u_2\{\Phi^{\prime,\prime}(a_1, u_2) - \Phi^{\prime,\prime}(a_2, u_1)\}, \quad \Delta' v_2 = -\tfrac{1}{2}u_1\{\Phi^{\prime,\prime}(a_1, u_2) - \Phi^{\prime,\prime}(a_2, u_1)\}; \tag{129}$$

so that, on account of the homogeneous form and dimension ($=3$) of S_3, the equation (124) gives

$$\begin{aligned} S_3 = -\frac{1}{24}\frac{u_1^2 + u_2^2}{\Phi''(u_1) + \Phi''(u_2)}\{\Phi^{\prime,\prime}(a_1, u_2) - \Phi^{\prime,\prime}(a_2, u_1)\}^2 \\ + \tfrac{1}{6}\{u_1^2 \Phi''(a_1) + 2u_1 u_2 \Phi^{\prime,\prime}(a_1, a_2) + u_2^2 \Phi''(a_2)\}, \end{aligned} \tag{130}$$

because in (124) we are to make

$$\Delta'^2\Phi = u_1^2\Phi''(a_1) + 2u_1u_2\Phi','(a_1, a_2) + u_2^2\Phi''(a_2). \tag{131}$$

We may substitute, if we choose, for the factor $\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}$ any one of the other forms (95) which are more simple but less symmetric, except indeed the form $-\frac{u_1u_2}{\Phi','(u_1, u_2)}$ which might be substituted with advantage.

We have then, to the accuracy of the 3rd order inclusive, for the case $i=2$, this expression for the principal function S:

$$\begin{aligned} S = \int_{(x_1=a_1,\ x_2=a_2)}^{(x_1=a_1+u_1,\ x_2=a_2+u_2)} \Phi(x_1, x_2, dx_1, dx_2) &= \Phi(a_1, a_2, u_1, u_2) + \tfrac{1}{2}\{u_1\Phi'(a_1) + u_2\Phi'(a_2)\} \\ &+ \tfrac{1}{6}\{u_1^2\Phi''(a_1) + 2u_1u_2\Phi','(a_1, a_2) + u_2^2\Phi''(a_2)\} \\ &- \frac{1}{24}\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}\{\Phi','(a_1, u_2) - \Phi','(a_2, u_1)\}^2, \end{aligned} \tag{132}$$

in which $-\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)} = \frac{u_1u_2}{\Phi','(u_1, u_2)}$.

[*Examples.*]

[7.] For example, if*
$$\Phi(x_1, x_2, dx_1, dx_2) = \frac{dx_1^2}{2dx_2} + f(x_1)\,dx_2, \tag{133}$$

then
$$\Phi(a_1, a_2, u_1, u_2) = \frac{u_1^2}{2u_2} + f(a_1)\,u_2 \tag{134}$$

and consequently
$$\begin{aligned} &\Phi'(a_1) = u_2f'(a_1), \quad \Phi'(a_2) = 0, \\ &\Phi''(a_1) = u_2f''(a_1), \quad \Phi','(a_1, a_2) = 0, \quad \Phi''(a_2) = 0, \\ &\Phi','(a_1, u_2) = f'(a_1), \quad \Phi','(a_2, u_1) = 0, \\ &\Phi''(u_1) = \frac{1}{u_2}, \quad \Phi''(u_2) = \frac{u_1^2}{u_2^3}; \end{aligned} \tag{135}$$

therefore the general approximate expression (132), for the case $i=2$, gives here

$$\begin{aligned} S &= \int_{(x_1=a_1,\ x_2=a_2)}^{(x_1=a_1+u_1,\ x_2=a_2+u_2)} \left\{\frac{dx_1^2}{2dx_2} + f(x_1)\,dx_2\right\} \\ &= \frac{u_1^2}{2u_2} + u_2f(a_1) + \tfrac{1}{2}u_1u_2f'(a_1) + \tfrac{1}{6}u_1^2u_2f''(a_1) - \tfrac{1}{24}u_2^3\{f'(a_1)\}^2. \end{aligned} \tag{136}$$

In this example the general differential equations (7) become

$$d\frac{dx_1}{dx_2} = f'(x_1)\,dx_2, \quad d\left\{\frac{dx_1^2}{2dx_2^2} - f(x_1)\right\} = 0; \tag{137}$$

equations which are obviously compatible with each other, and which concur in giving

$$\frac{dx_1^2}{dx_2^2} - 2f(x_1) = b_1^2 - 2f(a_1), \tag{138}$$

---

* [This is the dynamical problem of the linear motion of a particle of unit mass whose coordinate is x_1 at time x_2, the force potential being $-f(x_1)$.]

b_1 denoting, as in page 333, the initial value of $\frac{\delta\, dS}{\delta\, dx_1}$, which is here $\frac{dx_1}{dx_2}$; hence if we suppose $\frac{dx_1}{dx_2} > 0$, we shall have

$$dx_2 = \frac{dx_1}{\sqrt{2f(x_1) - 2f(a_1) + b_1^2}}, \tag{139}$$

and

$$S = \int_{a_1}^{a_1+u_1} \frac{2f(x_1) - f(a_1) + \frac{1}{2}b_1^2}{\sqrt{2f(x_1) - 2f(a_1) + b_1^2}}\, dx_1 \tag{140}$$

rigorously. Change here x_1 to $a_1 + u_1$ and we get *approximately*

$$f(x_1) = f(a_1 + u_1) = f(a_1) + u_1 f'(a_1) + \tfrac{1}{2}u_1^2 f''(a_1) \tag{141}$$

and therefore

$$\begin{aligned}\{2f(x_1) - 2f(a_1) + b_1^2\}^{-\frac{1}{2}} &= b_1^{-1}\left\{1 + 2u_1\frac{f'(a_1)}{b_1^2} + u_1^2\frac{f''(a_1)}{b_1^2}\right\}^{-\frac{1}{2}} \\ &= b_1^{-1} - b_1^{-3}\{u_1 f'(a_1) + \tfrac{1}{2}u_1^2 f''(a_1)\} + \tfrac{3}{2}b_1^{-5}u_1^2\{f'(a_1)\}^2,\end{aligned} \tag{142}$$

therefore, by (139),

$$\begin{aligned}u_2 = \int_{a_1}^{x_1}\{2f(x_1) - 2f(a_1) + b_1^2\}^{-\frac{1}{2}}\, dx_1 &= \int_0^{u_1}\{2f(a_1 + u_1) - 2f(a_1) + b_1^2\}^{-\frac{1}{2}}\, du_1 \\ &= b_1^{-1}u_1 - b_1^{-3}\{\tfrac{1}{2}u_1^2 f'(a_1) + \tfrac{1}{6}u_1^3 f''(a_1)\} + \tfrac{1}{2}b_1^{-5}u_1^3\{f'(a_1)\}^2,\end{aligned} \tag{143}$$

$$b_1 = \frac{u_1}{u_2} - \tfrac{1}{2}b_1^{-2}u_2^{-1}u_1^2 f'(a_1) - \tfrac{1}{6}b_1^{-2}u_2^{-1}u_1^3 f''(a_1) + \tfrac{1}{2}b_1^{-4}u_2^{-1}u_1^3\{f'(a_1)\}^2; \tag{144}$$

hence as a first approximation

$$b_1 = \frac{u_1}{u_2}; \tag{145}$$

as a second approximation

$$b_1 = \frac{u_1}{u_2} - \tfrac{1}{2}u_2 f'(a_1); \tag{146}$$

and as a third approximation

$$\begin{aligned}b_1 &= \frac{u_1}{u_2} - \tfrac{1}{2}u_2 f'(a_1)\left\{1 + \frac{u_2^2}{u_1}f'(a_1)\right\} - \tfrac{1}{6}u_1 u_2 f''(a_1) + \frac{1}{2}\frac{u_2^3}{u_1}\{f'(a_1)\}^2 \\ &= \frac{u_1}{u_2} - \tfrac{1}{2}u_2 f'(a_1) - \tfrac{1}{6}u_1 u_2 f''(a_1);\end{aligned} \tag{147}$$

also

$$2f(a_1 + u_1) - f(a_1) + \tfrac{1}{2}b_1^2 = \tfrac{1}{2}b_1^2 + f(a_1) + 2u_1 f'(a_1) + u_1^2 f''(a_1), \tag{148}$$

therefore

$$\begin{aligned}\frac{2f(a_1 + u_1) - f(a_1) + \frac{1}{2}b_1^2}{\sqrt{2f(x_1) - 2f(a_1) + b_1^2}} &= \tfrac{1}{2}b_1 + \frac{f(a_1)}{b_1} + u_1\frac{f'(a_1)}{b_1}\left\{2 - \frac{1}{2} - \frac{f(a_1)}{b_1^2}\right\} \\ &\quad + \frac{u_1^2}{b_1}\left[f''(a_1) - 2\left(\frac{f'(a_1)}{b_1}\right)^2 + \{\tfrac{1}{2}b_1^2 + f(a_1)\}\left\{-\frac{f''(a_1)}{2b_1^2} + \frac{3}{2}\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\}\right] \\ &= \tfrac{1}{2}b_1 + \frac{f(a_1)}{b_1} + u_1\frac{f'(a_1)}{b_1}\left\{\frac{3}{2} - \frac{f(a_1)}{b_1^2}\right\} + \frac{u_1^2}{b_1}\left\{\tfrac{3}{4}f''(a_1) - \frac{5}{4}\left(\frac{f'(a_1)}{b_1}\right)^2\right. \\ &\qquad \left. - \frac{f(a_1)f''(a_1)}{2b_1^2} + \tfrac{3}{2}f(a_1)\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\};\end{aligned} \tag{149}$$

hence

$$S=\int_0^{u_1}\frac{2f(a_1+u_1)-f(a_1)+\frac{1}{2}b_1^2}{\sqrt{2f(x_1)-2f(a_1)+b_1^2}}du_1=\left(\frac{b_1}{2}+\frac{f(a_1)}{b_1}\right)u_1+f'(a_1)\left\{\frac{3}{2b_1}-\frac{f(a_1)}{b_1^3}\right\}\frac{u_1^2}{2}$$
$$+\left\{\frac{3}{4}\frac{f''(a_1)}{b_1}-\frac{5}{4b_1}\left(\frac{f'(a_1)}{b_1}\right)^2-\frac{f(a_1)f''(a_1)}{2b_1^3}+\frac{3}{2}\frac{f(a_1)}{b_1}\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\}\frac{u_1^3}{3}; \quad (150)$$

in which, by (147),

$$\frac{b_1u_1}{2}=\frac{u_1^2}{2u_2}-\frac{u_1u_2}{4}f'(a_1)-\frac{u_1^2u_2}{12}f''(a_1), \quad (151)$$

$$\frac{f(a_1)}{b_1}u_1=u_2f(a_1)\left\{1-\frac{u_2^2}{2u_1}f'(a_1)-\frac{u_2^2}{6}f''(a_1)\right.^{-1}$$
$$=u_2f(a_1)\left\{1+\frac{u_2^2}{2u_1}f'(a_1)+\frac{u_2^2}{6}f''(a_1)+\frac{u_2^4}{4u_1^2}\{f'(a_1)\}\right. \quad (152)$$

$$\frac{3f'(a_1)}{4b_1}u_1^2=\frac{3}{4}u_1u_2f'(a_1)\left\{1+\frac{u_2^2}{2u_1}f'(a_1)\right\}, \quad (153)$$

$$\frac{f(a_1)f'(a_1)}{2b_1^3}u_1^2=-\frac{u_2^3}{2u_1}f(a_1)f'(a_1)\left\{1+\frac{3u_2^2}{2u_1}f'(a_1)\right\}, \quad (154)$$

$$\left\{\frac{f''(a_1)}{4b_1}-\frac{5\{f'(a_1)\}^2}{12b_1^3}-\frac{f(a_1)f''(a_1)}{6b_1^3}+\frac{f(a_1)\{f'(a_1)\}^2}{2b_1^5}\right\}u_1^3$$
$$=\frac{1}{4}u_1^2u_2f''(a_1)-\frac{1}{12}u_2^3[5\{f'(a_1)\}^2+2f(a_1)f''(a_1)]+\frac{1}{2}u_2^5u_1^{-2}f(a_1)\{f'(a_1)\}^2; \quad (155)$$

therefore, adding these last five expressions,

$$S=\frac{u_1^2}{2u_2}+u_2f(a_1)$$
$$-\frac{u_1u_2}{4}f'(a_1)+\frac{u_2^3u_1^{-1}}{2}f(a_1)f'(a_1)+\frac{3u_1u_2}{4}f'(a_1)-\frac{u_2^3u_1^{-1}}{2}f(a_1)f'(a_1)$$
$$-\frac{u_1^2u_2}{12}f''(a_1)+\frac{u_2^3}{6}f(a_1)f''(a_1)+\frac{u_2^5u_1^{-2}}{4}f(a_1)\{f'(a_1)\}^2+\frac{3u_2^3}{8}\{f'(a_1)\}^2-\frac{3u_2^5u_1^{-2}}{4}f(a_1)\{f'(a_1)\}^2$$
$$+\frac{u_1^2u_2}{4}f''(a_1)-\frac{u_2^3}{6}f(a_1)f''(a_1)-\frac{5u_2^3}{12}\{f'(a_1)\}^2+\frac{u_2^5u_1^{-2}}{2}f(a_1)\{f'(a_1)\}^2$$
$$=\frac{u_1^2}{2u_2}+u_2f(a_1)+\frac{u_1u_2}{2}f'(a_1)+\frac{u_1^2u_2}{6}f''(a_1)-\frac{u_2^3}{24}\{f'(a_1)\}^2, \quad (156)$$

as in (136).

This has been a complicated process: its most essential part, after the deduction of the rigorous intermediate integral equation (138), has been the approximate elimination of b_1 between the two rigorous expressions

$$u_2=\int_0^{u_1}\frac{du_1}{\sqrt{b_1^2+2f(a_1+u_1)-2f(a_1)}} \quad (143)$$

and

$$S=\int_0^{u_1}\frac{\frac{1}{2}b_1^2+2f(a_1+u_1)-f(a_1)}{\sqrt{b_1^2+2f(a_1+u_1)-2f(a_1)}}du_1, \quad (150)$$

giving the approximate result (136) through the medium of the approximate expression (147), deduced from (143).

In the present example we have, by (133),

$$y_1 = \frac{\delta dS}{\delta dx_1} = \frac{dx_1}{dx_2}; \quad y_2 = \frac{\delta dS}{\delta dx_2} = -\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2 + f(x_1); \tag{157}$$

so that the general equation (14), $0 = \Psi(y_1, \ldots, y_i, x_1, \ldots, x_i)$, may here be put under the form

$$0 = \tfrac{1}{2}y_1^2 + y_2 - f(x_1), \tag{158}$$

and the general partial differential equation (16), which may always be thus written

$$0 = \Psi\left(\frac{\delta S}{\delta u_1}, \ldots, \frac{\delta S}{\delta u_i}, a_1 + u_1, \ldots, a_i + u_i\right), \tag{159}$$

becomes in the present example

$$0 = \frac{1}{2}\left(\frac{\delta S}{\delta u_1}\right)^2 + \frac{\delta S}{\delta u_2} - f(a_1 + u_1), \tag{160}$$

and gives

$$\frac{\delta S}{\delta u_1} = \pm\sqrt{2f(a_1 + u_1) - 2\frac{\delta S}{\delta u_2}}. \tag{161}$$

If we take the upper sign, the complete and general integral of this partial differential equation (161) is given by the following equations:

$$S = \int_0^{u_1} \sqrt{2f(a_1 + u_1) - 2b_2}\,du_1 + b_2 u_2 + \phi(b_2), \quad 0 = u_2 - \int_0^{u_1} \frac{du_1}{\sqrt{2f(a_1 + u_1) - 2b_2}} + \phi'(b_2), \tag{162}$$

$\phi(b_2)$ being an arbitrary function of b_2 and $\phi'(b_2)$ being its derived function, but b_2 being treated as constant in effecting the two definite integrations; and in the present question this arbitrary function $\phi(b_2)$ and therefore also $\phi'(b_2)$ must be supposed to be identically equal to zero, because S vanishes with u_1 and u_2 independently of the auxiliary quantity b_2, which may easily be shown to be equal to $\frac{\delta S}{\delta u_2}$ and to be constant in the progression of $u_1\, u_2\, S$; we may then rigorously determine the form of the principal function S by eliminating b_2 between the two equations

$$u_2 = \int_0^{u_1} \frac{du_1}{\sqrt{2f(a_1 + u_1) - 2b_2}}, \quad S = \int_0^{u_1} \frac{2f(a_1 + u_1) - b_2}{\sqrt{2f(a_1 + u_1) - 2b_2}}\,du_1, \tag{163}$$

which may easily be seen to coincide with the equations (143) and (150).

(Jan. 23rd, 1836.)

As another example,* let

$$\Phi(x_1, x_2, dx_1, dx_2) = e^{2hx_1}\left(h\frac{dx_1^2}{dx_2} + g\,dx_2\right), \tag{164}$$

h and g being any arbitrary constants and e being the napierian base. Then

$$\Phi(a_1, a_2, u_1, u_2) = e^{2ha_1}\left(h\frac{u_1^2}{u_2} + gu_2\right); \tag{165}$$

$$\Phi'(a_1) = 2h\Phi, \quad \Phi'(a_2) = 0, \quad \Phi'(u_1) = 2h\frac{u_1}{u_2}e^{2ha_1}, \quad \Phi'(u_2) = e^{2ha_1}\left(g - h\frac{u_1^2}{u_2^2}\right); \tag{166}$$

---

* [Problem of the fall of a heavy body in a medium resisting as the square of the velocity.]

$$\Phi''(a_1) = 4h^2\Phi, \quad \Phi'{,}'(a_1, a_2) = 0, \quad \Phi''(a_2) = 0,$$

$$\Phi'{,}'(a_1, u_1) = 4h^2 \frac{u_1}{u_2} e^{2ha_1}, \quad \Phi'{,}'(a_1, u_2) = 2h\left(g - h\frac{u_1^2}{u_2^2}\right) e^{2ha_1}, \quad \Phi'{,}'(a_2, u_1) = 0, \quad \Phi'{,}'(a_2, u_2) = 0, \tag{167}$$

$$\Phi''(u_1) = \frac{2h}{u_2} e^{2ha_1}, \quad \Phi'{,}'(u_1, u_2) = -\frac{2hu_1}{u_2^2} e^{2ha_1}, \quad \Phi''(u_2) = \frac{2hu_1^2}{u_2^3} e^{2ha_1};$$

$$\frac{u_1^2 + u_2^2}{\Phi''(u_1) + \Phi''(u_2)} = -\frac{u_1 u_2}{\Phi'{,}'(u_1, u_2)} = \frac{u_2^3}{2h} e^{-2ha_1}, \tag{168}$$

and the general approximate expression (132) becomes

$$S = \int_{x_1 = a_1,\, x_2 = a_2}^{x_1 = a_1 + u_1,\, x_2 = a_2 + u_2} e^{2hx_1}\left(h\frac{dx_1^2}{dx_2} + g\,dx_2\right.$$
$$= (1 + hu_1 + \tfrac{2}{3}h^2u_1^2)\, e^{2ha_1}\left(h\frac{u_1^2}{u_2} + gu_2\right) - \frac{(gu_2^2 - hu_1^2)^2}{12u_2} he^{2ha_1}. \tag{169}$$

In this example the general differential equations (7) become

$$e^{2hx_1}\left(h\frac{dx_1^2}{dx_2} + g\,dx_2\right) = d\,.\,e^{2hx_1}\frac{dx_1}{dx_2}, \quad 0 = d\,.\,e^{2hx_1}\left(g - h\frac{dx_1^2}{dx_2^2}\right), \tag{170}$$

and both agree in giving

$$d\frac{dx_1}{dx_2} - g\,dx_2 + h\frac{dx_1^2}{dx_2} = 0 \tag{171}$$

as the ordinary differential equation of the second order between x_1 and x_2. The second equation (170) gives, as an intermediate integral,

$$e^{2hx_1}\left(g - h\frac{dx_1^2}{dx_2^2}\right) = b_2 = \text{const.}, \tag{172}$$

b_2 denoting as usual the initial value of $\Phi'(dx_2)$ or of $\frac{\delta\,dS}{\delta\,dx_2}$; therefore

$$\frac{dx_2}{dx_1} = \pm\sqrt{\frac{h}{g - b_2 e^{-hx_1}}}, \tag{173}$$

and hence, taking the upper sign,

$$u_2 = \sqrt{h}\int_0^{u_1} \frac{du_1}{\sqrt{g - b_2 e^{-2h(a_1 + u_1)}}}. \tag{174}$$

Also

$$\frac{dS}{dx_2} = e^{2hx_1}\left(g + h\frac{dx_1^2}{dx_2^2}\right) = 2ge^{2hx_1} - b_2, \tag{175}$$

therefore

$$S = -b_2u_2 + 2g\sqrt{h}\int_0^{u_1} \frac{e^{2h(a_1 + u_1)}\,du_1}{\sqrt{g - b_2 e^{-2h(a_1 + u_1)}}}, \tag{176}$$

that is, by the expression for u_2,

$$S = \sqrt{h}\int_0^{u_1} \frac{2ge^{2h(a_1 + u_1)} - b_2}{\sqrt{g - b_2 e^{-2h(a_1 + u_1)}}}\,du_1. \tag{177}$$

The equations (174), (177) are rigorous and the approximate elimination of b_2 between them ought to conduct to the expression (169).

To effect this approximate elimination, we shall first develope the reciprocal of the radical. We have

$$g-b_2e^{-2h(a_1+u_1)}=g-b_2e^{-2ha_1}e^{-2hu_1}=g-b_2e^{-2ha_1}(1-2hu_1+2h^2u_1^2), \tag{178}$$

therefore

$$\begin{aligned}\{g-b_2e^{-2h(a_1+u_1)}\}^{-\frac{1}{2}}&=e^{ha_1}\{ge^{2ha_1}-b_2+2b_2hu_1-2b_2h^2u_1^2\}^{-\frac{1}{2}}\\&=e^{ha_1}(ge^{2ha_1}-b_2)^{-\frac{1}{2}}\left\{1+\frac{2b_2hu_1}{ge^{2ha_1}-b_2}(1-h_1u_1)\right.^{-\frac{1}{2}}\\&=e^{ha_1}(ge^{2ha_1}-b_2)^{-\frac{1}{2}}\left\{1-\frac{b_2hu_1(1-hu_1)}{ge^{2ha_1}-b_2}+\frac{\frac{3}{2}b_2^2h^2u_1^2}{(ge^{2ha_1}-b_2)^2}\right\}\\&=(g-b_2e^{-2ha_1})^{-\frac{1}{2}}-u_1e^{ha_1}\,b_2h\,(ge^{2ha_1}-b_2)^{-\frac{3}{2}}\\&\qquad+u_1^2e^{ha_1}b_2h^2(ge^{2ha_1}+\tfrac{1}{2}b_2)(ge^{2ha_1}-b_2)^{-\frac{5}{2}};\end{aligned} \tag{179}$$

therefore

$$u_2=\frac{u_1\sqrt{h}}{\sqrt{g-b_2e^{-2ha_1}}}-\frac{u_1^2}{2}e^{-2ha_1}\,b_2\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{3}{2}}+\frac{u_1^3}{3}e^{-4ha_1}\,b_2\,(ge^{2ha_1}+\tfrac{1}{2}b_2)\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{5}{2}}; \tag{180}$$

hence, as a first approximation,

$$\frac{g-b_2e^{-2ha_1}}{h}=\frac{u_1}{u_2} \tag{181}$$

and so (cf. (172))

$$b_2=\left(g-h\frac{u_1^2}{u_2^2}\right)e^{2ha_1}; \tag{182}$$

and since in this approximation

$$\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{1}{2}}=\frac{u_2}{u_1}, \tag{183}$$

we have as a second approximation

$$u_1\sqrt{\frac{h}{g-b_2e^{-2ha_1}}}=u_2+\frac{(gu_2^2-hu_1^2)\,u_2}{2u_1}, \tag{184}$$

that is,

$$\left(\frac{u_2}{u_1}\right)^2\frac{g-b_2e^{-2ha_1}}{h}=1+hu_1-\frac{gu_2^2}{u_1}$$

or

$$b_2e^{-2ha_1}=g-h\frac{u_1^2}{u_2^2}\left\{1+hu_1-\frac{gu_2^2}{u_1}\right\}=\left(g-h\frac{u_1^2}{u_2^2}\right)(1+hu_1). \tag{185}$$

Consequently, as a third approximation,

$$\begin{aligned}\frac{u_1}{u_2}\sqrt{\frac{h}{g-b_2e^{-2ha_1}}}&=1+\frac{gu_2^2-hu_1^2}{2u_1}\left\{1+hu_1+\frac{3}{2u_1}(gu_2^2-hu_1^2)\right\}-\frac{1}{6u_1^2}(gu_2^2-hu_1^2)(3gu_2^2-hu_1^2)\\&=1+\frac{gu_2^2-hu_1^2}{2u_1}\left\{1-\frac{hu_1}{6}+\frac{gu_2^2}{2u_1}\right\},\end{aligned} \tag{186}$$

$$\begin{aligned}\frac{u_2^2}{u_1^2}\frac{g-b_2e^{-2ha_1}}{h}&=1+\frac{hu_1^2-gu_2^2}{u_1}\left\{1+hu_1+\frac{3}{2u_1}(gu_2^2-hu_1^2)\right\}+\frac{3}{4u_1^2}(gu_2^2-hu_1^2)^2\\&\qquad+\frac{1}{3u_1^2}(gu_2^2-hu_1^2)(3gu_2^2-hu_1^2),\end{aligned} \tag{187}$$

$$\frac{b_2e^{-2ha_1}u_2^2}{gu_2^2-hu_1^2}=1+hu_1+h^2u_1^2+\tfrac{3}{4}h\,(gu_2^2-hu_1^2)-\frac{h}{3}(3gu_2^2-hu_1^2)=1+hu_1+\tfrac{7}{12}h^2u_1^2-\tfrac{1}{4}ghu_2^2, \tag{188}$$

and
$$b_2 = e^{2ha_1}\left(g - h\frac{u_1^2}{u_2^2}\right)\{1 + hu_1 + \tfrac{7}{12}h^2u_1^2 - \tfrac{1}{4}ghu_2^2\}. \tag{189}$$

Again, we have by what precedes
$$\sqrt{\frac{h}{g-b_2e^{-2h(a_1+u_1)}}} = \sqrt{\frac{h}{g-b_2e^{-2ha_1}}} - u_1b_2e^{-2ha_1}\left(\frac{h}{g-b_2e^{-2ha_1}}\right)^{\frac{3}{2}}$$
$$+ u_1^2b_2e^{-2ha_1}(g + \tfrac{1}{2}b_2e^{-2ha_1})\left(\frac{h}{g-b_2e^{-2ha_1}}\right)^{\frac{5}{2}}; \tag{190}$$

also
$$2ge^{2hu_1} - b_2e^{-2ha_1} = 2g - b_2e^{-2ha_1} + 4ghu_1 + 4gh^2u_1^2; \tag{191}$$

therefore, by (177) and (180),
$$Se^{-2ha_1} = u_2(2g - b_2e^{-2ha_1}) + (2ghu_1^2 + \tfrac{4}{3}gh^2u_1^3)\left(\frac{h}{g-b_2e^{-2ha_1}}\right.$$
$$- \tfrac{4}{3}ghu_1^3b_2e^{-2ha_1}\left(\frac{h}{g-b_2e^{-2ha_1}}\right)^{\frac{3}{2}}, \tag{192}$$

in which
$$u_2(2g - b_2e^{-2ha_1}) = gu_2 + h\frac{u_1^2}{u_2} + \left(\frac{hu_1^2}{u_2} - gu_2\right)(hu_1 + \tfrac{7}{12}h^2u_1^2 - \tfrac{1}{4}ghu_2^2), \tag{193}$$
$$2ghu_1^2(1 + \tfrac{2}{3}hu_1)\sqrt{\frac{h}{g-b_2e^{-2ha_1}}} = 2ghu_1u_2\left(1 + \tfrac{1}{6}hu_1 + \tfrac{1}{2}g\frac{u_2^2}{u_1}\right), \tag{194}$$

and
$$-\tfrac{4}{3}ghu_1^3b_2e^{-2ha_1}\left(\frac{h}{g-b_2e^{-2ha_1}}\right)^{\frac{3}{2}} = -\tfrac{4}{3}ghu_2(gu_2^2 - hu_1^2). \tag{195}$$

Therefore, adding the three last expressions, we find
$$Se^{-2ha_1} = \left(\frac{hu_1^2}{u_2} + gu_2\right)(1 + hu_1) + \frac{7}{12}\frac{h^3u_1^4}{u_2} + \tfrac{5}{6}gh^2u_1^2u_2 - \tfrac{1}{12}g^2hu_2^3, \tag{196}$$
agreeing with (169).

It would however have been simpler, in this example, to have put the differential equation of the second order (171) under the form:
$$x_1'' = g - hx_1'^2, \tag{197}$$
and then to have deduced from it by differentiation
$$x_1''' = -2hx_1'x_1'' = -2hx_1'(g - hx_1'^2), \tag{198}$$
and therefore
$$a_1'' = g - ha_1'^2, \tag{199}$$
$$a_1''' = -2ha_1'(g - ha_1'^2), \tag{200}$$
x_1', x_1'', x_1''' being differential coefficients of x_1 considered as a function of x_2, and a_1', a_1'', a_1''' being their values when $x_2 = a_2$, $x_1 = a_1$. For thus we should obtain, by Taylor's theorem,
$$u_1 = a_1'u_2 + \tfrac{1}{2}a_1''u_2^2 + \tfrac{1}{6}a_1'''u_2^3, \tag{201}$$
that is,
$$u_1 = a_1'u_2 + \tfrac{1}{2}u_2^2(g - ha_1'^2)(1 - \tfrac{2}{3}ha_1'u_2), \tag{202}$$
which gives as a first approximation
$$a_1' = \frac{u_1}{u_2}; \tag{203}$$

as a second approximation

$$a_1' = \frac{u_1}{u_2} - \frac{u_2}{2}\left(g - h\frac{u_1^2}{u_2^2}\right); \tag{204}$$

therefore

$$g - ha_1'^2 = \left(g - h\frac{u_1^2}{u_2^2}\right)(1 + hu_1), \tag{205}$$

and

$$(1 + hu_1)(1 - \tfrac{2}{3}ha_1'u_2) = 1 + \tfrac{1}{3}hu_1. \tag{206}$$

As a third approximation

$$a_1' = \frac{u_1}{u_2} - \frac{u_2}{2}\left(g - h\frac{u_1^2}{u_2^2}\right)(1 + \tfrac{1}{3}hu_1). \tag{207}$$

Also

$$S = e^{2ha_1}\int_0 e^{2hu_1}(hx_1'^2 + g)\,du_2, \tag{208}$$

in which

$$x_1' = a_1' + a_1''u_2 + \tfrac{1}{2}a_1'''u_2^2 \tag{209}$$

and

$$e^{2hu_1} = 1 + 2hu_1 + 2h^2u_1^2 = 1 + 2ha_1'u_2 + (2h^2a_1'^2 + ha_1'')u_2^2; \tag{210}$$

therefore

$$e^{2hu_1}(hx_1'^2 + g) = (ha_1'^2 + g)\{1 + 2ha_1'u_2 + (2h^2a_1'^2 + ha_1'')u_2^2\} + 2ha_1'a_1''\{u_2 + 2ha_1'u_2^2\} + \\ + h(a_1''^2 + a_1'a_1''')u_2^2, \tag{211}$$

and

$$\begin{aligned} Se^{-2ha_1} &= (ha_1'^2 + g)\left\{u_2 + ha_1'u_2^2 + \frac{h}{3}(2ha_1'^2 + a_1'')u_2^3\right\} + ha_1'a_1''\{u_2^2 + \tfrac{4}{3}ha_1'u_2^3\} + \frac{h}{3}(a_1''^2 + a_1'a_1''')u_2^3 \\ &= (ha_1'^2 + g)u_2 + 2gha_1'u_2^2 + \tfrac{1}{3}hu_2^3\{(g + ha_1'^2)^2 + g^2 - h^2a_1'^4\} \\ &= (ha_1'^2 + g)u_2(1 + \tfrac{2}{3}ghu_2^2) + 2gha_1'u_2^2, \end{aligned} \tag{212}$$

in which, by (207),

$$(ha_1'^2 + g)u_2 = \frac{hu_1^2}{u_2} + gu_2 - hu_1u_2\left(g - h\frac{u_1^2}{u_2^2}\right)(1 + \tfrac{1}{3}hu_1) + \frac{hu_2^3}{4}\left(g - h\frac{u_1^2}{u_2^2}\right)^2,$$

$$\tfrac{2}{3}ghu_2^3(ha_1'^2 + g) = \tfrac{2}{3}ghu_2(hu_1^2 + gu_2^2), \tag{213}$$

$$2gha_1'u_2^2 = 2ghu_1u_2 - ghu_2^3\left(g - h\frac{u_1^2}{u_2^2}\right);$$

therefore, adding these three last expressions, we find for the principal function S this expression, agreeing with (196) and with (169):

$$S = \left\{\left(\frac{hu_1^2}{u_2} + gu_2\right)(1 + hu_1) + \frac{7}{12}\frac{h^3u_1^4}{u_2} + \tfrac{5}{6}gh^2u_1^2u_2 - \tfrac{1}{12}g^2hu_2^3\right\}e^{2ha_1}. \tag{214}$$

It is worth observing that the differential equation of the second order (197) is that of the fall of a heavy body in a medium which resists as the square of the velocity; so that the integral of this equation can be rigorously expressed by the method of the principal function.

As a third example* put

$$dS = \Phi(x_1, x_2, dx_1, dx_2) = \frac{dx_1^2}{2dx_2} + gx_1dx_2, \tag{215}$$

g being any arbitrary constant. Then

$$\Phi(a_1, a_2, u_1, u_2) = \frac{u_1^2}{2u_2} + ga_1u_2; \tag{216}$$

---

* [Particular case of example on page 348.]

therefore

$$\Phi'(a_1)=gu_2, \quad \Phi'(a_2)=0, \quad \Phi'(u_1)=\frac{u_1}{u_2}, \quad \Phi'(u_2)=ga_1-\frac{u_1^2}{2u_2^2}; \tag{217}$$

and

$$\begin{gathered}\Phi''(a_1)=0, \quad \Phi'{,}'(a_1, a_2)=0, \quad \Phi''(a_2)=0, \\ \Phi'{,}'(a_1, u_1)=0, \quad \Phi'{,}'(a_1, u_2)=g, \quad \Phi'{,}'(a_2, u_1)=0, \quad \Phi'{,}'(a_2, u_2)=0, \\ \Phi''(u_1)=\frac{1}{u_2}, \quad \Phi'{,}'(u_1, u_2)=-\frac{u_1}{u_2^2}, \quad \Phi''(u_2)=\frac{u_1^2}{u_2^3};\end{gathered} \tag{218}$$

hence

$$\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)} - - \frac{u_1 u_2}{\Phi'{,}'(u_1, u_2)}=u_2^3, \tag{219}$$

so that the general approximate expression (132) becomes in this example

$$S=\int_{x_1=a_1,\, x_2=a_2}^{x_1=a_1+u_1,\, x_2=a_2+u_2} \frac{dx_1^2}{2dx_2}+gx_1\,dx_2=\frac{u_1^2}{2u_2}+ga_1u_2+\tfrac{1}{2}gu_1u_2-\tfrac{1}{24}g^2u_2^3. \tag{220}$$

In this particular example,

$$y_1=\frac{\delta\, dS}{\delta\, dx_1}=\frac{dx_1}{dx_2}, \tag{221}$$

$$y_2=\frac{\delta\, dS}{\delta\, dx_2}=gx_1-\frac{dx_1^2}{2dx_2^2}, \tag{222}$$

also

$$\frac{\delta\, dS}{\delta x_1}=g\,dx_2, \tag{223}$$

and

$$\frac{\delta\, dS}{\delta x_2}=0; \tag{224}$$

thus the differential equations (7) become here

$$d\,\frac{dx_1}{dx_2}=g\,dx_2, \quad d\left(gx_1-\frac{1}{2}\frac{dx_1^2}{dx_2^2}\right)=0, \tag{225}$$

and they concur in giving as the complete integral with two arbitrary constants:

$$x_1=a_1+a_1'(x_2-a_2)+\tfrac{1}{2}g(x_2-a_2)^2, \tag{226}$$

that is,

$$u_1=a_1'u_2+\tfrac{1}{2}gu_2^2. \tag{227}$$

Hence, rigorously,

$$a_1'=\frac{u_1}{u_2}-\tfrac{1}{2}gu_2. \tag{228}$$

Also

$$\frac{dx_1}{dx_2}=a_1'+gu_2 \tag{229}$$

and

$$\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2+gx_1=ga_1+\tfrac{1}{2}a_1'^2+2ga_1'u_2+g^2u_2^2; \tag{230}$$

therefore

$$S=\int_0^{u_2}\left\{\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2+gx_1\right\}du_2=(ga_1+\tfrac{1}{2}a_1'^2)\,u_2+ga_1'u_2^2+\tfrac{1}{3}g^2u_2^3 \tag{231}$$

rigorously, or

$$\begin{aligned}S&=ga_1u_2+\tfrac{1}{2}u_2(a_1'+gu_2)^2-\tfrac{1}{6}g^2u_2^3\\&=ga_1u_2+\tfrac{1}{2}u_2\left(\frac{u_1}{u_2}+\tfrac{1}{2}gu_2\right)^2-\tfrac{1}{6}g^2u_2^3\\&=ga_1u_2+\frac{u_1^2}{2u_2}+\tfrac{1}{2}gu_1u_2-\tfrac{1}{24}g^2u_2^3\end{aligned} \tag{232}$$

rigorously, as deduced in (220) from the generally approximate expression (132), which in this example is more than approximate.

XIV.

CALCULUS OF PRINCIPAL RELATIONS

A NEW SERIES OF INVESTIGATIONS

[1836.]

[Note Book 42.]

[CONTENTS

		PAGE
1.	Statement of the theory for a single differential equation of the first order.	358
2.	Extension to a number of differential equations of any order.	360
3.	The case of one total differential equation of the first order resumed.	364
4.	Two or more differential equations of the first order.	368
5.	Differential equations of order higher than the first.	372
6.	Particular cases treated in greater detail.]	375

[*Statement of the theory for a single differential equation of the first order.*]

(Feb. 18th, 1836.)

[**1.**] Let the following be any proposed or original ordinary differential equation of the first order:

$$0=f(x, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'), \tag{1}$$

in which

$$x_1'=\frac{dx_1}{dx}, \quad x_2'=\frac{dx_2}{dx}, \qquad x_n'=\frac{dx_n}{dx}. \tag{2}$$

If $n=1$, the equation is simply

$$0=f(x, x_1, x_1'), \tag{3}$$

and then it has always an integral of the form

$$0=F(a, a_1, x, x_1), \tag{4}$$

a_1 being the value of x_1 which corresponds to the value a of x. But if $n>1$, the one original differential equation (1) is not in general sufficient to determine the forms of the two or more functions $x_1, \ldots, x_n$; however it will still conduct in general to what I call the *principal integral relation* between those functions and their initial values, of the form

$$0=F(a, a_1, \ldots, a_n, x, x_1, \ldots, x_n), \tag{5}$$

provided that we combine it with the following $n-1$ *principal supplementary differential equations* of the second order:

$$\frac{\dfrac{\delta f}{\delta x_1}-\dfrac{d}{dx}\dfrac{\delta f}{\delta x_1'}}{\dfrac{\delta f}{\delta x_1'}}=\ldots\ldots=\frac{\dfrac{\delta f}{\delta x_n}-\dfrac{d}{dx}\dfrac{\delta f}{\delta x_n'}}{\dfrac{\delta f}{\delta x_n'}}, \tag{6}$$

assigned by the Calculus of Variations. Since these supplementary equations enable us to choose λ so that

$$(7) \qquad \lambda\frac{\delta f}{\delta x_1}=\frac{d}{dx}\left(\lambda\frac{\delta f}{\delta x_1'}\right), \quad \ldots, \quad \lambda\frac{\delta f}{\delta x_n}=\frac{d}{dx}\left(\lambda\frac{\delta f}{\delta x_n'}\right),$$

we see that if we multiply by λ, thus chosen, the variation of the original differential equation (1), namely

$$(8) \qquad 0=\frac{\delta f}{\delta x}\delta x+\frac{\delta f}{\delta x_1}\delta x_1+\ldots+\frac{\delta f}{\delta x_n}\delta x_n+\frac{\delta f}{\delta x_1'}\delta x_1'+\ldots+\frac{\delta f}{\delta x_n'}\delta x_n',$$

in which

$$(9) \qquad 0=\frac{\delta f}{\delta x}+\frac{\delta f}{\delta x_1}x_1'+\ldots+\frac{\delta f}{\delta x_n}x_n'+\frac{\delta f}{\delta x_1'}x_1''+\ldots+\frac{\delta f}{\delta x'}x_n'',$$

we shall get

$$(10) \qquad 0=\left(\lambda\frac{\delta f}{\delta x_1'}\right)'(\delta x_1-x_1'\delta x)+\ldots+\left(\lambda\frac{\delta f}{\delta x_n'}\right)'(\delta x_n-x_n'\delta x)$$
$$+\lambda\frac{\delta f}{\delta x_1'}(\delta x_1'-x_1''\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n'-x_n''\delta x),$$

in which $\delta x_i'=\delta\frac{dx_i}{dx}=\frac{d}{dx}\delta x_i-x_i'\frac{d}{dx}\delta x$, and therefore

$$(11) \qquad \delta x_i'-x_i''\delta x=\frac{d}{dx}(\delta x_i-x_i'\delta x)=(\delta x_i-x_i'\delta x)'.$$

Hence

$$(12) \qquad 0=\Delta\left\{\lambda\frac{\delta f}{\delta x_1'}(\delta x_1-x_1'\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n-x_n'\delta x)\right\};$$

that is,

$$(13) \qquad 0=\lambda_x\left\{\frac{\delta f_x}{\delta x_1'}(\delta x_1-x_1'\delta x)+\ldots+\frac{\delta f_x}{\delta x_n'}(\delta x_n-x_n'\delta x)\right\}$$
$$-\lambda_a\left\{\frac{\delta f_a}{\delta a_1'}(\delta a_1-a_1'\delta a)+\ldots+\frac{\delta f_a}{\delta a_n'}(\delta a_n-a_n'\delta a)\right\},$$

if we denote by λ_x and f_x the final, and by λ_a and f_a the initial values of λ and f.

Comparing the equation (13), which is the *integral of the variation* of the original differential equation (1), with the *variation of the principal integral relation* (5), namely

$$(14) \qquad 0=\frac{\delta F}{\delta a}\delta a+\frac{\delta F}{\delta a_1}\delta a_1+\ldots+\frac{\delta F}{\delta a_n}\delta a_n+\frac{\delta F}{\delta x}\delta x+\frac{\delta F}{\delta x_1}\delta x_1+\ldots+\frac{\delta F}{\delta x_n}\delta x_n,$$

and observing that the coefficients of the one must be proportional to those of the other, we find the n final equations

$$(15) \qquad \left\{\begin{array}{l} 0=\dfrac{\delta F}{\delta x_1}\left(x_1'\dfrac{\delta f_x}{\delta x_1'}+\ldots+x_n'\dfrac{\delta f_x}{\delta x_n'}\right)+\dfrac{\delta F}{\delta x}\dfrac{\delta f_x}{\delta x_1'}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0=\dfrac{\delta F}{\delta x_n}\left(x_1'\dfrac{\delta f_x}{\delta x_1'}+\ldots+x_n'\dfrac{\delta f_x}{\delta x_n'}\right)+\dfrac{\delta F}{\delta x}\dfrac{\delta f_x}{\delta x_n'}; \end{array}\right.$$

the n initial equations

$$(16)\qquad \begin{cases} 0=\dfrac{\delta F}{\delta a_1}\left(a'_1\dfrac{\delta f_a}{\delta a'_1}+\ldots+a'_n\dfrac{\delta f_a}{\delta a'_n}\right)+\dfrac{\delta F}{\delta a}\dfrac{\delta f_a}{\delta a'_1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0=\dfrac{\delta F}{\delta a_n}\left(a'_1\dfrac{\delta f_a}{\delta a'_1}+\ldots+a'_n\dfrac{\delta f_a}{\delta a'_n}\right)+\dfrac{\delta F}{\delta a}\dfrac{\delta f_a}{\delta a'_n}; \end{cases}$$

and this other equation

$$(17)\qquad 0=\lambda_a\left(a'_1\frac{\delta f_a}{\delta a'_1}+\ldots+a'_n\frac{\delta f_a}{\delta a'_n}\right)\frac{\delta F}{\delta x}+\lambda_x\left(x'_1\frac{\delta f_x}{\delta x'_1}+\ldots+x'_n\frac{\delta f_x}{\delta x'_n}\right)\frac{\delta F}{\delta a}.$$

The n differential coefficients or derived functions $x'_1, \ldots, x'_n$ can in general be eliminated between the $n+1$ equations (1) and (15), and the result will be a partial differential equation of the first order and of the form

$$(18)\qquad 0=\Psi\left(\frac{\dfrac{\delta F}{\delta x_1}}{\dfrac{\delta F}{\delta x}}, \ldots, \frac{\dfrac{\delta F}{\delta x_n}}{\dfrac{\delta F}{\delta x}}, x, x_1, \ldots, x_n\right)$$

which the principal integral relation (5) must satisfy. In like manner, by eliminating $a'_1, \ldots, a'_n$ between the n equations (16) and the initial form of (1), namely

$$(19)\qquad 0=f(a, a_1, \ldots, a_n, a'_1, \ldots, a'_n),$$

we obtain this other partial differential equation to be satisfied also by (5),

$$(20)\qquad 0=\Psi\left(\frac{\dfrac{\delta F}{\delta a_1}}{\dfrac{\delta F}{\delta a}}, \ldots, \frac{\dfrac{\delta F}{\delta a_n}}{\dfrac{\delta F}{\delta a}}, a, a_1, \ldots, a_n\right)$$

Ψ being the same function as in (18).

When the form of F is known, as well as that of f, and when both these forms are substituted in the equations (16), those n equations along with the principal integral relation (5) reduce themselves to only n distinct equations, which however are in general sufficient to determine the forms of the n functions $x_1, \ldots, x_n$. These n functions involve the independent variable x and the $2n$ arbitrary constants $a_1, \ldots, a_n, a'_1, \ldots, a'_n$, which are not however all independent, being connected by the relation (19) when a is given or assumed. Thus the system of $n+1$ equations (5) and (16) is equivalent to a system of only n distinct equations and is a form for the complete integral, with $2n-1$ arbitrary constants, of the system of n equations (1) and (6).

[*Extension to a number of differential equations of any order.*]

[**2**.] More generally, let there be a proposed differential equation:

$$(21)\qquad 0=f(x, x_1, x'_1, \ldots, x_1^{(\omega_1)}, x_2, x'_2, \ldots, x_2^{(\omega_2)}, \ldots, x_n, x'_n, \ldots, x_n^{(\omega_n)});$$

and let it be combined with the following supplementary differential equations:

$$(22)\qquad \begin{cases} 0=\lambda\dfrac{\delta f}{\delta x_1}-\dfrac{d}{dx}\left(\lambda\dfrac{\delta f}{\delta x'_1}\right)+\ldots+(-1)^{\omega_1}\left(\dfrac{d}{dx}\right)^{\omega_1}\left(\lambda\dfrac{\delta f}{\delta x_1^{(\omega_1)}}\right), \\ \\ 0=\lambda\dfrac{\delta f}{\delta x_n}-\dfrac{d}{dx}\left(\lambda\dfrac{\delta f}{\delta x'_n}\right)+\ldots+(-1)^{\omega_n}\left(\dfrac{d}{dx}\right)^{\omega_n}\left(\lambda\dfrac{\delta f}{\delta x_n^{(\omega_n)}}\right). \end{cases}$$

We shall then have

$$(23)\qquad 0=\lambda(\delta f-f'\,\delta x)=\lambda\left\{\frac{\delta f}{\delta x_1}(\delta x_1-x_1'\,\delta x)+\ldots+\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1^{(\omega_1)}-x_1^{(\omega_1+1)}\,\delta x)\right\}$$

$$+\lambda\left\{\frac{\delta f}{\delta x_n}(\delta x_n-x_n'\,\delta x)+\ldots+\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n^{(\omega_n)}-x_n^{(\omega_n+1)}\delta x)\right\}$$

$$=\lambda\frac{\delta f}{\delta x_1'}(\delta x_1'-x_1''\delta x)+\left(\lambda\frac{\delta f}{\delta x_1'}\right)'(\delta x_1-x_1'\,\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1^{(\omega_1)}-x_1^{(\omega_1+1)}\,\delta x)$$

$$+(-1)^{\omega_1+1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)}(\delta x_1-x_1'\,\delta x)$$

$$+\ldots\ldots\ldots$$

$$+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n'-x_n''\,\delta x)+\left(\lambda\frac{\delta f}{\delta x_n'}\right)'(\delta x_n-x_n'\,\delta x)+\ldots$$

$$+\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n^{(\omega_n)}-x_n^{(\omega_n-1)}\,\delta x)+(-1)^{\omega_n-1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)}(\delta x_n-x_n'\,\delta x);$$

in which

$$(24)\qquad \delta x_i^{(\omega_i)}-x_i^{(\omega_i+1)}\,\delta x=(\delta x_i-x_i'\,\delta x)^{(\omega_i)}.$$

Also in general

$$(25)\qquad yz^{(\omega_i)}+(-1)^{\omega_i+1}y^{(\omega_i)}z=\{yz^{(\omega_i-1)}-y'z^{(\omega_i-2)}+y''z^{(\omega_i-3)}-\ldots+(-1)^{\omega_i+1}y^{(\omega_i-1)}z\}'.$$

Therefore

$$(26)\qquad 0=\lambda(\delta f-f'\,\delta x)$$

$$=\left[\left\{\lambda\frac{\delta f}{\delta x_1'}-\left(\lambda\frac{\delta f}{\delta x_1''}\right)'+\ldots+(-1)^{\omega_1-1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-1)}\right\}(\delta x_1-x_1'\,\delta x)\right]'$$

$$+\ldots\ldots$$

$$+\left[\left\{\lambda\frac{\delta f}{\delta x_n'}-\left(\lambda\frac{\delta f}{\delta x_n''}\right)'+\ldots+(-1)^{\omega_n-1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-1)}\right\}(\delta x_n-x_n'\,\delta x)\right]'$$

$$+\left[\left\{\lambda\frac{\delta f}{\delta x_1''}-\left(\lambda\frac{\delta f}{\delta x_1'''}\right)'+\ldots+(-1)^{\omega_1-2}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-2)}\right\}(\delta x_1-x_1'\,\delta x)'\right]'$$

$$+.$$

$$+\left[\left\{\lambda\frac{\delta f}{\delta x_n''}-\left(\lambda\frac{\delta f}{\delta x_n'''}\right)'+\ldots+(-1)^{\omega_n-2}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-2)}\right\}(\delta x_n-x_n'\,\delta x)'\right]'$$

$$+\&c.$$

$$+\left[\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1-x_1'\,\delta x)^{(\omega_1-1)}\right]'+\ldots+\left[\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n-x_n'\delta x)^{(\omega_n-1)}\right]';$$

so that, by integration,

$$
\begin{aligned}
(27)\qquad 0 &= \int_a^x \lambda\,(\delta f - f'\,\delta x)\,dx \\
&= \Delta\Bigg[\left\{\lambda\frac{\delta f}{\delta x_1'} - \left(\lambda\frac{\delta f}{\delta x_1''}\right)' + \ldots + (-1)^{\omega_1-1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-1)}\right\}(\delta x_1 - x_1'\,\delta x) \\
&\quad + \left\{\lambda\frac{\delta f}{\delta x_1''} - \left(\lambda\frac{\delta f}{\delta x_1'''}\right)' + \ldots + (-1)^{\omega_1-2}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-2)}\right\}(\delta x_1 - x_1'\,\delta x)' \\
&\quad + \ldots + \lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1 - x_1'\,\delta x)^{(\omega_1-1)} \\
&\quad + \&\text{c.} \\
&\quad + \left\{\lambda\frac{\delta f}{\delta x_n'} - \left(\lambda\frac{\delta f}{\delta x_n''}\right)' + \ldots + (-1)^{\omega_n-1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-1)}\right\}(\delta x_n - x_n'\,\delta x) \\
&\quad + \left\{\lambda\frac{\delta f}{\delta x_n''} - \left(\lambda\frac{\delta f}{\delta x_n'''}\right)' + \ldots + (-1)^{\omega_n-2}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-2)}\right\}(\delta x_n - x_n'\,\delta x)' \\
&\quad + \ldots + \lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n - x_n'\,\delta x)^{(\omega_n-1)}\Bigg].
\end{aligned}
$$

The general term is

$$\Delta\left[(-1)^{\alpha}\left(\lambda\frac{\delta f}{\delta x_m^{(\beta+1)}}\right)^{(\alpha)}(\delta x_m - x_m'\,\delta x)^{(\beta-\alpha)}\right],$$

where α may have any value $\ngtr \beta$ and β may have any value $< \omega_m$, m having any value > 0 but $\ngtr n$. Thus

$$
\begin{aligned}
(28)\qquad 0 = {} & \Sigma_{(\alpha)0}^{\beta}\,\Sigma_{(\beta)0}^{\omega_m-1}\,\Sigma_{(m)1}^{n}\left\{(-1)^{\alpha}\left(\lambda_x\frac{\delta f_x}{\delta x_m^{(\beta+1)}}\right)^{(\alpha)}(\delta x_m - x_m'\,\delta x)^{(\beta-\alpha)}\right. \\
& \left. - \Sigma_{(\alpha)0}^{\beta}\,\Sigma_{(\beta)0}^{\omega_m-1}\,\Sigma_{(m)1}^{n}\left\{(-1)^{\alpha}\left(\lambda_a\frac{\delta f_a}{\delta a_m^{(\beta+1)}}\right)^{(\alpha)}(\delta a_m - a_m'\,\delta a)^{(\beta-\alpha)}\right\}\right..
\end{aligned}
$$

More generally still, let

$$
(29)\qquad \begin{cases}
0 = f_1\,(x,\, x_1,\, x_1',\, \ldots,\, x_1^{(\omega_{1,1})},\, x_2,\, x_2',\, \ldots,\, x_2^{(\omega_{1,2})},\, \ldots,\, x_n,\, x_n',\, \ldots,\, x_n^{(\omega_{1,n})}), \\
0 = f_2\,(x,\, x_1,\, x_1',\, \ldots,\, x_1^{(\omega_{2,1})},\, x_2,\, x_2',\, \ldots,\, x_2^{(\omega_{2,2})},\, \ldots,\, x_n,\, x_n',\, \ldots,\, x_n^{(\omega_{2,n})}), \\
\ldots\ldots \\
0 = f_m\,(x,\, x_1,\, x_1',\, \ldots,\, x_1^{(\omega_{m,1})},\, x_2,\, x_2',\, \ldots,\, x_2^{(\omega_{m,2})},\, \ldots,\, x_n,\, x_n',\, \ldots,\, x_n^{(\omega_{m,n})})
\end{cases}
$$

be any m original differential equations between any n functions $x_1, \ldots, x_n$ of an independent variable x; and let them be combined with the n following supplemental differential equations between the same functions and m multipliers $\lambda_1, \ldots, \lambda_m$:

$$
(30)\qquad \begin{cases}
0 = \lambda_1\dfrac{\delta f_1}{\delta x_1} - \left(\lambda_1\dfrac{\delta f_1}{\delta x_1'}\right)' + \ldots + (-1)^{\omega_{1,1}}\left(\lambda_1\dfrac{\delta f_1}{\delta x_1^{(\omega_{1,1})}}\right)^{(\omega_{1,1})} \\[2ex]
\quad + \lambda_m\dfrac{\delta f_m}{\delta x_1} - \left(\lambda_m\dfrac{\delta f_m}{\delta x_1'}\right)' + \ldots + (-1)^{\omega_{m,1}}\left(\lambda_m\dfrac{\delta f_m}{\delta x_1^{(\omega_{m,1})}}\right)^{(\omega_{m,1})}, \\
\quad\ldots\ldots\ldots\ldots \\
0 = \lambda_1\dfrac{\delta f_1}{\delta x_n} - \left(\lambda_1\dfrac{\delta f_1}{\delta x_n'}\right)' + \ldots + (-1)^{\omega_{1,n}}\left(\lambda_1\dfrac{\delta f_1}{\delta x_n^{(\omega_{1,n})}}\right)^{(\omega_{1,n})} \\
\quad + \ldots\ldots \\
\quad + \lambda_m\dfrac{\delta f_m}{\delta x_n} - \left(\lambda_m\dfrac{\delta f_m}{\delta x_n'}\right)' + \ldots + (-1)^{\omega_{m,n}}\left(\lambda_m\dfrac{\delta f_m}{\delta x_n^{(\omega_{m,n})}}\right)^{(\omega_{m,n})}
\end{cases}
$$

Then

$$(31)\qquad \lambda_1(\delta f_1 - f_1'\delta x) + \ldots + \lambda_m(\delta f_m - f_m'\delta x)$$

is an exact derived function, or total differential coefficient, namely the differential coefficient of the expression $\Sigma L_{i,k,l}$, in which

$$(32)\qquad L'_{i,k,l} = \lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}(\delta x_k^{(l+1)} - x_k^{(l+2)}\delta x) + (-1)^l \left(\lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}\right)^{(l+1)} (\delta x_k - x_k'\delta x),$$

and

$$(33)\qquad L_{i,k,l} = \lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}(\delta x_k^{(l)} - x_k^{(l+1)}\delta x) - \ldots + (-1)^l \left(\lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}\right)^{(l)} (\delta x_k - x_k'\delta x);$$

and hence, by integration,*

$$(34)\qquad 0 = \Sigma_{(\alpha)0}^{\beta}\, \Sigma_{(\beta)0}^{\omega_{\mu,\nu}-1}\, \Sigma_{(\mu)1}^{m}\, \Sigma_{(\nu)1}^{n} \left\{ (-1)^\alpha \left(\lambda_{\mu,x} \frac{\delta f_{\mu,x}}{\delta x_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} \right.$$
$$\left. - (-1)^\alpha \left(\lambda_{\mu,a} \frac{\delta f_{\mu,a}}{\delta a_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta a_\nu - a_\nu'\delta a)^{(\beta-\alpha)} \right\}.$$

(Feb. 19th.)

So far the Calculus of Variations conducts. But, proceeding to the Calculus of Principal Relations, it may be shown that the auxiliary or supplementary system of differential equations (30) conducts to a *principal integral* of the original system (29) of the form†

$$(35)\qquad 0 = F\begin{pmatrix} x,\ x_1,\ x_1',\ \ldots,\ x_1^{(\omega_1-1)},\ x_2,\ x_2',\ \ldots,\ x_2^{(\omega_2-1)},\ \ldots,\ x_n,\ x_n',\ \ldots,\ x_n^{(\omega_n-1)} \\ a,\ a_1,\ a_1',\ \ldots,\ a_1^{(\omega_1-1)},\ a_2,\ a_2',\ \ldots,\ a_2^{(\omega_2-1)},\ \ldots,\ a_n,\ a_n',\ \ldots,\ a_n^{(\omega_n-1)} \end{pmatrix};$$

ω_1 being the greatest exponent of the series $\omega_{1,1}, \omega_{2,1}, \ldots, \omega_{m,1}$, or more precisely an exponent of that series not less than any other exponent of the same series; ω_2 being an exponent similarly selected from the series $\omega_{1,2}, \omega_{2,2}, \ldots, \omega_{m,2}$, and so on. Hence, taking the variation of the principal integral (35), we find

$$(36)\qquad 0 = \frac{\delta F}{\delta x}\delta x + \frac{\delta F}{\delta x_1}\delta x_1 + \ldots + \frac{\delta F}{\delta x_1^{(\omega_1-1)}}\delta x_1^{(\omega_1-1)} + \ldots + \frac{\delta F}{\delta x_n}\delta x_n + \ldots + \frac{\delta F}{\delta x_n^{(\omega_n-1)}}\delta x_n^{(\omega_n-1)}$$
$$+ \frac{\delta F}{\delta a}\delta a + \frac{\delta F}{\delta a_1}\delta a_1 + \ldots + \frac{\delta F}{\delta a_1^{(\omega_1-1)}}\delta a_1^{(\omega_1-1)} + \ldots + \frac{\delta F}{\delta a_n}\delta a_n + \ldots + \frac{\delta F}{\delta a_n^{(\omega_n-1)}}\delta a_n^{(\omega_n-1)};$$

and the fundamental theorem of the Calculus of Principal Relations, for the integration of total differential equations, is that the coefficients of this last variation are proportional to the coefficients of the formula (34), or that

$$(37)\qquad \delta F = \lambda\, \Sigma_{(\alpha)0}^{\beta}\, \Sigma_{\beta(0)}^{\omega_{\mu,\nu}-1}\, \Sigma_{(\mu)1}^{m}\, \Sigma_{(\nu)1}^{n} (-1)^\alpha \left\{ \left(\lambda_{\mu,x} \frac{\delta f_{\mu,x}}{\delta x^{(\beta+1)}}\right)^{(\alpha)} (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} \right.$$
$$\left. - \left(\lambda_{\mu,a} \frac{\delta f_{\mu,a}}{\delta a_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta a_\nu - a_\nu'\delta a)^{(\beta-\alpha)} \right\}.$$

We have also, by the Calculus of Variations

$$(38)\qquad (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} = \delta x_\nu^{(\beta-\alpha)} - x_\nu^{(\beta-\alpha+1)}\delta x,$$

---

* [$\lambda_{\mu,x}$, $f_{\mu,x}$ and $\lambda_{\mu,a}$, $f_{\mu,a}$ are the final and initial values of λ_μ, f_μ.]

† See subsequent investigations of these pages, respecting the general existence of this relation.

and

$$(\delta a_\nu - a'_\nu \delta a)^{(\beta-\alpha)} = \delta a_\nu^{(\beta-\alpha)} - a_\nu^{(\beta-\alpha+1)} \delta a; \tag{39}$$

so that the fundamental theorem of the Calculus of Principal Relations may be thus written:

(40)

[*The case of one total differential equation of the first order resumed.*]

[3.] As an example of the application of this fundamental formula of the Calculus of Principal Relations, let $m=1$, $\omega_{1,1} = \omega_{1,2} = \dots = \omega_{1,n} = 1$. Then $\omega_{\mu,\nu} = 1$, $\beta = 0$, $\alpha = 0$, $\mu = 1$, and the formula (40) becomes

$$\delta F = \lambda\, \Sigma_{(\nu)1}^{n} \left\{ \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_\nu} (\delta x_\nu - x'_\nu \delta x) - \lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_\nu} (\delta a_\nu - a'_\nu \delta a) \right\}; \tag{41}$$

and resolves itself into the $2n+2$ separate equations following:

$$\frac{\delta F}{\delta x} = -\lambda\lambda_{1,x} \Sigma_{(\nu)1}^{n} \left(x'_\nu \frac{\delta f_{1,x}}{\delta x'_\nu} \right), \quad \frac{\delta F}{\delta x_1} = \lambda\lambda \quad \delta f_{1,x} \qquad \delta F \qquad \delta f_{1,x} \tag{42}$$

and

$$\frac{\delta F}{\delta a} = \lambda\lambda_{1,a} \Sigma_{(\nu)1}^{n} \left(a'_\nu \frac{\delta f_{1,a}}{\delta a'_\nu} \right), \quad \frac{\delta F}{\delta a_1} = -\lambda\lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_1}, \quad \dots, \quad \frac{\delta F}{\delta a_n} = -\lambda\lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_n} \tag{43}$$

In this example there is only one original differential equation of the form

$$0 = f_1(x, x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n) = f_{1,x}, \tag{44}$$

and the n supplementary differential equations are of the form

$$0 = \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x_1} - \left(\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_1} \right)', \quad \dots, \quad 0 = \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x_n} - \left(\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_n} \right)'; \tag{45}$$

they may also be thus written,

$$\frac{\lambda'_{1,x}}{\lambda_{1,x}} = \frac{\dfrac{\delta f_{1,x}}{\delta x_1} - \left(\dfrac{\delta f_{1,x}}{\delta x'_1} \right)'}{\dfrac{\delta f_{1,x}}{\delta x'_1}} = \frac{\dfrac{\delta f_{1,x}}{\delta x_2} - \left(\dfrac{\delta f_{1,x}}{\delta x'_2} \right)'}{\dfrac{\delta f_{1,x}}{\delta x'_2}} = \dots = \frac{\dfrac{\delta f_{1,x}}{\delta x_n} - \left(\dfrac{\delta f_{1,x}}{\delta x'_n} \right)'}{\dfrac{\delta f_{1,x}}{\delta x'_n}}, \tag{46}$$

and they give in general $n-1$ supplementary differential equations of the second order between the n functions $x_1, \dots, x_n$. If we join to these the equation

$$0 = f'_{1,x}, \tag{47}$$

which is the differential coefficient of the original equation, we shall have n equations of the second order between the n functions $x_1, \dots, x_n$, which can thus in general be found in terms of the final and initial values of the variable x and of the $2n$ initial data or constants $a_1, a'_1, a_2, a'_2, \dots, a_n, a'_n$. But these initial constants are not all arbitrary and independent, being connected by the given integral relation

$$0 = f_{1,a}. \tag{48}$$

We may therefore in general conceive the n constants $a_1', a_2', \ldots, a_n'$ eliminated by this relation (48) between the n expressions of $x_1, x_2, \ldots, x_n$, and thus a relation obtained of the form

$$(49) \qquad 0 = F(x, x_1, x_2, \ldots, x_n, a, a_1, a_2, \ldots, a_n),$$

which is in this example the *principal integral relation* between the n functions x_1, x_n, or the *principal integral of the original differential equation* (44).

The $n+1$ equations (42) give, by elimination of $\lambda\lambda_{1,x}$, the n equations following:

$$(50) \qquad 0 = \frac{\delta F}{\delta x}\frac{\delta f_{1,x}}{\delta x_1'} + \frac{\delta F}{\delta x_1}\Sigma_{(\nu)1}^{n}\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right), \quad \ldots, \quad 0 = \frac{\delta F}{\delta x}\frac{\delta f_{1,x}}{\delta x_n'} + \frac{\delta F}{\delta x_n}\Sigma_{(\nu)1}^{n}\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right);$$

and if between the n equations (50) and the original differential equation (44) we eliminate $x_1', \ldots, x_n'$, we obtain a partial differential equation of the first order, of the form

$$(51) \qquad 0 = \Psi\left(\frac{\delta F}{\delta x}, \frac{\delta F}{\delta x_1}, \ldots, \frac{\delta F}{\delta x_n}, x, x_1, \ldots, x_n\right),$$

which the function F must satisfy. In like manner, the $n+1$ equations (43) give, by elimination of $\lambda\lambda_{1,a}$,

$$(52) \qquad 0 = \frac{\delta F}{\delta a}\frac{\delta f_{1,a}}{\delta a_1'} + \frac{\delta F}{\delta a_1}\Sigma_{(\nu)1}^{n}\left(a_\nu'\frac{\delta f_{1,a}}{\delta a_\nu'}\right), \quad \ldots, \quad 0 = \frac{\delta F}{\delta a}\frac{\delta f_{1,a}}{\delta a_n'} + \frac{\delta F}{\delta a_n}\Sigma_{(\nu)1}^{n}\left(a_\nu'\frac{\delta f_{1,a}}{\delta a_\nu'}\right);$$

and if between these and equation (48) we eliminate $a_1', \ldots a_n'$, we find this other partial differential equation of the first order

$$(53) \qquad 0 = \Psi\left(\frac{\delta F}{\delta a}, \frac{\delta F}{\delta a_1}, \ldots, \frac{\delta F}{\delta a_n}, a, a_1, \ldots, a_n\right),$$

which the function F must satisfy. In these two partial differential equations the form of Ψ is the same and is homogeneous of dimension 0 with respect to the $n+1$ partial differential coefficients, final or initial, of the function F: so that in the notation of derived functions

$$(54) \qquad 0 = \frac{\delta F}{\delta x}\Psi'\left(\frac{\delta F}{\delta x}\right) + \frac{\delta F}{\delta x_1}\Psi'\left(\frac{\delta F}{\delta x_1}\right) + \ldots + \frac{\delta F}{\delta x_n}\Psi'\left(\frac{\delta F}{\delta x_n}\right),$$

and

$$(55) \qquad 0 = \frac{\delta F}{\delta a}\Psi'\left(\frac{\delta F}{\delta a}\right) + \frac{\delta F}{\delta a_1}\Psi'\left(\frac{\delta F}{\delta a_1}\right) + \ldots + \frac{\delta F}{\delta a_n}\Psi'\left(\frac{\delta F}{\delta a_n}\right).$$

The n equations (52) and equation (49) are equivalent by (53) only to n distinct equations; but these are in general sufficient, when the form of F is known, to determine the forms of the n functions $x_1, \ldots, x_n$ in terms of the independent variable x, its assumed initial value a and any $2n-1$ of the $2n$ initial data $a_1, a_1', a_2, a_2', \ldots, a_n, a_n'$, which are themselves connected by the one equation (48). The system (49)–(52) is therefore a form for the complete integral of the system of the original differential equation (44) and the $n-1$ supplementary differential equations deduced by elimination from (45), and this integration of a system of several total differential equations, by means of one principal integral relation, is the chief use of such principal relations and the reason for giving them that name.

We have, by differentiating the principal integral relation

$$0 = F' = \frac{\delta F}{\delta x} + x_1' \frac{\delta F}{\delta x_1} + \dots + x_n' \frac{\delta F}{\delta x_n}; \tag{56}$$

so that if we put, for abbreviation,

$$\frac{\delta F}{\delta x_1} = t_1 \frac{\delta F}{\delta x}, \qquad \frac{\delta F}{\delta x_n} = t_n \frac{\delta F}{\delta x}, \tag{57}$$

we shall have

$$0 = 1 + x_1' t_1 + \dots + x_n' t_n. \tag{58}$$

At the same time, by (50),

$$\frac{\delta f_{1,x}}{\delta x_1'} = -t_1 \Sigma^n_{(\nu)1} \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'} \right), \quad \dots, \quad \frac{\delta f_{1,x}}{\delta x_n'} = -t_n \Sigma^n_{(\nu)1} \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'} \right). \tag{59}$$

The variation of (58) gives

$$x_1' \delta t_1 + \dots + x_n' \delta t_n = -t_1 \delta x_1' - \dots - t_n \delta x_n'. \tag{60}$$

Therefore by (59)

$$\begin{aligned} (x_1' \delta t_1 + \dots + x_n' \delta t_n) \Sigma^n_{(\nu)1} \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'} \right) &= \frac{\delta f_{1,x}}{\delta x_1'} \delta x_1' + \dots + \frac{\delta f_{1,x}}{\delta x_n'} \delta x_n' \\ &= -\frac{\delta f_{1,x}}{\delta x} \delta x - \frac{\delta f_{1,x}}{\delta x_1} \delta x_1 - \dots - \frac{\delta f_{1,x}}{\delta x_n} \delta x_n; \end{aligned} \tag{61}$$

but also

$$\begin{aligned} x_1' \delta t_1 + \dots + x_n' \delta t_n &= x_1' \delta \frac{\dfrac{\delta F}{\delta x_1}}{\dfrac{\delta F}{\delta x}} + \dots + x_n' \delta \frac{\dfrac{\delta F}{\delta x_n}}{\dfrac{\delta F}{\delta x}} \\ &= \left(\frac{\delta F}{\delta x} \right)^{-1} \left\{ \delta \frac{\delta F}{\delta x} + x_1' \delta \frac{\delta F}{\delta x_1} + \dots + x_n' \delta \frac{\delta F}{\delta x_n} \right\}; \end{aligned} \tag{62}$$

therefore

$$\begin{aligned} 0 = \left(\delta \frac{\delta F}{\delta x} + x_1' \delta \frac{\delta F}{\delta x_1} + \dots + x_n' \delta \frac{\delta F}{\delta x_n} \right) \Sigma^n_{(\nu)1} \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'} \right) \\ + \frac{\delta F}{\delta x} \left(\frac{\delta f_{1,x}}{\delta x} \delta x + \frac{\delta f_{1,x}}{\delta x_1} \delta x_1 + \dots + \frac{\delta f_{1,x}}{\delta x_n} \delta x_n \right). \end{aligned} \tag{63}$$

Comparing this with the variation of the partial differential equation (51), that is, with the following formula:

$$\begin{aligned} 0 = \Psi' \left(\frac{\delta F}{\delta x} \right) \delta \frac{\delta F}{\delta x} + \Psi' \left(\frac{\delta F}{\delta x_1} \right) \delta \frac{\delta F}{\delta x_1} + \dots + \Psi' \left(\frac{\delta F}{\delta x_n} \right) \delta \frac{\delta F}{\delta x_n} \\ + \Psi'(x) \delta x + \Psi'(x_1) \delta x_1 + \dots + \Psi'(x_n) \delta x_n, \end{aligned} \tag{64}$$

we find

$$\Psi' \left(\frac{\delta F}{\delta x_1} \right) = x_1' \Psi' \left(\frac{\delta F}{\delta x} \right), \quad \dots, \quad \Psi' \left(\frac{\delta F}{\delta x_n} \right) = x_n' \Psi' \left(\frac{\delta F}{\delta x} \right), \tag{65}$$

$$\Psi'(x_1) \frac{\delta f_{1,x}}{\delta x} = \Psi'(x) \frac{\delta f_{1,x}}{\delta x_1}, \quad \dots, \quad \Psi'(x_n) \frac{\delta f_{1,x}}{\delta x} = \Psi'(x) \frac{\delta f_{1,x}}{\delta x_n}, \tag{66}$$

and

$$\frac{\delta F}{\delta x} \Psi' \left(\frac{\delta F}{\delta x} \right) \frac{\delta f_{1,x}}{\delta x} = \Psi'(x) \Sigma^n_{(\nu)1} \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'} \right). \tag{67}$$

The equations (65) agree evidently with (54) and (56).

Let $F+\Delta F$ be a function nearly equal to F, and let it be substituted instead of F in the partial differential equation (51) and the result developed by Taylor's theorem as far as the first powers of the coefficients of the small function ΔF. We get

$$(68)\qquad \Psi\left(\frac{\delta F}{\delta x}+\frac{\delta\Delta F}{\delta x}, \frac{\delta F}{\delta x_1}+\frac{\delta\Delta F}{\delta x_1}, \ldots, \frac{\delta F}{\delta x_n}+\frac{\delta\Delta F}{\delta x_n}, x, x_1, \ldots, x_n\right)=\Psi+\Delta\Psi=\Delta\Psi$$

$$=\Psi'\left(\frac{\delta F}{\delta x}\right)\frac{\delta\Delta F}{\delta x}+\Psi'\left(\frac{\delta F}{\delta x_1}\right)\frac{\delta\Delta F}{\delta x_1}+\ldots+\Psi'\left(\frac{\delta F}{\delta x_n}\right)\frac{\delta\Delta F}{\delta x_n},$$

that is, by (65),

$$(69)\qquad \Delta\Psi=\Psi'\left(\frac{\delta F}{\delta x}\right)\left(\frac{\delta\Delta F}{\delta x}+x_1'\frac{\delta\Delta F}{\delta x_1}+\ldots+x_n'\frac{\delta\Delta F}{\delta x_n}\right)=\Psi'\left(\frac{\delta F}{\delta x}\right)\frac{d\Delta F}{dx};$$

and therefore

$$(70)\qquad \Delta F=\int\left\{\Psi'\left(\frac{\delta F}{\delta x}\right)\right\}^{-1}\Delta\Psi\,dx.$$

And if we choose that F and ΔF shall vanish when $x=a$ (which we are always at liberty to do), we may then take a and x for the limits of the integration and write

$$(71)\qquad \Delta F=\int_a^x\left\{\Psi'\left(\frac{\delta F}{\delta x}\right)\right\}^{-1}\Delta\Psi\,dx.$$

This theorem is very important in the Calculus of Principal Relations for it enables us theoretically to express and often practically to calculate the small correction of an approximate form $F+\Delta F=0$ for the principal relation (40) by means of a definite integral.*

The important equations (65) might have been obtained more simply by observing that

$$(72)\qquad 0=d_x\delta_x F-\delta_x d_x F=d\left(\frac{\delta F}{\delta x}\delta x+\frac{\delta F}{\delta x_1}\delta x_1+\ldots+\frac{\delta F}{\delta x_n}\delta x_n\right)$$

$$-\delta\left(\frac{\delta F}{\delta x}dx+\frac{\delta F}{\delta x_1}dx_1+\ldots+\frac{\delta F}{\delta x_n}dx_n\right)$$

$$=d\frac{\delta F}{\delta x}.\delta x+d\frac{\delta F}{\delta x_1}.\delta x_1+\ldots+d\frac{\delta F}{\delta x_n}.\delta x_n-\delta\frac{\delta F}{\delta x}.dx-\delta\frac{\delta F}{\delta x_1}.dx_1-\ldots-\delta\frac{\delta F}{\delta x_n}.dx_n,$$

that is,

$$(73)\qquad 0=\delta\frac{\delta F}{\delta x}+x_1'\delta\frac{\delta F}{\delta x_1}+\ldots+x_n'\delta\frac{\delta F}{\delta x_n}-\left(\frac{\delta F}{\delta x}\right)'\delta x-\left(\frac{\delta F}{\delta x_1}\right)'\delta x_1-\ldots-\left(\frac{\delta F}{\delta x_n}\right)'\delta x_n.$$

Comparing this with (64) we find not only equations (65) but also

$$(74)\qquad \Psi'(x)=-\left(\frac{\delta F}{\delta x}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),\quad \Psi'(x_1)=-\left(\frac{\delta F}{\delta x_1}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),\quad \ldots,$$

$$\Psi'(x_n)=-\left(\frac{\delta F}{\delta x_n}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),$$

equations simpler than (66) and (67).

With respect to the logic of (72), we may observe that $x, x_1, \ldots, x_n$ are not all arbitrary when $a, a_1, \ldots, a_n$ are given but are then connected by the principal integral relation $F=0$, if the functions $x_1, \ldots, x_n$ are to satisfy the original differential equation and the principal supplementary equations assigned above. But, consistently with these differential equations and data, we may vary $x, x_1, \ldots, x_n$ provided that we make the variations satisfy the equation

---

* [This method of approximating to the principal function is exactly the same as Hamilton used in the Second Essay on Dynamics. Cf. equation (F), p. 170.]

$$0 = \delta_x F = \frac{\delta F}{\delta x}\delta x + \frac{\delta F}{\delta x_1}\delta x_1 + \ldots + \frac{\delta F}{\delta x_n}\delta x_n, \tag{75}$$

omitting squares and products of the variations. At the same time $x_1', \ldots, x_n'$ will in general vary but must satisfy (among others) the following equation, deduced by differentiation from (75),

$$\begin{aligned} 0 &= \frac{d}{dx}\left\{\frac{\delta F}{\delta x_1}(\delta x_1 - x_1'\delta x) + \ldots + \frac{\delta F}{\delta x_n}(\delta x_n - x_n'\delta x)\right\} \\ &= \left(\frac{\delta F}{\delta x_1}\right)'(\delta x_1 - x_1'\delta x) + \ldots + \left(\frac{\delta F}{\delta x_n}\right)'(\delta x_n - x_n'\delta x) \\ &\qquad + \frac{\delta F}{\delta x_1}(\delta x_1' - x_1''\delta x) + \ldots + \frac{\delta F}{\delta x_n}(\delta x_n' - x_n''\delta x) \\ &= \left(\frac{\delta F}{\delta x}\right)'\delta x + \left(\frac{\delta F}{\delta x_1}\right)'\delta x_1 + \ldots + \left(\frac{\delta F}{\delta x_n}\right)'\delta x_n + \frac{\delta F}{\delta x_1}\delta x_1' + \ldots + \frac{\delta F}{\delta x_n}\delta x_n'; \end{aligned} \tag{76}$$

and also the following, deduced by variation from (56),

$$0 = \delta\frac{\delta F}{\delta x} + x_1'\delta\frac{\delta F}{\delta x_1} + \ldots + x_n'\delta\frac{\delta F}{\delta x_n} + \frac{\delta F}{\delta x_1}\delta x_1' + \ldots + \frac{\delta F}{\delta x_n}\delta x_n'. \tag{77}$$

Comparing therefore (76) and (77) we find that we can eliminate between these two equations all the n variations $\delta x_1', \ldots, \delta x_n'$, and that the equation (73) results.

[*Two or more differential equations of the first order.*]

[4.] As another example of the application of the formulae (29)–(40), let there be two original differential equations of the first order

$$\begin{cases} 0 = f_1(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n'), \\ 0 = f_2(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n'), \end{cases} \tag{78}$$

so that now $m = 2$, $\omega_{\mu,\nu} = 1$, $\beta = 0$, $\alpha = 0$, and equation (40) becomes

$$\delta F = \lambda \Sigma_{(\mu)1}^{2} \Sigma_{(\nu)1}^{n}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}(\delta x_\nu - x_\nu'\delta x) - \lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}(\delta a_\nu - a_\nu'\delta a)\right\}. \tag{79}$$

It resolves itself into $2n+2$ separate equations

$$\frac{\delta F}{\delta x} = -\lambda\Sigma_{(\mu)1}^{2}\Sigma_{(\nu)1}^{n}\left(\lambda_{\mu,x}x_\nu'\frac{\delta f_{\mu,x}}{\delta x_\nu'}\right), \quad \frac{\delta F}{\delta x_1} = \lambda\Sigma_{(\mu)1}^{2}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right), \quad \ldots, \quad \frac{\delta F}{\delta x_n} = \lambda\Sigma_{(\mu)1}^{2}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right), \tag{80}$$

and

$$\frac{\delta F}{\delta a} = \lambda\Sigma_{(\mu)1}^{2}\Sigma_{(\nu)1}^{n}\left(\lambda_{\mu,a}a_\nu'\frac{\delta f_{\mu,a}}{\delta a_\nu'}\right), \quad \frac{\delta F}{\delta a_1} = -\lambda\Sigma_{(\mu)1}^{2}\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_1'}\right), \quad \ldots, \quad \frac{\delta F}{\delta a_n} = -\lambda\Sigma_{(\mu)1}^{2}\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_n'}\right). \tag{81}$$

(Feb. 20th.)

The supplementary differential equations are now

$$0 = \Sigma_{(\mu)1}^{2}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1} - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'\right\}, \quad \ldots, \quad 0 = \Sigma_{(\mu)1}^{2}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n} - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'\right\}. \tag{82}$$

They are n in number but they involve 2 unknown multipliers $\lambda_{1,x}$, $\lambda_{2,x}$ and their first differential coefficients $\lambda'_{1,x}$, $\lambda'_{2,x}$. They may be differentiated any number i of times, and thus we may obtain in new equations of the form

$$(83)\quad \begin{cases} 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1}\right)'-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x'_1}\right)''\right\},\ \ldots,\ 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n}\right)'-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x'_n}\right)''\right\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1}\right)^{(i)}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x'_1}\right)^{(i+1)}\right\},\ \ldots, \\ \qquad\qquad 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n}\right)^{(i)}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x'_n}\right)^{(i+1)}\right\}; \end{cases}$$

but these involve the $2i$ additional differential coefficients $\lambda''_{1,x}, \lambda''_{2,x}, \ldots, \lambda^{(i+1)}_{1,x}, \lambda^{(i+1)}_{2,x}$. If we take $i>0$, then $n+in$ will be greater than $3+2i$ (n being supposed greater than 2), and then by eliminating the $3+2i$ ratios of the $4+2i$ multipliers $\lambda_{1,x}, \lambda_{2,x}, \lambda'_{1,x}, \lambda'_{2,x}, \ldots, \lambda^{(i+1)}_{1,x}, \lambda^{(i+1)}_{2,x}$ between the $n+in$ equations (82) and (83) we shall obtain a number $i(n-2)+n-3$ of resulting differential equations of the order $i+2$ between the functions $x_1, \ldots, x_n$. Even if we take $i=0$, we shall thus have $n-3$ equations of the second order between these n functions by eliminating the 3 ratios of $\lambda_{1,x}, \lambda_{2,x}, \lambda'_{1,x}, \lambda'_{2,x}$ between the n equations (82); but we must join to these another equation, which will be of the third order, in order to determine the forms of the n functions $x_1, \ldots, x_n$; and then we shall have two equations of the first order, $n-3$ of the second and one of the third, giving for the sum of the exponents of the orders $2\times 1+(n-3)\times 2+1\times 3=2n-1$: so that the complete expressions of the n functions $x_1, \ldots, x_n$ will involve $2n-1$ independent constants and, by eliminating $n-1$ of them, we can in general obtain a relation of the form (49), which I have called the principal integral relation between the final and initial values of the n functions $x_1, \ldots, x_n$ and of the independent variable x, namely,

$$(49)\qquad 0=F(x, x_1, \ldots, x_n, a, a_1, \ldots, a_n).$$

Comparing its variation with (79), we find (80) and (81). Any three of equations (82) enable us to express $\dfrac{\lambda_{2,x}}{\lambda_{1,x}}$ in terms of $x, x_1, \ldots, x_n, x'_1, \ldots, x'_n, x''_1, \ldots, x''_n$, and therefore $\dfrac{\lambda_{2,a}}{\lambda_{1,a}}$ in terms of $a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$. Thus the n equations obtained by eliminating λ between (81) will be n equations between

$$\frac{\dfrac{\delta F}{\delta a_1}}{\dfrac{\delta F}{\delta a}}, \ldots, \frac{\dfrac{\delta F}{\delta a_n}}{\dfrac{\delta F}{\delta a}}, a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n,$$

when this value of $\dfrac{\lambda_{2,a}}{\lambda_{1,a}}$ is substituted. Also the $n-3$ supplementary differential equations of the second order, combined with the differentials of the two original differential equations (78) of the first order and with those two original equations themselves, will give, when referred to initial values, $n-1$ equations between $a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$ and two equations between $a, a_1, \ldots, a_n, a'_1, \ldots a'_n$; so that we can in general eliminate the $2n$ quantities $a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$ between the $n+1$ equations last mentioned and the n equations obtained from (81), and shall thus be conducted in general to a partial differential equation of the form (53), which will be subject to the conditions (55). The rigorous equations (65) will also hold and therefore the formula of approximation (71), as connected with a partial differential equation of the form (51) subject to the condition (54), which may be obtained by eliminating the $2n+4$ quantities

$$x_1', \ldots, x_n', x_1'', \ldots, x_n'', \lambda, \frac{\lambda_{2,x}}{\lambda_{1,x}}, \frac{\lambda_{1,x}'}{\lambda_{1,x}}, \frac{\lambda_{2,x}'}{\lambda_{1,x}}$$

between the $n+1$ equations (80), the n equations (82), the two equations (78) and the two differentials of these. Also the equations (81) or those deduced from them by elimination of λ, combined if necessary with the equation (49), are forms for the integrals of the system of the original and supplementary differential equations of the question.

As a more general example, let the original differential system be

$$(84)\quad 0=f_1(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n')=f_{1,x}, \ldots, 0=f_m(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n')=f_{m,x};$$

so that the supplementary equations are

$$(85)\quad 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'\right\}, \ldots, 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'\right\}.$$

If $n \nless 2m$, we can eliminate the $2m-1$ ratios of the $2m$ functions $\lambda_{\mu,x}$ and $\lambda_{\mu,x}'$ between these n supplementary equations; and so obtain $n-2m+1$ equations of the second order between the n functions $x_1, \ldots, x_n$, to be combined with the m original equations (84). There will then remain $m-1$ equations to be assigned between the same n functions, and these will be of the third order and will be had by elimination from any $2m-1$ of the n equations obtained by differentiating (85). And the sum of the exponents of the orders of the n equations between the n functions $x_1, \ldots, x_n$ will be $m \times 1+\overline{n-2m+1} \times 2+\overline{m-1} \times 3=2n-1$; so that we shall still have a principal integral relation of the form (49) and its variation will be of the form

$$(86)\quad \delta F=\lambda\Sigma_{(\mu)1}^{m}\Sigma_{(\nu)1}^{n}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}(\delta x_\nu-x_\nu'\delta x)-\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}(\delta a_\nu-a_\nu'\delta a)\right.$$

Hence

$$(87)\quad \frac{\delta F}{\delta x}=-\lambda\Sigma_{(\mu)1}^{m}\Sigma_{(\nu)1}^{n}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}x_\nu'\right), \quad \frac{\delta F}{\delta x_1}=\lambda\Sigma_{(\mu)1}^{m}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right),$$
$$\frac{\delta F}{\delta x_n}=\lambda\Sigma_{(\mu)1}^{m}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right),$$

and

$$(88)\quad \frac{\delta F}{\delta a}=\lambda\Sigma_{(\mu)1}^{m}\Sigma_{(\nu)1}^{n}\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}a_\nu'\right), \quad \frac{\delta F}{\delta a_1}=-\lambda\Sigma_{(\mu)1}^{m}\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_1'}\right), \quad \ldots,$$
$$\frac{\delta F}{\delta a_n}=-\lambda\Sigma_{(\mu)1}^{m}\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_n'}\right).$$

If we eliminate the $2m+2n$ quantities

$$\lambda, \frac{\lambda_{2,x}}{\lambda_{1,x}}, \ldots, \frac{\lambda_{m,x}}{\lambda_{1,x}}, \frac{\lambda_{1,x}'}{\lambda_{1,x}}, \frac{\lambda_{2,x}'}{\lambda_{1,x}}, \ldots, \frac{\lambda_{m,x}'}{\lambda_{1,x}} \quad x_1', \ldots, x_n', x_1'', \ldots, x_n''$$

between the $n+1$ equations (87), the $m+n$ equations (84), (85) and the m differentials of (84), we are in general conducted to a partial differential equation of the form (51); and an analogous elimination conducts to another partial differential equation (53)—together with conclusions similar to those drawn in pages 364–368. In particular, the equations (88), when the form of F

is known, will give the complete integrals of the system of the n differential equations between the n functions $x_1, \ldots, x_n$.

If $n < 2m$ but $2n \nless 3m$, we can then eliminate the $3m-1$ ratios of $\lambda_{\mu,x}, \lambda'_{\mu,x}, \lambda''_{\mu,x}$ between the n equations (85) and their n differentials, and so obtain $2n-3m+1$ equations of the third order between the functions $x_1, \ldots, x_n$, to be combined with the m original equations of the first order and with $2m-n-1$ equations of the fourth order (observing that $2m-n-1 < n-m$, because $2n+1 > 3m$ and therefore $2m+n-(2n+1) < 2m+n-3m$). Thus the sum of the exponents of the orders is $(m \times 1) + (\overline{2n-3m+1} \times 3) + (\overline{2m-n-1} \times 4) = 2n-1$. In this case, therefore, as in the last, we are conducted to a principal integral of the form (49) and to conclusions altogether similar.

In general the n equations (85) and their in differentials as far as the ith order contain $(i+2)m-1$ ratios of $\lambda_{\mu,x}, \lambda'_{\mu,x}, \ldots, \lambda^{(i+1)}_{\mu,x}$, and

$$n + in - (i+2)m + 1 = n(i+1) - m(i+2) + 1 = n - 2m + 1 + i(n-m).$$

If i be the least integer which makes this last remainder > 0 and k the corresponding value of the remainder, so that

$$k = i(n-m) + n - 2m + 1 \tag{89}$$

is > 0 but $\ngtr n-m$, we shall then have k supplementary equations of order $i+2$ and $n-m-k$ equations of order $(i+3)$, to be combined with the m original differential equations of the first order. The sum of the exponents of all these n differential equations between the n functions $x_1, \ldots, x_n$ is therefore

$$(m \times 1) + (k \times \overline{i+2}) + (\overline{n-m-k} \times \overline{i+3}) = n(i+3) - m(i+2) - k = 2n-1;$$

conclusions therefore follow in general of the same kind as those deduced for the particular cases above.

i is the least integer which makes $n(i+1) \nless m(i+2)$, that is, $\dfrac{n}{m} \nless \dfrac{i+2}{i+1}$, or

$$i + 1 \nless \frac{m}{n-m} \tag{90}$$

Therefore i is the next less integer to

$$\frac{m}{n-m} = \frac{\text{number of original differential equations}}{\text{number of supplementary differential equations}},$$

the supplementary differential equations being between the n functions $x_1, x_2, \ldots, x_n$ only and not involving the multipliers $\lambda_{\mu,x}$ nor their differential coefficients, which are supposed to have been eliminated. And $k = (i+1)(n-m) - m + 1$.

Thus, let $n = 10$ and let $m = 1, 2, \ldots, 9$ successively. We shall have

m	i	k	$n-m-k$	Original equations	Supplementary equations	Sum of exponents
1	0	9	0	1 of 1st order	9 of 2nd order	1+9.2 =19
2	0	7	1	2 ,, ,, ,,	7 ,, ,, ,, , 1 of 3rd order	2+7.2+1.3=19
3	0	5	2	3 ,, ,, ,,	5 ,, ,, ,, , 2 ,, ,, ,,	3+5.2+2.3=19
4	0	3	3	4 ,, ,, ,,	3 ,, ,, ,, , 3 ,, ,, ,,	4+3.2+3.3=19
5	0	1	4	5 ,, ,, ,,	1 ,, ,, ,, , 4 ,, ,, ,,	5+1.2+4.3=19
6	1	3	1	6 ,, ,, ,,	3 ,, 3rd ,, , 1 ,, 4th ,,	6+3.3+1.4=19
7	2	3	0	7 ,, ,, ,,	3 ,, 4th ,,	7+3.4 =19
8	3	1	1	8 ,, ,, ,,	1 ,, 5th ,, , 1 ,, 6th ,,	8+1.5+1.6=19
9	8	1	0	9 ,, ,, ,,	1 ,, 10th ,,	9+1.10 =19

[Differential equations of order higher than the first.]

[**5.**] If the original differential system be of the second order,

(91) $0=f_{1,x}=f_1(x, x_1, x_1', x_1'', \ldots, x_n, x_n', x_n''), \ldots, 0=f_{m,x}=f_m(x, x_1, x_1', x_1'', \ldots, x_n, x_n', x_n'')$,

then the n supplementary equations between the n original functions $x_1, \ldots, x_n$ and the m supplementary functions $\lambda_{1,x}, \ldots, \lambda_{m,x}$ are

$$(92)\quad \begin{aligned} 0&=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'+\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1''}\right)''\right\}=\sigma_{1,x},\\ 0&=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'+\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n''}\right)''\right\}=\sigma_{n,x}.\end{aligned}$$

With these we may join their derived equations up to any order i,

$$(93)\qquad 0=\sigma_{1,x}'=\sigma_{2,x}'=\ldots=\sigma_{n,x}', \ldots, 0=\sigma_{1,x}^{(i)}=\sigma_{2,x}^{(i)}=\ldots=\sigma_{n,x}^{(i)};$$

and if we take i large enough, it will always be possible to eliminate between these $n+in$ equations (92), (93) the $m(i+3)-1$ ratios of the $m(i+3)$ functions $\lambda_{\mu,x}, \lambda_{\mu,x}', \ldots, \lambda_{\mu,x}^{(i+2)}$, and so to get a number k of equations of the order $(i+4)$ between the n functions $x_1, \ldots, x_n$, where

$$(94)\qquad k=n(i+1)-m(i+3)+1=(n-m)(i+3)-2n+1.$$

Let i be the least integer which makes k positive, so that i is the least integer which gives

$$(95)\qquad i+3 \nless \frac{2n}{n-m},$$

that is, $i+2$ is the next less integer to $\frac{2n}{n-m}$ or i is the next less integer to $\frac{2m}{n-m}$. Then

$$(96)\qquad k>0, \quad k \ngtr n-m,$$

and the supplementary $n-m$ equations between the functions $x_1, \ldots, x_n$ will be k of the order $i+4$ and $n-m-k$ of the order $i+5$. The total sum of exponents will be

$$(97)\qquad (m.2)+(k.\overline{i+4})+(\overline{n-m-k}.\overline{i+5})=n(i+5)-m(i+3)-k=4n-1.$$

This therefore will be the total number of independent and arbitrary constants (besides a) in the expressions of the n functions $x_1, \ldots, x_n$, and only the same constants will enter into the n expressions of $x_1', \ldots, x_n'$. Eliminating therefore $2n-1$ constants between the $2n$ expressions of $x_1, \ldots, x_n, x_1', \ldots, x_n'$, we shall in general obtain a *principal integral* relation of the form

$$(98)\qquad 0=F(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n', a, a_1, a_1', a_2, a_2', \ldots, a_n, a_n').$$

If the original differential system be of order ω,

$$(99)\qquad \begin{cases} 0=f_{1,x}=f_1(x, x_1, x_1', \ldots, x_1^{(\omega)}, x_2, x_2', \ldots, x_2^{(\omega)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega)}),\\ \cdots\cdots\\ 0=f_{m,x}=f_m(x, x_1, x_1', \ldots, x_1^{(\omega)}, x_2, x_2', \ldots, x_2^{(\omega)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega)}),\end{cases}$$

then the n supplementary equations are

$$(100)\quad \begin{cases} 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'+\ldots+(-1)^{\omega}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1^{(\omega)}}\right)^{(\omega)}\right\}=\sigma_{1,x},\\ 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'+\ldots+(-1)^{\omega}\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n^{(\omega)}}\right)^{(\omega)}\right\}=\sigma_{n,x},\end{cases}$$

to which we may join in equations of the form (93), making in all $n(i+1)$ supplementary equations between the $m(i+\omega+1)-1$ ratios of the $m(i+\omega+1)$ functions $\lambda_{\mu,x}, \lambda'_{\mu,x}, \ldots, \lambda^{(\omega)}_{\mu,x}$. Let i be the least integer which makes

$$(101) \qquad i+\omega+1 \not< \frac{n\omega}{n-m}$$

and therefore

$$(102) \qquad n(i+1) \not< m(i+\omega+1).$$

Putting

$$(103) \qquad k = n(i+1) - m(i+\omega+1) + 1,$$

we have

$$(104) \qquad k > 0, \quad k \not> n-m;$$

we have also m original equations of the ωth order, k supplementary of order $i+2\omega$ and $n-m-k$ of order $i+2\omega+1$ between $x_1, x_2, \ldots, x_n$. The sum of all the exponents of these n equations is

$$(105) \qquad (m\,.\,\omega) + (k\,.\,i+2\omega) + (n-m-k\,.\,i+2\omega+1)$$
$$= n(i+2\omega+1) - m(i+\omega+1) - k = 2\omega n - 1.$$

This therefore is the total number of independent constants in the expressions of $x_1, x'_1, \ldots, x_1^{(\omega-1)}, x_2, x'_2, \ldots, x_2^{(\omega-1)}, \ldots, x_n, x'_n, \ldots, x_n^{(\omega-1)}$, and by eliminating $\omega n-1$ of the constants we can deduce finally a principal integral relation of the form

$$(106) \qquad 0 = F\begin{pmatrix} x, x_1, x'_1, \ldots, x_1^{(\omega-1)}, x_2, x'_2, \ldots, x_2^{(\omega-1)}, \ldots, x_n, x'_n, \ldots, x_n^{(\omega-1)} \\ a, a_1, a'_1, \ldots, a_1^{(\omega-1)}, a_2, a'_2, \ldots, a_2^{(\omega-1)}, \ldots, a_n, a'_n, \ldots, a_n^{(\omega-1)} \end{pmatrix}.$$

(Feb. 23rd.)

Let there be one original differential equation of the form

$$(107) \qquad 0 = f(x, x_1, x'_1, \ldots, x_1^{(\omega_1)}, x_2, x'_2, \ldots, x_2^{(\omega_2)}, \ldots, x_n, x'_n, \ldots, x_n^{(\omega_n)}) = f_x.$$

The n supplementary equations are then

$$(108) \qquad \left\{\begin{aligned} 0 &= \lambda\frac{\delta f}{\delta x_1} - \left(\lambda\frac{\delta f}{\delta x'_1}\right)' + \ldots + (-1)^{\omega_1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)} = \sigma_{1,x}, \\ &\ldots\ldots \\ 0 &= \lambda\frac{\delta f}{\delta x_n} - \left(\lambda\frac{\delta f}{\delta x'_n}\right)' + \ldots + (-1)^{\omega_n}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)} = \sigma_{n,x}. \end{aligned}\right.$$

We shall suppose $\omega_1 \not< 1, \omega_2 \not< \omega_1, \ldots, \omega_n \not< \omega_{n-1}$. The $n+1$ equations (107) and (108), which may be written for abridgment thus

$$(109) \qquad 0 = f,\ 0 = \sigma_1, \ldots, 0 = \sigma_n,$$

may be joined with the $(n+1)\,\omega_n - \Sigma\omega$ derived equations following:

$$(110) \qquad 0 = f', \ldots, 0 = f^{(\omega_n)},\ 0 = \sigma'_1, \ldots\ 0 = \sigma_1^{(\omega_n-\omega_1)}, \ldots, 0 = \sigma'_{n-1}, \ldots, 0 = \sigma_{n-1}^{(\omega_n-\omega_{n-1})},$$

and with these other $(n+1)\,i$ derived equations

$$(111) \qquad 0 = f^{(\omega_n+1)}, \ldots, 0 = f^{(\omega_n+i)},\ 0 = \sigma_1^{(\omega_n-\omega_1+1)}, \ldots, 0 = \sigma_1^{(\omega_n-\omega_1+i)}, \ldots,$$
$$0 = \sigma_{n-1}^{(\omega_n-\omega_{n-1}+1)}, \ldots, 0 = \sigma_{n-1}^{(\omega_n-\omega_{n-1}+i)},\ 0 = \sigma'_n, \ldots, \sigma_n^{(i)}.$$

Thus we shall have in all $(n+1)(1+\omega_n+i)-\Sigma\omega$ equations between the ω_n+i ratios of $\frac{\lambda'}{\lambda}, \ldots, \frac{\lambda^{(\omega_n+i)}}{\lambda}$, the independent variable x, the $1+\omega_n+\omega_1+i$ functions $x_1, x_1', \ldots, x_1^{(\omega_n+\omega_1+i)}$, the $1+\omega_n+\omega_2+i$ functions $x_2, x_2', \ldots, x_2^{(\omega_n+\omega_2+i)}$, ..., and finally the $1+2\omega_n+i$ functions $x_n, x_n', \ldots, x_n^{(2\omega_n+i)}$. The number of these latter functions, when we leave out $x_1, x_1', \ldots, x_1^{(\omega_n+\omega_1+i)}$, is $(n-1)(1+\omega_n+i)+\Sigma\omega-\omega_1$; and, when added to the number ω_n+i of ratios of $\frac{\lambda'}{\lambda}$ &c. it gives $n(1+\omega_n+i)+\Sigma\omega-\omega_1-1$, which will be less than the number

$$(n+1)(1+\omega_n+i)-\Sigma\omega$$

of equations provided that

$$i=2\Sigma\omega-\omega_1-\omega_n-2+k, \quad k>0. \tag{112}$$

When i is taken sufficiently large to satisfy this condition, we shall have, by elimination, k equations between $x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2+k)}$; we shall therefore have one equation between $x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-1)}$, of which differential equation the integral will contain $2\Sigma\omega-1$ arbitrary constants. These will be the only constants which will enter into the complete expressions of the n functions $x_1, \ldots, x_n$ and of all their differential coefficients. They can therefore in general be eliminated between the $2\Sigma\omega$ expressions

$$x_1, x_1', \ldots, x_1^{(\omega_1-1)}, x_2, x_2', \ldots, x_2^{(\omega_2-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n-1)},$$
$$a_1, a_1', \ldots, a_1^{(\omega_1-1)}, a_2, a_2', \ldots, a_2^{(\omega_2-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega_n-1)},$$

and thus a principal integral relation obtained of the form

$$0=F\begin{pmatrix}x, x_1, x_1', \ldots, x_1^{(\omega_1-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n-1)}\\ a, a_1, a_1', \ldots, a_1^{(\omega_1-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega_n-1)}\end{pmatrix}. \tag{113}$$

Let there be an original differential system

$$\left\{\begin{array}{l}0=f_1\,(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)})=f_{1,x},\\ \ldots\ldots\\ 0=f_m(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)})=f_{m,x},\end{array}\right. \tag{114}$$

which is to be combined with the supplementary system

$$\begin{array}{l}0=\lambda_1\frac{\delta f_1}{\delta x_1}-\left(\lambda_1\frac{\delta f_1}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_1}\left(\lambda_1\frac{\delta f_1}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)}\\ \quad +\ldots\ldots\\ \quad +\lambda_m\frac{\delta f_m}{\delta x_1}-\left(\lambda_m\frac{\delta f_m}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_1}\left(\lambda_m\frac{\delta f_m}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)}=\sigma_{1,x},\\ \\ \quad\lambda_1\frac{\delta f_1}{\delta x_n}-\left(\lambda_1\frac{\delta f_1}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_n}\left(\lambda_1\frac{\delta f_1}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)}\\ \quad +\ldots\ldots\\ \quad +\lambda_m\frac{\delta f_m}{\delta x_n}-\left(\lambda_m\frac{\delta f_m}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_n}\left(\lambda_m\frac{\delta f_m}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)}=\sigma_{n,x}.\end{array} \tag{115}$$

Let it still be supposed that

$$\omega_1 \not< 1,\ \omega_2 \not< \omega_1,\ \ldots,\ \omega_n \not< \omega_{n-1}. \tag{116}$$

The equation $\sigma_{1,x}=0$ contains

$$\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_1)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_1)}, x_1, x_1', \ldots, x_1^{(2\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_1+\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_1+\omega_n)};$$

the equation $\sigma_{2,x}=0$ contains

$$\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_2)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_2)}, x_1, x_1', \ldots, x_1^{(\omega_2+\omega_1)}, x_2, x_2', \ldots, x_2^{2\omega_2}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_2+\omega_n)};$$

and so on for the remaining $n-2$ equations of (115). Hence the equations $0=\sigma_{1,x}^{(\omega_n-\omega_1)}$, $0=\sigma_{2,x}^{(\omega_n-\omega_2)}$, &c., contain each the same things as $\sigma_{n,x}=0$; and by taking i large enough we can in general eliminate the $m(1+\omega_n+i)-1$ ratios of $\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_n+i)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_n+i)}$ and also the $(n-1)(1+\omega_n+i)-\Sigma\omega-\omega_1$ functions $x_2, \ldots, x_2^{(\omega_n+\omega_2+i)}, \ldots, x_n, \ldots, x_n^{(2\omega_n+i)}$, being in all $(n+m-1)(1+\omega_n+i)+\Sigma\omega-\omega_1-1$ things, between the $(n+m)(1+\omega_n+i)-\Sigma\omega$ equations

$$\begin{cases} 0=f_{1,x}=f_{1,x}'=\ldots=f_{1,x}^{(\omega_n+i)},\ 0=f_{2,x}=f_{2,x}'=\ldots=f_{2,x}^{(\omega_n+i)},\ \ldots,\ 0=f_{m,x}=f_{m,x}'=\ldots=f_{m,x}^{(\omega_n+i)}, \\ 0=\sigma_{1,x}=\sigma_{1,x}'=\ldots=\sigma_{1,x}^{(\omega_n-\omega_1+i)},\ 0=\sigma_{2,x}=\sigma_{2,x}'=\ldots=\sigma_{2,x}^{(\omega_n-\omega_2+i)},\ \ldots, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0=\sigma_{n,x}=\sigma_{n,x}'=\ldots=\sigma_{n,x}^{(i)}, \end{cases} \tag{117}$$

and thus obtain $\omega_1+\omega_n+i+2-2\Sigma\omega$ equations between $x, x_1, x_1', \ldots, x_1^{(\omega_1+\omega_n+i)}$. If then we take

$$i=2\Sigma\omega-\omega_1-\omega_n-1, \tag{118}$$

we shall have one differential equation of the order $2\Sigma\omega-1$ to determine the function x_1, and thus we are conducted as before to a principal integral relation of the form (113).*

Thus, from the system of the $m+n$ equations (114), (115), we can deduce by differentiation $(m+n)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ other equations and then by elimination transform the system of all these $(m+n)(2\Sigma\omega-\omega_1)-\Sigma\omega$ equations into a system of the following form:

$$x_1^{(2\Sigma\omega-1)}=\phi(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)}); \tag{119}$$

$$x_2=\phi_{2,0}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)}), \ldots, x_n=\phi_{n,0}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)}); \tag{120}$$

$$\lambda_1'=\lambda_1\psi_{1,1}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)}); \tag{121}$$

$$\lambda_2=\lambda_1\psi_{2,0}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_m=\lambda_1\psi_{m,0}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}); \tag{122}$$

$$\begin{cases} x_2'=\phi_{2,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ x_2^{(2\Sigma\omega+\omega_2-\omega_1-1)}=\phi_{2,2\Sigma\omega+\omega_2-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}), \\ \cdots\cdots \\ x_n'=\phi_{n,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ x_n^{(2\Sigma\omega+\omega_n-\omega_1-1)}=\phi_{n,2\Sigma\omega+\omega_n-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}); \end{cases} \tag{123}$$

$$\lambda_1''=\lambda_1\psi_{1,2}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_1^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{1,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}); \tag{124}$$

$$\begin{cases} \lambda_2'=\lambda_1\psi_{2,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_2^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{2,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}), \\ \cdots\cdots \\ \lambda_m'=\lambda_1\psi_{m,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_m^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{m,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}). \end{cases} \tag{125}$$

Now the equations (123) ought to be deducible by differentiation from (119) and (120); the equations (124) from (119) and (121); and the equations (125) from (119), (121) and (122).*

[*Particular cases treated in greater detail.*]

[**6.**] To illustrate this point let us consider some less general cases, and first let $m=1$, $\omega_n=1$ (and therefore $\Sigma\omega=n$), so that we have the case of equation (1), page 358. Thus for (114) and (115) we have now the $n+1$ equations (1) and (7),

$$0=f,\quad 0=\sigma_1=\sigma_2=\ldots=\sigma_n; \tag{126}$$

---

* Consult p. 384.

to which we are to join these other $2n^2 - n - 2$ derived equations

$$(127) \qquad 0 = f' = \ldots = f^{(2n-2)}, \quad 0 = \sigma_1' = \ldots = \sigma_1^{(2n-3)}, \ \ldots, \quad 0 = \sigma_n' = \ldots = \sigma_n^{(2n-3)}.$$

Then by elimination we are to deduce the $n+1$ equations

$$(128) \qquad x_1^{(2n-1)} = \phi(x, x_1, x_1', \ldots, x_1^{(2n-2)}),$$

$$(129) \qquad x_2 = \phi_{2,0}(x, \ldots, x_1^{(2n-2)}), \ \ldots, \ x_n = \phi_{n,0}(x, \ldots, x_1^{(2n-2)}),$$

$$(130) \qquad \lambda' = \lambda\psi_1(x, \ldots, x_1^{(2n-2)}),$$

and these other $2n^2 - n - 2$ equations

$$(131) \qquad \begin{cases} x_2' = \phi_{2,1}(x, \ldots, x_1^{(2n-2)}), \ \ldots, \ x_2^{(2n-1)} = \phi_{2,2n-1}(x, \ldots, x_1^{(2n-2)}), \\ \cdots\cdots \\ x_n' = \phi_{n,1}(x, \ldots, x_1^{(2n-2)}), \ \ldots, \ x_n^{(2n-1)} = \phi_{n,2n-1}(x, \ldots, x_1^{(2n-2)}), \end{cases}$$

$$(132) \qquad \lambda'' = \lambda\psi_2(x, \ldots, x_1^{(2n-2)}), \ \ldots, \ \lambda^{(2n-2)} = \lambda\psi_{2n-2}(x, \ldots, x_1^{(2n-2)}).$$

The equations (131) ought to be deducible by differentiation from the equations (128) and (129), and the equations (132) from (128) and (130).*

To particularise still further, let $n = 2$. We have then the three equations

$$(133) \quad 0 = f(x, x_1, x_1', x_2, x_2') = f, \quad 0 = \lambda\frac{\delta f}{\delta x_1} - \left(\lambda\frac{\delta f}{\delta x_1'}\right)' = \sigma_1, \quad 0 = \lambda\frac{\delta f}{\delta x_2} - \left(\lambda\frac{\delta f}{\delta x_2'}\right)' = \sigma_2,$$

and the four derived equations

$$(134) \qquad 0 = f' = f'', \quad 0 = \sigma_1', \quad 0 = \sigma_2',$$

in which

$$(135) \qquad f' = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta x_1}x_1' + \frac{\delta f}{\delta x_1'}x_1'' + \frac{\delta f}{\delta x}x_2' + \frac{\delta f}{\delta x'}x_2'',$$

and

$$(136) \qquad \begin{aligned} f'' = {} & \frac{\delta^2 f}{\delta x^2} + 2\frac{\delta^2 f}{\delta x\,\delta x_1}x_1' + 2\frac{\delta^2 f}{\delta x\,\delta x_1'}x_1'' + 2\frac{\delta^2 f}{\delta x\,\delta x_2}x_2' + 2\frac{\delta^2 f}{\delta x\,\delta x_2'}x_2'' \\ & + \frac{\delta^2 f}{\delta x_1^2}x_1'^2 + 2\frac{\delta^2 f}{\delta x_1\,\delta x_1'}x_1'x_1'' + 2\frac{\delta^2 f}{\delta x_1\,\delta x_2}x_1'x_2' + \frac{\delta^2 f}{\delta x_1\,\delta x_2'}x_1'x_2'' \\ & + \frac{\delta^2 f}{\delta x_1'^2}x_1''^2 + 2\frac{\delta^2 f}{\delta x_1'\,\delta x_2}x_1''x_2' + 2\frac{\delta^2 f}{\delta x_1'\,\delta x_2'}x_1''x_2'' \\ & + \frac{\delta^2 f}{\delta x_2^2}x_2'^2 + 2\frac{\delta^2 f}{\delta x_2\,\delta x'}x_2'x_2'' + \frac{\delta^2 f}{\delta x_2'^2}x_2''^2 \\ & + \frac{\delta f}{\delta x_1}x_1'' + \frac{\delta f}{\delta x_1'}x_1''' + \frac{\delta f}{\delta x_2}x_2'' + \frac{\delta f}{\delta x_2'}x_2'''. \end{aligned}$$

Also

$$(137) \qquad \sigma_1 = \lambda\left\{\frac{\delta f}{\delta x_1} - \left(\frac{\delta f}{\delta x_1'}\right)'\right\} - \lambda'\frac{\delta f}{\delta x_1'}, \quad \sigma_2 = \lambda\left\{\frac{\delta f}{\delta x_2} - \left(\frac{\delta f}{\delta x_2'}\right)'\right\} - \lambda'\frac{\delta f}{\delta x_2'},$$

and*

$$(138 \qquad \begin{aligned} \sigma_1' &= \lambda\left\{\left(\frac{\delta f}{\delta x_1}\right)' - \left(\frac{\delta f}{\delta x_1'}\right)''\right\} + \lambda'\left\{\frac{\delta f}{\delta x_1} - 2\left(\frac{\delta f}{\delta x_1'}\right)'\right\} - \lambda''\frac{\delta f}{\delta x_1'}, \\ \sigma_2' &= \lambda \qquad\qquad\qquad\qquad + \lambda' \qquad\qquad\qquad\qquad - \lambda''\frac{\delta f}{\delta x_2'}. \end{aligned}$$

---

* See p. 383.

(Feb. 24th, 1836.)

We may simplify these equations by supposing that the original relation between x, x_1, x_1', x_2, x_2' has been resolved with respect to x_2', and thus put under the form

$$(139)\qquad 0=\chi(x,\ x_1,\ x_1',\ x_2)-x_2'=f.$$

We shall then have

$$(140)\quad \frac{\delta f}{\delta x_2'}=-1,\quad \frac{\delta^2 f}{\delta x\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_1\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_1'\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_2\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_2'^2}=0,\quad \left(\frac{\delta f}{\delta x_2'}\right)'=0;$$

$$(141)\qquad x_2'=\chi(x,\ x_1,\ x_1',\ x_2)=f+x_2';$$

$$(142)\qquad x_2''=\frac{\delta\chi}{\delta x}+\frac{\delta\chi}{\delta x_1}x_1'+\frac{\delta\chi}{\delta x_1'}x_1''+\frac{\delta\chi}{\delta x_2}x_2'=f'+x_2'';$$

$$(143)\qquad x_2'''=\frac{\delta^2\chi}{\delta x^2}+2\frac{\delta^2\chi}{\delta x\,\delta x_1}x_1'+2\frac{\delta^2\chi}{\delta x\,\delta x_1'}x_1''+2\frac{\delta^2\chi}{\delta x\,\delta x_2}x_2'+\frac{\delta^2\chi}{\delta x_1^2}x_1'^2+2\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'x_1''$$
$$+2\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1'x_2'+\frac{\delta^2\chi}{\delta x_1'^2}x_1''^2+2\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1''x_2'+\frac{\delta^2\chi}{\delta x_2^2}x_2'^2+\frac{\delta\chi}{\delta x_1}x_1''+\frac{\delta\chi}{\delta x_1'}x_1'''+\frac{\delta\chi}{\delta x_2}x_2'';$$

$$(144)\qquad 0=\sigma_1=\lambda\left\{\frac{\delta\chi}{\delta x_1}-\left(\frac{\delta\chi}{\delta x_1'}\right)'\right\}-\lambda'\frac{\delta\chi}{\delta x_1'};$$

$$(145)\qquad 0=\sigma_2=\lambda\frac{\delta\chi}{\delta x_2}+\lambda';$$

therefore

$$(146)\qquad 0=\frac{1}{\lambda}\left(\sigma_1+\sigma_2\frac{\delta\chi}{\delta x_1'}\right)=\frac{\delta\chi}{\delta x_1}-\left(\frac{\delta\chi}{\delta x_1'}\right)'+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'},$$

that is,

$$(147)\qquad 0=\frac{\delta\chi}{\delta x_1}-\frac{\delta^2\chi}{\delta x\,\delta x_1'}+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'}-\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'-\frac{\delta^2\chi}{\delta x_1'^2}x_1''-\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\chi;$$

$$(148)\qquad 0=\sigma_1'=\lambda\left\{\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''\right\}+\lambda'\left\{\frac{\delta\chi}{\delta x_1}-2\left(\frac{\delta\chi}{\delta x_1'}\right)'\right\}-\lambda''\frac{\delta\chi}{\delta x_1'};$$

$$(149)\qquad 0=\sigma_2'=\lambda\left(\frac{\delta\chi}{\delta x_2}\right)'+\lambda'\frac{\delta\chi}{\delta x_2}+\lambda'';$$

therefore

$$(150)\quad 0=\sigma_1'+\sigma_2'\frac{\delta\chi}{\delta x_1'}=\lambda\left\{\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta\chi}{\delta x_2}\right)'\right\}+\lambda'\left\{\frac{\delta\chi}{\delta x_1}-2\left(\frac{\delta\chi}{\delta x_1'}\right)'+\frac{\delta\chi}{\delta x_1'}\frac{\delta\chi}{\delta x_2}\right\};$$

and

$$(151)\quad 0=\frac{1}{\lambda}\left(\sigma_1'+\sigma_2'\frac{\delta\chi}{\delta x_1'}\right)-\frac{\lambda'}{\lambda^2}\left(\sigma_1+\sigma_2\frac{\delta\chi}{\delta x_1'}\right)+\frac{\sigma_2}{\lambda}\left(\frac{\delta\chi}{\delta x_1'}\right)'=\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''$$
$$+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta\chi}{\delta x_2}\right)'+\frac{\delta\chi}{\delta x_2}\left(\frac{\delta\chi}{\delta x_1'}\right)'.$$

We can eliminate x_2' and x_2'' between the four equations (141), (142), (146) and (151), and thus obtain two equations between x, x_1, x_1', x_1'', x_1''', x_2; namely, the equation (147) between x, x_1, x_1',

x_1'', x_2 and another equation deduced from (151) between $x, x_1, x_1', x_1'', x_1''', x_2$. To develope this last equation, we have

$$(152)\qquad \left(\frac{\delta\chi}{\delta x_1}\right)' = \frac{\delta^2\chi}{\delta x\,\delta x_1} + \frac{\delta^2\chi}{\delta x_1^2}x_1' + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_2',$$

$$(153)\qquad \frac{\delta\chi}{\delta x_2}\Big)' = \frac{\delta^2\chi}{\delta x\,\delta x_2} + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1'' + \frac{\delta^2\chi}{\delta x_2^2}x_2',$$

$$(154)\qquad \frac{\delta\chi}{\delta x_1'}\Big)' = \frac{\delta^2\chi}{\delta x\,\delta x_1'} + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1' + \frac{\delta^2\chi}{\delta x_1'^2}x_1'' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_2'$$

$$(155)\qquad \left(\frac{\delta\chi}{\delta x_1'}\right)'' = \frac{\delta^3\chi}{\delta x^2\,\delta x_1'} + 2\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1'' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}x_2' + \frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2$$
$$+ 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1'' + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'x_2' + \frac{\delta^3\chi}{\delta x_1'^3}x_1''^2 + 2\frac{\delta^3\chi}{\delta x_1'\,\delta x_2}x_1''x_2' + \frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}x_2'^2$$
$$+ \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1'^2}x_1''' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_2'';$$

therefore

$$(156)\qquad 0 = \frac{\delta^2\chi}{\delta x\,\delta x_1} + \frac{\delta^2\chi}{\delta x_1^2}x_1' + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}\chi + \frac{\delta\chi}{\delta x_1'}\left(\frac{\delta^2\chi}{\delta x\,\delta x_2} + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1'' + \frac{\delta^2\chi}{\delta x_2^2}\chi\right)$$
$$+ \frac{\delta\chi}{\delta x_2}\left(\frac{\delta^2\chi}{\delta x\,\delta x_1'} + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1' + \frac{\delta^2\chi}{\delta x_1'^2}x_1''\right) - \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\left(\frac{\delta\chi}{\delta x} + \frac{\delta\chi}{\delta x_1}x_1' + \frac{\delta\chi}{\delta x_1'}x_1''\right)$$
$$-\left(\frac{\delta^3\chi}{\delta x^2\,\delta x_1'} + 2\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1'' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}\chi + \frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2 + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1''\right.$$
$$\left. + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'\chi + \frac{\delta^3\chi}{\delta x_1'^3}x_1''^2 + 2\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''\chi + \frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}\chi^2 + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1'^2}x_1\right).$$

The system of the three equations, (145), (147) and (156), conducts by elimination to expressions of x_1''', x_2 and $\frac{\lambda'}{\lambda}$ as functions of x, x_1, x_1', x_1'', which may be thus denoted:

$$(157)\qquad x_1''' = \phi(x, x_1, x_1', x_1''),$$

$$(158)\qquad x_2 = \phi_{2,0}(x, x_1, x_1', x_1''),$$

$$(159)\qquad \frac{\lambda'}{\lambda} = \psi_1(x, x_1, x_1', x_1'').$$

But the system of six equations (141), (142), (144), (145), (148) and (149) gives also three other expressions:

$$(160)\qquad x_2' = \phi_{2,1}(x, x_1, x_1', x_1''),$$

$$(161)\qquad x_2'' = \phi_{2,2}(x, x_1, x_1', x_1''),$$

$$(162)\qquad \frac{\lambda''}{\lambda} = \psi_2(x, x_1, x_1', x_1'');$$

and we wish to show that these last three equations can be deduced by differentiation and elimination from the equations (157)–(159), or, which will come to the same thing, that the two equations (141), (144) can be deduced from the three (145), (147) and (156). To prove this, we shall show first that (141) can be deduced by differentiation and elimination from (147) and (156). Differentiating (147), we find

$$(163)\qquad 0=\frac{\delta^2\chi}{\delta x\,\delta x_1}+\frac{\delta^2\chi}{\delta x_1^2}x_1'+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1''+\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_2'+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta^2\chi}{\delta x\,\delta x_2}+\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1'+\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1''+\frac{\delta^2\chi}{\delta x_2^2}x_2'\right)$$
$$+\frac{\delta\chi}{\delta x_2}\left(\frac{\delta^2\chi}{\delta x\,\delta x_1'}+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'+\frac{\delta^2\chi}{\delta x_1'^2}x_1''\right)-\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\left(\frac{\delta\chi}{\delta x}+\frac{\delta\chi}{\delta x_1}x_1'+\frac{\delta\chi}{\delta x_1'}x_1''\right.$$
$$-\left(\frac{\delta^3\chi}{\delta x^2\,\delta x_1'}+2\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1'+2\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1''+\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2\right.$$
$$+2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1''+\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1'^3}x_1''^2+\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}x_2'\chi$$
$$\left.+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1''+\frac{\delta^2\chi}{\delta x_1'^2}x_1'''\right),$$

and subtracting (156) from this to eliminate x_1''' we have

$$(164)\qquad 0=L\,(x_2'-\chi),$$

in which the coefficient L is

$$(165)\qquad L=\frac{\delta^2\chi}{\delta x_1\,\delta x_2}+\frac{\delta\chi}{\delta x_1'}\frac{\delta^2\chi}{\delta x_2^2}-\left(\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}+\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'+\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}\chi\right.$$
$$=\frac{\delta^2\chi}{\delta x_1\,\delta x_2}-\left(\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\right)'+\frac{\delta\chi}{\delta x_1'}\frac{\delta^2\chi}{\delta x_2^2}+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}(x_2'-\chi).$$

Now this process of differentiating (147) and then eliminating x_1''' by (156) is evidently the process to be employed in order to get an equation between x, x_1, x_1', x_1'', x_2, x_2', which when combined with (147) shall give x_2' as a function of the form (160); and we now see that this process gives $x_2'=\chi$, that is (141), because the coefficient L does not in general vanish. Having thus concluded (141), we can conclude also (146) from (147) and (156), and then (145) will enable us to conclude (144) also. We have then deduced, as we proposed, the equations (141) and (144), and can therefore deduce (142), (148) and (149) from (145), (147) and (156) or from the equivalent system (157), (158) and (159); so that this latter system conducts ultimately by differentiation and elimination to the system (160)–(162), which was the thing to be proved. (It might be useful to consider separately the cases in which the coefficient L, (165), vanishes.)

The argument may also be thus stated (without λ, λ', λ''). The original equation (141), being combined with the supplementary equation (146), which may be written

$$(166)\qquad x_2'=\psi(x,\,x_1,\,x_1',\,x_1'',\,x_2)=\left(\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\right)^{-1}\left\{\frac{\delta\chi}{\delta x_1}+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'}-\frac{\delta^2\chi}{\delta x\,\delta x_1'}-\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'-\frac{\delta^2\chi}{\delta x_1'^2}x_1''\right\},$$

and with its own differential (142) and the differential of (166), namely

$$(167)\qquad =\frac{\delta\psi}{\delta x}+\frac{\delta\psi}{\delta x_1}x_1'+\frac{\delta\psi}{\delta x_1'}x_1''+\frac{\delta\psi}{\delta x_1''}x_1'''+\frac{\delta\psi}{\delta x_2}x_2',$$

gives the two following equations:

$$(168) \qquad \chi(x, x_1, x_1', x_2) = \psi(x, x_1, x_1', x_1'', x_2),$$

$$(169) \qquad \frac{\delta\chi}{\delta x} + \frac{\delta\chi}{\delta x_1} x_1' + \frac{\delta\chi}{\delta x_1'} x_1'' + \frac{\delta\chi}{\delta x_2} \chi = \frac{\delta\psi}{\delta x} + \frac{\delta\psi}{\delta x_1} x_1' + \frac{\delta\psi}{\delta x_1'} x_1'' + \frac{\delta\psi}{\delta x_1''} x_1''' + \frac{\delta\psi}{\delta x_2} \chi.$$

These two equations combined give, by elimination, two others of the forms (157) and (158), and reciprocally we may consider these two last equations as conducting to (168) and (169). Now the differential of (168), when (169) is subtracted from it, gives

$$(170) \qquad \frac{\delta\chi}{\delta x_2} - \frac{\delta\psi}{\delta x_2}\Big) (x_2' - \chi) = 0,$$

and, since in general

$$(171) \qquad \frac{\delta\chi}{\delta x_2} \mp \frac{\delta\psi}{\delta x_2},$$

we conclude that the equations (168) and (169) give (141) and therefore also (166), (142) and (167). The equations therefore of the forms (160) and (161), which result along with (157) and (158) by elimination from the four equations (141), (142), (166) and (167), are consequences of equations (157) and (158) and can be deduced from them by differentiation and elimination.

When the original equation is left under the form

$$(133) \qquad 0 = f(x, x_1, x_1', x_2, x_2') = f,$$

and the supplementary equation (freed from λ) is of the form

$$(172) \qquad 0 = -\frac{}{\frac{\delta f}{\delta x_1'}} \cdot \frac{}{\lambda \frac{\delta f}{\delta x_2'}} - \frac{\frac{\delta f}{\delta x_1} - \left(\frac{\delta f}{\delta x_1'}\right)'}{\frac{\delta f}{\delta x_1'}} \quad \frac{\frac{\delta f}{\delta x_2} - \left(\frac{\delta f}{\delta x_2'}\right)'}{\frac{\delta f}{\delta x_2'}} = \psi(x, x_1, x_1', x_1'', x_2, x_2', x_2''),$$

the function ψ being linear with respect to x_1'' and x_2'', then these two equations, combined with the three derived equations

$$(173) \qquad 0 = f', \quad 0 = f'', \quad 0 = \psi',$$

will in general enable us to deduce by elimination x_1'''', x_2, x_2', x_2'', x_2''' as functions of x, x_1, x_1', x_1'' of the forms (157), (158), (160), (161) and

$$(174) \qquad x_2''' = \phi_{2,3}(x, x_1, x_1', x_1'').$$

We may propose to show that equations (160), (161) and (174) can be deduced from (157) and (158). We may begin by eliminating x_2'' and x_2''' between the four equations (172), (173), and so obtain two equations between x, x_1, x_1', x_1'', x_1''', x_2, x_2' to be combined with (133). Then it must be shown that if x_1^{iv} be eliminated between the differentials of the three last mentioned equations the two resulting equations will both be satisfied by and will give the equation (172). Eliminating x_2''' between the equations $0 = f''$, $0 = \psi'$, we find

$$(175) \qquad 0 = \frac{\delta\psi}{\delta x_2''} f'' - \frac{\delta f}{\delta x_2'} \psi' = \text{funct}(x, x_1, x_1', x_1'', x_1''', x_2', x_2''),$$

quadratic with respect to x_2''; also eliminating x_2'' between the equations $0 = f'$, $0 = \psi$, we find

$$(176) \qquad 0 = \frac{\delta\psi}{\delta x_2''} f' - \frac{\delta f}{\delta x_2'} \psi = \text{funct}(x, x_1, x_1', x_1'', x_2, x_2');$$

and another equation of the form

$$(177) \qquad 0=\text{funct}\,(x,\ x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2')$$

will be obtained by eliminating x_2'' from (175) with the help of either $0=f'$ or $0=\psi$. The equations (133), (176) and (177) are the three above alluded to, which give x_1''', x_2, x_2' as functions of x, x_1, x_1', x_1''. Differentiating them all we should get three new expressions for x_1^{iv}, x_2', x_2'' as functions of the same four quantities x, x_1, x_1', x_1''. We must show that the two expressions for x_2' agree with each other and that the expressions for x_2, x_2', x_2'' agree with the equation $0=\psi$. We must show therefore that, on eliminating x_1''', x_1^{iv} between (133), (176), (177) and their three differentials, the four resulting equations between x, x_1, x_1', x_1'', x_2, x_2', x_2'' are equivalent to $0=f$, $0=f'$, $0=\psi$. This comes to showing that the two equations $0=f$, $0=f'$, combined with the three equations (175), (176) and the differential of (176), give by the elimination of x_1''' only the three equations $0=f$, $0=f'$, $0=\psi$. And this is easily seen to be true.

(Feb. 25th.)

We may also present the argument as follows. The system of five equations

$$(178) \qquad 0=f, \quad 0=f', \quad 0=f'', \quad 0=\psi, \quad 0=\psi'$$

may be conceived to determine x_1''', x_2, x_2', x_2'' and x_2''' as functions of x, x_1, x_1', x_1'', expressed by five equations of the forms (157), (158), (160), (161), (174). If we differentiate these five functions, making

$$(179) \qquad dx_1=x_1'\,dx, \quad dx_1'=x_1''\,dx, \quad dx_1''=x_1'''\,dx,$$

we shall get expressions for the five first differential coefficients

$$\frac{dx_1'''}{dx}, \quad \frac{dx_2}{dx}, \quad \frac{dx_2'}{dx}, \quad \frac{dx_2''}{dx}, \quad \frac{dx_2'''}{dx}.$$

If we differentiate in the same way the five equations (178), without that previous elimination which would conduct to the five separate expressions (157), (158), (160), (161), (174), we shall still obtain five linear equations to determine the same five differential coefficients, namely

$$(180) \qquad 0=df, \quad 0=df', \quad 0=df'', \quad 0=d\psi, \quad 0=d\psi';$$

and a subsequent elimination between these five linear equations will give these five coefficients. In order then to show that the two equations (157), (158) conduct in general to the equations (160), (161) and (174), it is necessary and sufficient to show that the equations (178), when combined with the three equations

$$(181) \qquad 0=\frac{df}{dx}, \quad 0=\frac{df'}{dx}, \quad 0=\frac{d\psi}{dx}$$

$\left(\text{which result from (180) by the elimination of } \frac{dx_1'''}{dx}, \frac{dx_2'''}{dx}\right)$ and with (179), conduct to the following relations:

$$(182) \qquad \frac{dx_2}{dx}=x_2', \quad \frac{dx_2'}{dx}=x_2'', \quad \frac{dx_2''}{dx}=x_2'''.$$

Accordingly equations (178), (179) and (181) give

$$(183)\quad \begin{cases} 0=\dfrac{df}{dx}-f'=\dfrac{\delta f}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta f}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right), \\ 0=\dfrac{df'}{dx}-f''=\dfrac{\delta f'}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta f'}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right)+\dfrac{\delta f'}{\delta x_2''}\left(\dfrac{dx_2''}{dx}-x_2'''\right), \quad \left[\dfrac{\delta f'}{\delta x_2''}=\dfrac{\delta f}{\delta x_2'}\right], \\ 0=\dfrac{d\psi}{dx}-\psi'=\dfrac{\delta\psi}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta\psi}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right)+\dfrac{\delta\psi}{\delta x_2''}\left(\dfrac{dx_2''}{dx}-x_2'''\right), \end{cases}$$

and these three linear equations conduct in general to the three relations (182).

Had we retained the λ's we might have argued thus. The equations (133), namely $0=f$, $0=\sigma_1$, $0=\sigma_2$, with their four derived equations

$$(134)\qquad 0=f',\quad 0=f'',\quad 0=\sigma_1',\quad 0=\sigma_2',$$

may be conceived to conduct, by elimination, to seven separate expressions of the forms (157), (158), (160), (161), (174), (159) and (162) for x_1''', x_2, x_2', x_2'', x_2''', $\dfrac{\lambda'}{\lambda}$, $\dfrac{\lambda''}{\lambda}$ as functions ϕ, $\phi_{2,0}$, $\phi_{2,1}$, $\phi_{2,2}$, $\phi_{2,3}$, ψ_1, ψ_2 of x, x_1, x_1', x_1''. If these seven functions had been actually found by performing this conceived elimination, we should then be able to deduce by differentiation and substitution expressions for the following differential coefficients:

$$(184)\qquad \frac{dx_1'''}{dx},\quad \frac{dx_2}{dx},\quad \frac{dx_2'}{dx},\quad \frac{dx_2''}{dx},\quad \frac{dx_2'''}{dx},$$

and also for the following ratios:

$$(185)\qquad \frac{1}{\lambda}\frac{d\lambda'}{dx},\quad \frac{1}{\lambda}\frac{d\lambda''}{dx},$$

provided that we substitute for dx_1, dx_1', dx_1'' their values (179) and also make

$$(186)\qquad d\lambda=\lambda' dx.$$

Or, instead of thus previously eliminating between the seven equations (133) and (134), we may at once differentiate these equations changing dx, dx_1', dx_1'', $d\lambda$ to their values (179), (186) and then determine the seven ratios (184), (185) by elimination between seven linear equations. And of these seven linear equations we need use only the four following:

$$(187)\qquad 0=\frac{df}{dx},\quad 0=\frac{df'}{dx},\quad 0=\frac{d\sigma_1}{dx},\quad 0=\frac{d\sigma_2}{dx},$$

(in combination with (133) and (134)) if we only want to deduce the four ratios

$$(188)\qquad \frac{dx_2}{dx},\quad \frac{dx_2'}{dx},\quad \frac{dx_2''}{dx},\quad \frac{1}{\lambda}\frac{d\lambda'}{dx},$$

and to examine whether they satisfy the equations (182) and also the following:

$$(189)\qquad \frac{1}{\lambda}\frac{d\lambda'}{dx}=\frac{\lambda''}{\lambda};$$

which four equations are the conditions requisite in order that the equations (160), (161), (174), (162) may result from the equations (157), (158), (159). Now the two first equations (187) combined with the two first equations (134) and with (182) give the two first equations (183), and the two last equations (187) combined with the two last equations (134) and with (182), (189) give in like manner the two following equations:

$$(190)\quad \begin{aligned} 0&=\frac{d\sigma_1}{dx}-\sigma_1'=\frac{\delta\sigma_1}{\delta x_2}\left(\frac{dx_2}{dx}-x_2'\right)+\frac{\delta\sigma_1}{\delta x_2'}\left(\frac{dx_2'}{dx}-x_2''\right)+\frac{\delta\sigma_1}{\delta x_2''}\left(\frac{dx_2''}{dx}-x_2'''\right)+\lambda\frac{\delta\sigma_1}{\delta\lambda'}\left(\frac{1}{\lambda}\frac{d\lambda'}{dx}-\frac{\lambda''}{\lambda}\right),\\ 0&=\frac{d\sigma_2}{dx}-\sigma_2'=\frac{\delta\sigma_2}{\delta x_2}\left(\frac{dx_2}{dx}-x_2'\right)+\dots \end{aligned}$$

And accordingly these equations (190) combined with the two first equations (183) conduct in general, by the ordinary process of elimination between four equations of the first degree, to the equations (182) and (189).

In the case of the system of $2n^2-1$ equations (126), (127), pages 375, 376, we may seek the $(n-1)(2n-1)$ differential coefficients

$$(191)\quad \frac{dx_2}{dx},\ \frac{dx_2'}{dx},\ \dots,\ \frac{dx_2^{(2n-2)}}{dx},\ \dots,\ \frac{dx_n}{dx},\ \frac{dx_n'}{dx},\quad \frac{dx_n^{(2n-2)}}{dx}$$

and the $2n-3$ ratios

$$(192)\quad \frac{1}{\lambda}\frac{d\lambda'}{dx},\ \frac{1}{\lambda}\frac{d\lambda''}{dx},\ \dots,\ \frac{1}{\lambda}\frac{d\lambda^{(2n-3)}}{dx}$$

by elimination between the $2n^2-n-2$ equations

$$(193)\quad 0=\frac{df}{dx},\ 0=\frac{df'}{dx},\ \dots,\ 0=\frac{df^{(2n-3)}}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_1}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_1'}{dx},\quad 0=\frac{1}{\lambda}\frac{d\sigma_1^{(2n-4)}}{dx},$$
$$0=\frac{1}{\lambda}\frac{d\sigma_n}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_n'}{dx},\ \dots,\ 0=\frac{1}{\lambda}\frac{d\sigma_n^{(2n-4)}}{dx},$$

in which we are to substitute the values

$$(194)\quad \frac{dx_1}{dx}=x_1',\ \dots,\ \frac{dx_1^{(2n-2)}}{dx}=x_1^{(2n-1)}$$

and the value

$$\frac{1}{\lambda}\frac{d\lambda}{dx}=\frac{\lambda'}{\lambda},$$

and with which we may combine the equations (126) and (127) themselves. In this manner we obtain $2n^2-n-2$ equations of the first degree of the forms

$$(195)\quad 0=\frac{df}{dx}-f',\ \dots,\ 0=\frac{df^{(2n-3)}}{dx}-f^{(2n-2)},\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_1}{dx}-\sigma_1'\right),\ \dots,$$
$$0=\frac{1}{\lambda}\left(\frac{d\sigma_1^{(2n-4)}}{dx}-\sigma_1^{(2n-3)}\right),\ \dots,\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_n}{dx}-\sigma_n'\right),\ \dots,\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_n^{(2n-4)}}{dx}-\sigma_n^{(2n-3)}\right),$$

which give

$$(196)\quad \frac{dx_2}{dx}=x_2',\quad \frac{dx_2^{(2n-2)}}{dx}=x_2^{(2n-1)},\ \dots,\ \frac{dx_n}{dx}=x_n',\ \dots,\ \frac{dx_n^{(2n-2)}}{dx}=x_n^{(2n-1)},$$
$$\frac{1}{\lambda}\frac{d\lambda'}{dx}=\frac{\lambda'}{\lambda},\ \dots,\ \frac{1}{\lambda}\frac{d\lambda^{(2n-3)}}{dx}=\frac{\lambda^{(2n-2)}}{\lambda}.$$

This shows that the $2n^2-n-2$ equations (131), (132) result by differentiation and elimination from the $n+1$ equations (128)–(130).

In the more general case of the system of $(n+m)(2\Sigma\omega-\omega_1)-\Sigma\omega$ equations (117), which may be written (on account of the value (118) of i)

$$(197)\quad \begin{cases} 0=f_1,\ 0=f_1',\ \ldots,\ 0=f_1^{(2\Sigma\omega-\omega_1-1)},\ \ \ldots,\ 0=f_m,\ 0=f_m',\ \ldots,\ 0=f_m^{(2\Sigma\omega-\omega_1-1)}, \\ 0=\sigma_1,\ 0=\sigma_1',\ \ldots,\ 0=\sigma_1^{(2\Sigma\omega-2\omega_1-1)},\ \ldots,\ 0=\sigma_n,\ 0=\sigma_n',\ \ldots,\ 0=\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-1)}, \end{cases}$$

if we establish the relations

$$(198)\qquad \frac{dx_1}{dx}=x_1',\ \frac{dx_1'}{dx}=x_1'',\ \ldots,\ \frac{dx_1^{(2\Sigma\omega-2)}}{dx}=x_1^{(2\Sigma\omega-1)},\ \frac{1}{\lambda_1}\frac{d\lambda_1}{dx}=\frac{\lambda_1'}{\lambda_1},$$

then the following $(n+m)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ linear equations between $\frac{dx_2}{dx}$ &c., namely

$$(199)\qquad \begin{aligned} &=\frac{df_1}{dx}-f_1',\ \ldots,\ 0=\frac{df_1^{(2\Sigma\omega-\omega_1-2)}}{dx}-f_1^{(2\Sigma\omega-\omega_1-1)},\ \ldots,\ 0=\frac{df_m}{dx}-f_m',\ \ldots, \\ &\qquad\qquad 0=\frac{df_m^{(2\Sigma\omega-\omega_1-2)}}{dx}-f_m^{(2\Sigma\omega-\omega_1-1)}, \\ &0=\frac{1}{\lambda_1}\left(\frac{d\sigma_1}{dx}-\sigma_1'\right),\ \ldots,\ 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_1^{(2\Sigma\omega-2\omega_1-2)}}{dx}-\sigma_1^{(2\Sigma\omega-2\omega_1-1)}\right),\ \ldots, \\ &\qquad\qquad 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_n}{dx}-\sigma_n'\right),\ \ldots,\ 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-2)}}{dx}-\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-1)}\right), \end{aligned}$$

will give by elimination

$$(200)\qquad \begin{aligned} &\frac{dx_2}{dx}=x_2',\quad \frac{dx_2^{(2\Sigma\omega+\omega_2-\omega_1-2)}}{dx}=x_2^{(2\Sigma\omega+\omega_2-\omega_1-1)},\ \ldots,\ \frac{dx_n}{dx}=x_n',\ \ldots, \\ &\qquad\qquad \frac{dx_n^{(2\Sigma\omega+\omega_n-\omega_1-2)}}{dx}=x_n^{(2\Sigma\omega+\omega_n-\omega_1-1)}, \\ &\frac{1}{\lambda_1}\frac{d\lambda_1'}{dx}=\frac{\lambda_1''}{\lambda_1},\quad \frac{1}{\lambda_1}\frac{d\lambda_1^{(2\Sigma\omega-\omega_1-2)}}{dx}=\frac{\lambda_1^{(2\Sigma\omega-\omega_1-1)}}{\lambda_1},\ \ldots,\ \frac{1}{\lambda_1}\frac{d\lambda_m}{dx}=\frac{\lambda_m'}{\lambda_1},\ \ldots, \\ &\qquad\qquad \frac{1}{\lambda_1}\frac{d\lambda_m^{(2\Sigma\omega-\omega_1-2)}}{dx}=\frac{\lambda_m^{(2\Sigma\omega-\omega_1-1)}}{\lambda_1}. \end{aligned}$$

Therefore the $(n+m)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ equations (123)–(125) result by differentiation and elimination from the $n+m$ equations (119)–(122).

Thus the system of these $n+m$ equations (119)–(122) is not only a necessary consequence of the system of the $n+m$ equations (114), (115) but also, reciprocally, the latter equations follow from the former. And the proof of this reciprocity was necessary, in strictness, in order to show that the complete expressions of the functions $x_1, \ldots, x_n$ in the equations (114), (115) involve *so many* as $2\Sigma\omega-1$ arbitrary constants. For the mere deduction of the system (119), (120) (with or without the equations (121), (122)) as a necessary consequence of (114), (115) only entitled us to infer that the complete expressions of $x_1, \ldots, x_n$, deduced from this last mentioned system, *could not contain more* than $2\Sigma\omega-1$ arbitrary constants; and left it doubtful whether some other combination of the equations of that system might not conduct to some new differential equation between x_1 and x *of an order lower than* $2\Sigma\omega-1$, in which case the number of arbitrary constants would be less than $2\Sigma\omega-1$, being always equal to the exponent of the lowest order in any given set of total differential equations between a function x_1 and a variable x which are supposed to be mutually compatible.

If we include the λ's, then the complete integral of the system of the $m+n$ equations (114), (115) will involve in all $2\Sigma\omega$ arbitrary constants, an additional constant being introduced by the integration of the equation (121) or by the circumstance that all the m multipliers λ may be together multiplied by any common constant without altering the relations expressed by the differential equations.

What if we had two original equations of the forms

$$(201)\qquad 0=f_1(x,\, x_1,\, x_1',\, x_2,\, x_2',\, x_3,\, x_3')=f_1, \quad 0=f_2(x,\, x_1,\, x_1',\, x_2,\, x_2',\, x_3,\, x_3',\, x_3'')=f_2?$$

Could we then apply the same reasoning as in the case of a system of the form (114), in which all the equations were of some common order (ω_1 for x_1, ω_2 for x_2, &c.) for each of the n functions though this order might change in passing from one function to another, whereas here the first equation is only of the first order with respect to x_3 while the second is of the second order with respect to the same function?

We have now the case included in the general form (29), in which $m=2$, $n=3$, $\omega_{1,1}=1$, $\omega_{1,2}=1$, $\omega_{1,3}=1$, $\omega_{2,1}=1$, $\omega_{2,2}=1$, $\omega_{2,3}=2$. And the supplementary equations (30) become now

$$(202)\qquad \left\{\begin{aligned} 0&=\lambda_1\frac{\delta f_1}{\delta x_1}-\left(\lambda_1\frac{\delta f_1}{\delta x_1'}\right)'+\lambda_2\frac{\delta f_2}{\delta x_1}-\left(\lambda_2\frac{\delta f_2}{\delta x_1'}\right)'=\sigma_1,\\ 0&=\lambda_1\frac{\delta f_1}{\delta x_2}-\left(\lambda_1\frac{\delta f_1}{\delta x_2'}\right)'+\lambda_2\frac{\delta f_2}{\delta x_2}-\left(\lambda_2\frac{\delta f_2}{\delta x_2'}\right)'=\sigma_2,\\ 0&=\lambda_1\frac{\delta f_1}{\delta x_3}-\left(\lambda_1\frac{\delta f_1}{\delta x_3'}\right)'-\lambda_2\frac{\delta f_2}{\delta x_3}-\left(\lambda_2\frac{\delta f_2}{\delta x_3'}\right)'+\left(\lambda_2\frac{\delta f_2}{\delta x_3''}\right)''=\sigma_3.\end{aligned}\right.$$

If the circumstance of f_1 not containing x_3'' be not sufficient to interrupt the application of our recent process, we are to consider $\omega_1=\omega_2=1$, $\omega_3=2$, that is, $\Sigma\omega=4$; and we are to form the system of 31 equations:

$$(203)\qquad \left\{\begin{array}{lllllll} 0=f_1, & 0=f_1', & 0=f_1'', & 0=f_1''', & 0=f_1^{\text{iv}}, & 0=f_1^{\text{v}}, & 0=f_1^{\text{vi}},\\ 0=f_2, & 0=f_2', & & & & , & 0=f_2^{\text{vi}},\\ 0=\sigma_1, & 0=\sigma_1', & 0=\sigma_1'', & 0=\sigma_1''', & 0=\sigma_1^{\text{iv}}, & 0=\sigma_1^{\text{v}}, & \\ 0=\sigma_2, & 0=\sigma_2', & & & , & 0=\sigma_2^{\text{v}}, & \\ 0=\sigma_3, & 0=\sigma_3', & & , & 0=\sigma_3^{\text{iv}}. & & \end{array}\right.$$

Then, if possible, we are to eliminate the 26 quantities

$$(204)\qquad x_2',\, x_2'',\, \ldots,\, x_2^{\text{vii}},\, x_3',\, \ldots,\, x_3^{\text{viii}},\, \frac{\lambda_1''}{\lambda_1},\, \frac{\lambda_1'''}{\lambda_1},\, \ldots,\, \frac{\lambda_1^{\text{vi}}}{\lambda_1},\, \frac{\lambda_2'}{\lambda_1},\, \frac{\lambda_2''}{\lambda_1},\qquad \frac{\lambda_2^{\text{vi}}}{\lambda_1}$$

between the 31 equations (203) and so to obtain a new system of five equations between the following 13 quantities

$$(205)\qquad x,\, x_1,\, x_1',\, x_1'',\, \ldots,\, x_1^{\text{vii}},\, x_2,\, x_3,\, \frac{\lambda_1'}{\lambda_1},\, \frac{\lambda_2}{\lambda_1},$$

enabling us to express

$$(206)\qquad x_1^{\text{vii}},\, x_2,\, x_3,\, \frac{\lambda_1'}{\lambda_1},\, \frac{\lambda_2}{\lambda_1}$$

as functions of x, x_1, x_1', x_1'', x_1''', x_1^{iv}, x_1^{v} and x_1^{vi}. We must now endeavour to show that the deduction of equations (205) from equations (203), or ultimately from equations (201), (202), is possible; and that, reciprocally, these five equations (201), (202) can be deduced by differentiation and elimination from (205).

The equation $0 = f_1^{(i)}$ involves

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+1)},\ x_2,\ x_2',\qquad x_2^{(i+1)},\ x_3,\ x_3',\ \ldots,\ x_3^{(i+1)}$$

and the equation $0 = f_2^{(i)}$ involves

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+1)},\ x_2,\ x_2',\ \ldots,\ x_2^{(i+1)},\ x_3,\ x_3',\ \ldots,\ x_3^{(i+2)}.$$

The equation $0 = \sigma_1$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_2,\ x_2',\ x_2'',\ x_3,\ x_3',\ x_3'',\ x_3''',\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},$$

the equation $0 = \sigma_2$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_2,\ x_2',\ x_2'',\ x_3,\ x_3',\ x_3'',\ x_3''',\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1}$$

and the equation $0 = \sigma_3$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2',\ x_2'',\ x_2''',\ x_3,\ x_3',\ x_3'',\ x_3''',\ x_3^{\text{iv}},\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},\ \frac{\lambda_2''}{\lambda_1}.$$

Hence the equations $0 = \sigma_1^{\text{v}}$, $0 = \sigma_2^{\text{v}}$, $0 = \sigma_3^{\text{iv}}$ involve each the quantities

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{\text{vii}},\ x_2,\ x_2',\ \ldots,\ x_2^{\text{vii}},\ x_3,\ x_3',\ \ldots,\ x_3^{\text{viii}},\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_1''}{\lambda_1},\ \ldots,\ \frac{\lambda_1^{\text{vi}}}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},\qquad \frac{\lambda_2^{\text{vi}}}{\lambda_1},$$

except that $0 = \sigma_3^{\text{iv}}$ does not involve $\dfrac{\lambda_1^{\text{vi}}}{\lambda_1}$. Generally the three equations

$$0 = \sigma_1^{(i+1)},\quad 0 = \sigma_2^{(i+1)},\quad 0 = \sigma_3^{(i)}$$

contain all the same quantities, namely

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+3)},\ x_2,\ \ldots,\ x_2^{(i+3)},\ x_3,\ \ldots,\ x_3^{(i+4)},\ \frac{\lambda_1'}{\lambda_1},\ \ldots,\ \frac{\lambda_1^{(i+2)}}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \ldots,\ \frac{\lambda_2^{(i+2)}}{\lambda_1},$$

except that $0 = \sigma_3^{(i)}$ does not contain $\dfrac{\lambda_1^{(i+3)}}{\lambda_1}$. This last mentioned circumstance, however, does not hinder us to eliminate $\dfrac{\lambda_1^{\text{vi}}}{\lambda_1}, \dfrac{\lambda_2^{\text{vi}}}{\lambda_1}, x_2^{\text{vii}}, x_3^{\text{viii}}$ between the five equations $0 = f_1^{\text{vi}} = f_2^{\text{vi}} = \sigma_1^{\text{v}} = \sigma_2^{\text{v}} = \sigma_3^{\text{vi}}$, nor does the circumstance that $0 = f_1^{\text{vi}}$ does not involve x_3^{viii}. And generally the circumstance that some of the 26 things (204) to be eliminated do not appear in some of the 31 equations (203) does not hinder us to effect the elimination supposed above, but rather tends to facilitate it. Since then the equations (203) contain altogether only the 26 things (204) and the 13 things (205) and are in general 31 distinct and independent relations between these 39 things *so far as elimination is concerned* (although, *with respect to differentiation*, only 5 of these relations are independent and the 26 others are deduced from them), we can deduce by elimination *at least five* equations between the 13 things (205).

Indeed, a doubt might be felt whether we could not in general deduce *more than five* equations between these 13 things; or, in other words, whether we could in general return by differentiation and elimination from the five equations thus deduced to the system of the 31 equations (203). But if we suppose (as, setting aside singular exceptions, we may and ought to do) that *for purposes of elimination* the 31 relations (203) are distinct and independent and therefore suffice to determine the 31 functions (204), (205), we can conclude that the expressions of the 26 functions

(204) result by differentiation and elimination from those of the five functions (206); namely, by showing that the following 26 linear equations between $\frac{dx_2}{dx}$, &c.,

$$(207)\quad \begin{aligned} &0=\frac{df_1}{dx}-f_1', \ \ldots,\ 0=\frac{df_1^{\mathrm{v}}}{dx}-f_1^{\mathrm{vi}},\ 0=\frac{df_2}{dx}-f_2',\ \ldots,\ 0=\frac{df_2^{\mathrm{v}}}{dx}-f_2^{\mathrm{vi}}, \\ &0=\frac{d\sigma_1}{dx}-\sigma_1',\ \ldots,\ 0=\frac{d\sigma_1^{\mathrm{iv}}}{dx}-\sigma_1^{\mathrm{v}},\ 0=\frac{d\sigma_2}{dx}-\sigma_2', \qquad 0=\frac{d\sigma_2^{\mathrm{iv}}}{dx}-\sigma_2^{\mathrm{v}}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0=\frac{d\sigma_3}{dx}-\sigma_3', \qquad 0=\frac{d\sigma_3'''}{dx}-\sigma_3^{\mathrm{iv}}, \end{aligned}$$

conduct by elimination and by supposing

$$(208)\qquad \frac{dx_1}{dx}=x_1',\ \ldots\ \frac{dx_1^{\mathrm{vi}}}{dx}=x_1^{\mathrm{vii}},\ \frac{1}{\lambda_1}\frac{d\lambda_1}{dx}=\frac{\lambda_1'}{\lambda_1}$$

to the following 26 expressions

$$(209)\qquad \begin{aligned} &\frac{dx_2}{dx}=x_2', \qquad \frac{dx_2^{\mathrm{vi}}}{dx}=x_2^{\mathrm{vii}},\ \frac{dx_3}{dx}=x_3',\ \ldots,\ \frac{dx_3^{\mathrm{vii}}}{dx}=x^{\mathrm{viii}} \\ &\frac{d\lambda_1'}{dx}=\lambda_1'', \qquad \frac{d\lambda_1^{\mathrm{v}}}{dx}=\lambda_1^{\mathrm{vi}},\ \frac{d\lambda_2}{dx}=\lambda_2',\ \ldots,\ \frac{d\lambda_2^{\mathrm{v}}}{dx}=\lambda_2^{\mathrm{vi}}. \end{aligned}$$

(Feb. 26th.)

The five equations (201), (202), $0=f_1=f_2=\sigma_1=\sigma_2=\sigma_3$, might be proposed to be integrated by five series of the forms

$$(210)\qquad \begin{cases} x_1+\Delta x_1=x_1+x_1'\Delta x+\tfrac{1}{2}x_1''\Delta x^2+\ldots, & \Delta x_2=x_2'\Delta x+\tfrac{1}{2}x_2''\Delta x^2+\ldots, \\ & \Delta x_3=x_3'\Delta x+\tfrac{1}{2}x_3''\Delta x^2+\ldots, \\ \lambda_1+\Delta\lambda_1=\lambda_1+\lambda_1'\Delta x+\tfrac{1}{2}\lambda_1''\Delta x^2+\ldots, & \Delta\lambda_2=\lambda_2'\Delta x+\tfrac{1}{2}\lambda_2''\Delta x^2+\ldots, \end{cases}$$

by substitution of which series we are to satisfy, independently of Δx, the five equations

$$(211)\qquad 0=f_1+\Delta f_1,\quad 0=f_2+\Delta f_2,\quad 0=\sigma_1+\Delta\sigma_1,\quad 0=\sigma_2+\Delta\sigma_2,\quad 0=\sigma_3+\Delta\sigma_3,$$

that is,

$$(212)\qquad \begin{cases} 0=f_1+f_1'\Delta x+\tfrac{1}{2}f_1''\Delta x^2+\ldots, & 0=f_2+f_2'\Delta x+\tfrac{1}{2}f_2''\Delta x^2+\ldots, \\ 0=\sigma_1+\sigma_1'\Delta x+\tfrac{1}{2}\sigma_1''\Delta x^2+\ldots, & 0=\sigma_2+\sigma_2'\Delta x+\tfrac{1}{2}\sigma_2''\Delta x^2+\ldots, \\ & 0=\sigma_3+\sigma_3'\Delta x+\tfrac{1}{2}\sigma_3''\Delta x^2+\ldots. \end{cases}$$

It is therefore necessary and, if Δx be small enough to allow the convergence of the three developments, it is in general sufficient to satisfy the five indefinite series of conditions:

$$(213)\qquad \begin{cases} 0=f_1,\ 0=f_1',\ 0=f_1'',\ \ldots,\ 0=f_2,\ 0=f_2',\ 0=f_2'',\ \ldots, \\ 0=\sigma_1,\ 0=\sigma_1',\ \ldots,\ 0=\sigma_2,\ 0=\sigma_2',\ \ldots,\ 0=\sigma_3,\ 0=\sigma_3',\ \ldots, \end{cases}$$

for *any one particular value of* x in order to satisfy the equations (211) by the developments (210) *for any near value of* x, and so to obtain the five integrals of (211), at least in series. The question, "how many arbitrary constants does the system of these five integrals contain?," is therefore reducible to the question, "how many functions of the five following series

$$(214)\qquad x_1,\ x_1',\ x_1'',\ \ldots,\ x_2,\ x_2',\ x_2'',\ \ldots,\ x_3,\ x_3',\ x_3'',\ \ldots,\ \lambda_1,\ \lambda_1',\ \lambda_1'',\ \ldots,\ \lambda_2,\ \lambda_2',\ \lambda_2'',\ \ldots$$

may have arbitrary values assigned to them, for an assumed particular value of x, so as to satisfy all the conditions of the five series (213)?"

The five conditions

$$0=f_1,\ 0=f_2,\ 0=\sigma_1,\ 0=\sigma_2,\ 0=\sigma_3 \tag{215}$$

involve only the 18 functions

$$x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2',\ x_2'',\ x_2''',\ x_3,\ x_3',\ x_3'',\ x_3''',\ x_3^{\text{iv}},\ \lambda_1,\ \lambda_1',\ \lambda_2,\ \lambda_2',\ \lambda_2'' \tag{216}$$

besides the variable x. The six additional conditions

$$0=f_1',\ 0=f_1'',\ 0=f_2',\ 0=f_2'',\ 0=\sigma_1',\ 0=\sigma_2' \tag{217}$$

involve only one additional function, namely

$$\lambda_1''. \tag{218}$$

Thus the 11 conditions (215), (217) involve only the variable x and the 19 functions (216), (218) and leave only 8, at most, of these functions arbitrary in value when the value of x is assumed. We might have eliminated λ_1'' between the two last of the six conditions (217) and then we should have five relations between the 18 functions (216) to combine with the five conditions (215), leaving thus only 8, at most, of those functions arbitrary in value for any one assumed value of x. Reciprocally, no new relation, deducible from the conditions (213), restricts the number of these arbitrary values so as to reduce it to be less than 8. For every new condition or set of conditions, taken out of the series (213), introduces a new function or functions which cannot in general be eliminated by means of those new conditions. Thus the condition $0=f_1'''$ introduces the two new functions x_1^{iv}, x_2^{iv}, the condition $0=f_2'''$ introduces x_1^{iv}, x_2^{iv}, x_3^{v}, the condition $0=\sigma_1''$ introduces x_1^{iv}, x_2^{iv}, x_3^{v}, λ_1''', λ_2''', the condition $0=\sigma_2''$ introduces the same five functions as $0=\sigma_1''$, and the condition $0=\sigma_3'$ introduces x_1^{iv}, x_2^{iv}, x_3^{v}, λ_2'''. These five conditions are indeed sufficient to *determine*, but not in general sufficient to *eliminate*, the five new functions x_1^{iv}, x_2^{iv}, x_3^{v}, λ_1''', λ_2''' when the 19 functions (216) and (218) are known. So that the five new conditions

$$0=f_1''',\ 0=f_2''',\ 0=\sigma_1'',\ 0=\sigma_2'',\ 0=\sigma_3' \tag{219}$$

can in general all be satisfied, but only in one way, by choosing suitable values for these five new functions

$$x_1^{\text{iv}},\ x_2^{\text{iv}},\ x_3^{\text{v}},\ \lambda_1''',\ \lambda_2''', \tag{220}$$

after 8 of the 19 functions (216), (218) have had their values arbitrarily assumed and after the remaining 11 values have been determined so as to satisfy the 11 conditions (215), (217). In like manner, the five new conditions

$$0=f_1^{\text{iv}},\ 0=f_2^{\text{iv}},\ 0=\sigma_1''',\ 0=\sigma_2''',\ 0=\sigma_3'' \tag{221}$$

can be satisfied by one but by only one set of values of the five new functions

$$x_1^{\text{v}},\ x_2^{\text{v}},\ x_3^{\text{vi}},\ \lambda_1^{\text{iv}},\ \lambda_2^{\text{iv}}, \tag{222}$$

and so on indefinitely. We may therefore conclude, by this mode of reasoning, that the five indefinite series of conditions (213) can all be satisfied together by one set of values of the functions (214), of which values eight and only eight are arbitrary after the value of x has been assumed. But of these eight arbitrary values, one is introduced by the consideration of the two series

$$\lambda_1,\ \lambda_1',\ \lambda_1'',\ \ldots,\ \lambda_2,\ \lambda_2',\ \lambda_2'',\ \ldots \tag{223}$$

and only seven belong to the three series

$$x_1, x_1', x_1'', \ldots, x_2, x_2', x_2'', \ldots, x_3, x_3', x_3'', \tag{224}$$

For if the 11 conditions (215), (217) be distributed into these two new groups

$$0=f_1,\ 0=f_1',\ 0=f_1'',\ 0=f_2,\ 0=f_2',\ 0=f_2'' \tag{225}$$

and

$$0=\sigma_1,\ 0=\sigma_1',\ 0=\sigma_2,\ 0=\sigma_2',\ 0=\sigma_3, \tag{226}$$

the six equations of the group (225) contain only the 13 functions

$$x_1, x_1', x_1'', x_1''', x_2, x_2', x_2'', x_2''', x_3, x_3', x_3'', x_3''', x_3^{\mathrm{iv}}, \tag{227}$$

besides the variable x which is always understood. They leave therefore only 7 of these 13 values arbitrary and the eighth arbitrary value is that of any one of the six functions

$$\lambda_1, \lambda_1', \lambda_1'', \lambda_2, \lambda_2', \lambda_2'', \tag{228}$$

of which the five equations of the group (226) determine only the five ratios.

It is important however to observe that we may not assume *any* seven of the values of the functions (227), although we have only the six equations (225) between them. Thus the first equation of this group, namely $0=f_1$, prevents us from assuming arbitrary values for all the functions $x_1, x_1', x_2, x_2', x_3, x_3'$, and therefore if the seven arbitrary values contain five of these they cannot contain the sixth. With this exception, however, we seem to be at liberty to select any seven of the functions (227) because the equation $0=f_2$ contains the same six functions as $0=f_1$ along with the new function x_3'' and because the differentials of these equations contain each more than seven different functions.

When the arbitrary values of some seven of the functions (227) have been assumed (not more than five belonging to the group $x_1, x_1', x_2, x_2', x_3, x_3'$ for the reason just now assigned) and when the other six of these functions have been determined by (225), we can then in general deduce the values of all the other functions of the series (224), without yet assuming that eighth arbitrary constant which is connected with the passage from the ratios to the absolute values of the functions (223). For the ratios only, and not the absolute values, of those multiplier-functions enter into the conditions (213).

We see, therefore, in this new way, that the complete expressions for the three functions x_1, x_2, x_3 (deduced from the differential equations (201) and (202) by differential elimination of λ_1, λ_2 and by integration) contain in general seven arbitrary constants and no more. We can therefore deduce a principal integral relation of the following form between those functions and x, x_3' and their initial values a_1, a_2, a_3, a, a_3':

$$0=F(x, x_1, x_2, x_3, x_3', a, a_1, a_2, a_3, a_3'). \tag{229}$$

Can we generalise the foregoing investigation so as to show, by a similar process, that a principal integral relation of the form (35) results from the two systems (29) and (30)?

In the first place we may observe that when equations of the forms (114) and (115) are given they are in number $m+n$, and they contain, taken altogether, the $(n+m)(\omega_n+1)+\Sigma\omega$ functions following (besides the independent variable x):

$$\begin{gathered} x_1, x_1', \ldots, x_1^{(\omega_1+\omega_n)}, x_2, x_2', \ldots, x_2^{(\omega_2+\omega_n)}, \ldots, x_n, x_n', \ldots, x_n^{(2\omega_n)}, \\ \lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_n)}, \lambda_2, \lambda_2', \ldots, \lambda_2^{(\omega_n)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_n)}, \end{gathered} \tag{230}$$

which functions, however, and no others will (at most) be contained in the following additional equations,

$$(231)\quad \begin{cases} 0=f'_1,\ 0=f''_1,\ \ldots,\ 0=f_1^{(\omega_n)},\ \ldots,\ 0=f'_m,\ 0=f''_m,\ \ldots,\ 0=f_m^{(\omega_n)}, \\ 0=\sigma'_1,\ 0=\sigma''_1,\ \ldots,\ 0=\sigma_1^{(\omega_n-\omega_1)},\ \ldots,\ 0=\sigma'_{n-1},\ 0=\sigma''_{n-1},\ \ldots,\ 0=\sigma_{n-1}^{(\omega_n-\omega_{n-1})}, \end{cases}$$

being in number $(n+m)\,\omega_n-\Sigma\omega$. Thus we have $(n+m)(\omega_n+1)-\Sigma\omega$ equations between $(n+m)(\omega_n+1)+\Sigma\omega$ functions and therefore, so far, $2\Sigma\omega$ of these functions remain arbitrary in value. Nor will any new derived equations of the forms

$$(232)\qquad 0=f_1^{(\omega_n+i)},\ \ldots,\ 0=f_m^{(\omega_n+i)},\ 0=\sigma_1^{(\omega_n-\omega_1+i)},\ 0=\sigma_2^{(\omega_n-\omega_2+i)},\ \ldots,\ 0=\sigma_n^{(i)}$$

give any new relations between these arbitrary values. They remain therefore ultimately arbitrary and the complete integrals of the $m+n$ differential equations (114), (115) contain in general $2\Sigma\omega$ arbitrary constants. But of these only $2\Sigma\omega-1$ enter into the expressions of the functions $x_1, x_2, \ldots, x_n$, the other being introduced by the circumstance that the m multiplier-functions $\lambda_1, \ldots, \lambda_m$ may be multiplied by any arbitrary constant without disturbing the relations of the question.

XV.

INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS BY THE CALCULUS OF VARIATIONS*

[1836.]

[Note Book 42.]

[CONTENTS

		PAGE
1.	The partial differential equation of the first order.	391
2.	The partial differential equation of the second order.	397
3.	Examples.]	403

[*The partial differential equation of the first order.*]

(April 22, 1836.)

[1.] (A) $$0 = F(x, y, z, p, q),$$

(B) $$z' = px' + qy';$$

(C) $$\left\{\begin{aligned} 0 = \lambda\delta(px' + qy' - z') + \mu\,\delta F = {} & \mu F'(x)\,\delta x + \lambda p\,\delta x' + \mu F'(p)\,\delta p + \lambda x'\delta p \\ & + \mu F'(y)\,\delta y + \lambda q\,\delta y' + \mu F'(q)\,\delta q + \lambda y'\delta q \\ & + \mu F'(z)\,\delta z - \lambda\,\delta z' \\ = {} & (\lambda p\,\delta x + \lambda q\,\delta y - \lambda\,\delta z)', \end{aligned}\right.$$

if

(1) $$\mu F'(p) + \lambda x' = 0,$$

(2) $$\mu F'(q) + \lambda y' = 0,$$

(3) $$\mu F'(x) = (\lambda p)',$$

(4) $$\mu F'(y) = (\lambda q)',$$

(5) $$\mu F'(z) = -\lambda'.$$

The two first conditions give

(6) $$0 = \mu\{pF'(p) + qF'(q)\} + \lambda z'.$$

Therefore

(7) $$\frac{x'}{z'} = \frac{F'(p)}{pF'(p) + qF'(q)},$$

(8) $$\frac{y'}{z'} = \frac{F'(q)}{pF'(p) + qF'(q)};$$

also (3) and (4) give

$$\mu\{F'(x)\,F'(p) + F'(y)\,F'(q)\} = \lambda'\{pF'(p) + qF'(q)\} + \lambda\{p'F'(p) + q'F'(q)\};$$

---

* [Hamilton's method for solving partial differential equations of the first order is very similar to Cauchy's method of characteristics. See Appendix, Note 10, p. 634.]

and

$$0 = F' = x'F'(x) + y'F'(y) + z'F'(z) + p'F'(p) + q'F'(q). \tag{9}$$

Hence

$$\lambda' \{pF'(p) + qF'(q)\} = \lambda z'F'(z), \tag{10}$$

agreeing with (5) and (6). Consequently from (3), (6) and (10)

$$p' \qquad -\frac{F'(x) + pF'(z)}{pF'(p) + qF'(q)}, \tag{11}$$

and similarly*

$$\frac{q'}{z'} = -\frac{F'(y) + qF'(z)}{pF'(p) + qF'(q)}. \tag{12}$$

Let x, y, p be considered as three functions of z, to be determined by any three of the differential equations of the first order (7), (8), (11), (12), q being previously eliminated by (A). The expressions of these three functions will involve three arbitrary constants, which may be chosen so as to be the *initial* values a, b, j of the same three functions, corresponding to some assumed initial value c of z; and, if γ denote the corresponding initial value of λ, we can express λ by (10) as follows:

$$\lambda = \gamma e^{\int_c^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}. \tag{13}$$

Then (C) will give by integration

$$\lambda(p\,\delta x + q\,\delta y - \delta z) = \gamma(j\,\delta a + k\,\delta b - \delta c), \tag{14}$$

k being the initial value of q, so that

$$0 = F(a, b, c, j, k). \tag{15}$$

Therefore, by (13), we have

$$p\,\delta x + q\,\delta y - \delta z = e^{-\int_c^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}} (j\,\delta a + k\,\delta b - \delta c); \tag{16}$$

and, if we suppose for simplicity $c = 0$, $\delta c = 0$, we shall then have the expressions

$$p\,\delta x + q\,\delta y - \delta z = e^{-\int_0^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}} (j\,\delta a + k\,\delta b) \tag{D}$$

and

$$0 = F(a, b, 0, j, k). \tag{17}$$

Here a, b, j, k, q may be regarded as known functions of x, y, z, p; or, reciprocally, x, y, p, q, k may be regarded as known functions of a, b, j, z; k indeed being a function of a, b, j alone. Considering then x and y as functions of a, b, j, z, we have by (D), between their partial differential coefficients, the relations

$$\begin{cases} px'(a) + qy'(a) = je^{-\int_0^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}, & px'(b) + qy'(b) = ke^{-\int_0^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}, \\ px'(j) + qy'(j) = 0, & px'(z) + qy'(z) = 1; \end{cases} \tag{E}$$

so that, if we substitute for x and y and also for p and q their values as functions of a, b, j, z in the equation (B), we shall get

$$0 = ja' + kb'. \tag{F}$$

---

* [Equations (7), (8), (11), (12) are of course the differential equations of the characteristics.]

More generally, by (D) we have

$$\text{(G)} \qquad px' + qy' - z' = (ja' + kb')\, e^{-\int_0^z \frac{F'(z)\,dz}{pF'(p)+qF'(q)}}.$$

If then we establish between the two known functions a and b of x, y, z, p any arbitrary relation

$$\text{(H)} \qquad b = \psi(a),$$

and put, besides,

$$\text{(I)} \qquad -\frac{j}{k} = \psi'(a),$$

we shall reproduce the equations (B) and (A); and consequently the system (H), (I) is the general integral for the partial differential equation (A). A particular integral with two arbitrary constants may be deduced from (F) by treating a and b as each separately constant and then eliminating p.*

As an example, let the proposed partial differential equation be

$$0 = p + \tfrac{1}{2}q^2 - gy = F.$$

Here the three auxiliary total differential equations (7), (8), (11) become

$$x' = \frac{{}'}{p+q^2}, \quad y' = \frac{qz'}{p+q^2}, \quad p' = 0,$$

to which we may join, as a combination of the same, the equation (12),

$$q' = \frac{gz'}{p+q^2}.$$

Hence

$$q' = gx', \quad qq' = gy'.$$

Therefore, by integration,

$$p = j, \quad q - gx = k - ga, \quad \tfrac{1}{2}q^2 - gy = \tfrac{1}{2}k^2 - gb = -j;$$

also

$$y' = qx' = gxx' + (k - ga)\,x',$$

and so

$$y - b = \tfrac{1}{2}g\,(x^2 - a^2) + (k - ga)\,(x - a) = k\,(x - a) + \tfrac{1}{2}g\,(x - a)^2.$$

Also from

$$z' = px' + qy' = jx' + {}^{q^2q'}$$

we have

$$z = j\,(x - a) + \frac{q^3 - k^3}{3g}.$$

Thus, to determine x, y, p, q as functions of a, b, j, k and z, we have the four equations

$$p = j, \quad q - k = g\,(x - a), \quad y - b = k\,(x - a) + \tfrac{1}{2}g\,(x - a)^2, \quad z = j\,(x - a) + \frac{q^3 - k^3}{3g},$$

to which we may join

$$0 = p + \tfrac{1}{2}q^2 - gy, \quad 0 = j + \tfrac{1}{2}k^2 - gb.$$

We may also conveniently consider the four equations just mentioned as serving to determine

---

* The logic is more fully explained in pages 395, 396.

p, q, y and z as functions of j, k, a, b, x; j being itself a known function of k and b. If now we treat j, k, a, b as variables, these equations give

$$y' = b' + k(x' - a') + (x - a)k' + g(x - a)(x' - a'), \quad z' = j(x' - a') + (x - a)j' + \frac{q^2q' - k^2k'}{g},$$

$$\begin{aligned} px' + qy' - z' &= jx' + q^2(x' - a') + \{k + g(x - a)\}\{b' + (x - a)k'\} - j(x' - a') \\ &\qquad + (x - a)(kk' - gb') - q^2(x' - a') + \frac{k^2 - q^2}{g}k' \\ &= ja' + kb' + k'\left\{2k(x - a) + g(x - a)^2 + \frac{k^2 - q^2}{g}\right\} = ja' + kb' = (gb - \tfrac{1}{2}k^2)a' + kb'; \end{aligned}$$

therefore

$$(gy - \tfrac{1}{2}q^2)x' + qy' - z' = (gb - \tfrac{1}{2}k^2)a' + kb'$$

identically, by the nature of the relations

$$q - k = g(x - a), \quad y - b = k(x - a) + \tfrac{1}{2}g(x - a)^2, \quad z = (gb - \tfrac{1}{2}k^2)(x - a) + \frac{q^3 - k^3}{3g},$$

from which three last equations we must conceive a, b, k deduced as functions of x, y, z and q. If, then, between the functions so deduced, we establish the two relations

$$b = \psi(a), \quad \tfrac{1}{2}k - \frac{gb}{k} = \psi'(a),$$

we shall have, as a consequence of them, this other:

$$z' = qy' + (gy - \tfrac{1}{2}q^2)x',$$

in which, by the two relations established between a and b, z and q may be considered as functions of the two independent variables x and y. Hence the partial differential coefficients of z, of the first order, are $gy - \frac{1}{2}q^2$ and q, and therefore the proposed partial differential equation is satisfied whatever the form of the arbitrary function ψ may be. A less general integral, but containing two arbitrary constants, may be obtained by supposing a and b to be both constants and by eliminating q and k between the three relations which connect a, b, k, x, y, z, q. This last corresponds to the *principal integral* of the total differential equation

$$z' = gyx' + \frac{y'^2}{2x'},$$

and is

$$z = \tfrac{1}{2}g(y + b)(x - a) + \frac{(y - b)^2}{2(x - a)} - \tfrac{1}{24}g^2(x - a)^3.$$

(April 23, 1836.)

Let

(1) $$q = f(x, y, z, p)$$

be any proposed partial differential equation of the first order. This equation gives

(2) $$z' = px' + f(x, y, z, p)y',$$

x', y', z' being the total derivatives of x, y, z, considered as functions of any one independent variable, of which p also may be considered as a function. Reciprocally, if we can discover any two such relations between the four variables x, y, z, p as to satisfy the total differential equation (2), independently of any third relation between the same four variables, we shall then, by eliminating p, deduce a relation between x, y, z which will be an integral of the partial differential equation (1). In this manner we are led to consider the modes of satisfying the total differential equation (2); and therefore (according to the general spirit of analysis) we are led to consider the consequences of that total differential equation.

One consequence is that which is obtained by taking the variation of the equation, namely,

$$0 = p\,\delta x' + f(x, y, z, p)\,\delta y' - \delta z' + x'\delta p + y'\{f'(x)\,\delta x + f'(y)\,\delta y + f'(z)\,\delta z + f'(p)\,\delta p\}. \tag{3}$$

The meaning and necessity of this consequence may be explained as follows. Since the equation (2) is not by itself sufficient to determine the four variables x, y, z, p as functions of any one independent variable, or even to determine any three of those variables as a function of the fourth, we may, in general, conceive that besides some one set of four variables x, y, z, p, which satisfies the equation, another set also, such as $x + \epsilon\xi$, $y + \epsilon\eta$, $z + \epsilon\zeta$, $p + \epsilon\varpi$, satisfies the same equation, ϵ being a small and constant, but arbitrary, multiplier. Then, along with the equation (2) itself, we shall have also the varied equation:

$$0 = (p + \epsilon\varpi)(x' + \epsilon\xi') + f(x + \epsilon\xi,\ y + \epsilon\eta,\ z + \epsilon\zeta,\ p + \epsilon\varpi) \,.\, (y' + \epsilon\eta') - (z' + \epsilon\zeta'), \tag{4}$$

which, when developed according to powers of ϵ, reduced by (2) itself, divided by ϵ, and made to tend to a limiting state by making ϵ tend to zero, becomes ultimately

$$0 = p\xi' + f(x, y, z, p)\,\eta' - \zeta' + x'\varpi + y'\{f'(x)\,\xi + f'(y)\,\eta + f'(z)\,\zeta + f'(p)\,\varpi\}; \tag{5}$$

and this is the consequence which with only a different notation is expressed by (3).

If we multiply (5) by a new function λ and integrate by parts, we find

$$\begin{aligned} 0 = \{\lambda p\xi + \lambda f(x, y, z, p)\,\eta - \lambda\zeta\}' + \{\lambda y'f'(x) - (\lambda p)'\}\,\xi + \{\lambda y'f'(y) - (\lambda f)'\}\,\eta \\ + \{\lambda y'f'(z) + \lambda'\}\,\zeta + \lambda\{x' + y'f'(p)\}\,\varpi; \end{aligned} \tag{6}$$

which will reduce itself to

$$0 = (\lambda p\xi + \lambda f\eta - \lambda\zeta)', \tag{7}$$

if we can put, consistently with (2),

$$x' = -y'f'(p), \tag{8}$$

$$\lambda' = -\lambda y'f'(z), \tag{9}$$

$$(\lambda f)' = \lambda y'f'(y), \tag{10}$$

$$(\lambda p)' = \lambda y'f'(x). \tag{11}$$

Expanding these equations, we find by (9) and (10)

$$0 = x'f'(x) + z'f'(z) + p'f'(p) - f \,.\, y' \,.\, f'(z); \tag{12}$$

also by (9) and (11)

$$0 = p' - y'f'(x) - p \,.\, y' \,.\, f'(z), \tag{13}$$

which latter equation also results by elimination of x', z' between the three equations (2), (8), (12). For the equations (2) and (8) give

$$z' = \{f - pf'(p)\}\,y', \tag{14}$$

and when we substitute these expressions (8) and (14) for x', z' in (12) and divide by $f'(p)$, we come to equation (13). Reciprocally (12) results from (2), (8) and (13). Thus we shall satisfy all the five equations (2), (8), (9), (10), (11) if we satisfy the four equations (8), (9), (13), (14). These four differential equations may in general be conceived to give, by integration, x, y, p and

λ/λ_0 as functions of z, a, b, j, if a, b, j and λ_0 denote the initial values of x, y, p and λ, corresponding to some assumed initial value, such as 0, of z; a, b, j, λ_0 being here treated as constants in forming the expressions of x', y', p', λ', which are to satisfy the differential equations. Now, just as (4) was formed from (2), so we must form three other analogously varied differential equations from (8), (9) and (13); and, integrating these, we must obtain $x+\epsilon\xi$, $y+\epsilon\eta$, $z+\epsilon\zeta$, $p+\epsilon\varpi$ and $\dfrac{\lambda+\epsilon\omega}{\lambda_0+\epsilon\omega_0}$ as functions of $z+\epsilon\zeta$, $a+\epsilon\alpha$, $b+\epsilon\beta$, $j+\epsilon\iota$, in which $\alpha, \beta, \iota, \omega_0$ are the initial values of $\xi, \eta, \varpi, \omega$, corresponding to the initial value 0 of $z+\epsilon\zeta$; which four functions will be precisely the same in form as those which expressed the dependence of x, y, p and $\dfrac{\lambda}{\lambda_0}$ on z, a, b, j. And the limiting equation (7) will give, by integration, (at the limit $\epsilon=0$)

$$\lambda(p\xi+f\eta-\zeta)=\lambda_0(j\alpha+k\beta), \tag{15}$$

in which

$$k=f_0=f(a, b, 0, j); \tag{16}$$

also, for the reason just mentioned,

$$\xi=x'(z)\zeta+x'(a)\alpha+x'(b)\beta+x'(j)\iota, \quad \eta=y'(z)\zeta+y'(a)\alpha+y'(b)\beta+y'(j)\iota, \tag{17}$$

$x'(z), x'(a), x'(b), x'(j), y'(z), y'(a), y'(b), y'(j)$ being the partial differential coefficients of the first order of x and y considered as functions of z, a, b, j. Therefore, since $\alpha, \beta, \iota, \zeta$ are independent, the equation (15) resolves itself into the four following:

$$px'(z)+fy'(z)=1, \quad px'(j)+fy'(j)=0, \quad px'(a)+fy'(a)=\frac{\lambda_0}{\lambda}j, \quad px'(b)+fy'(b)=\frac{\lambda_0}{\lambda}k. \tag{18}$$

Having discovered these four relations (18) between the partial differential coefficients of x and y considered as functions of z, a, b, j, we may now deduce the following relation between the total differential coefficients of the same two functions, treating a, b, j as variables,

$$px'+fy'=z'+\frac{\lambda_0}{\lambda}(ja'+kb'), \tag{19}$$

which shows that (2) gives

$$0=ja'+kb'. \tag{20}$$

Reciprocally, if in the two following equations,

$$b=\psi(a), \quad -\frac{j}{k}=\psi'(a), \tag{21}$$

we consider a, b and j as known functions of x, y, z, p, deduced by elimination from the integrals of the equations (8), (13), (14), we shall have solved the problem proposed, namely, that of discovering two relations between x, y, z, p, which are sufficient to conduct to the total differential equation (2), and which therefore give, by elimination of p, an integral of the partial differential equation (1). This integral system (21) contains an arbitrary function ψ; another integral of the same partial differential equation, containing no arbitrary function but containing two arbitrary constants, may be had by eliminating p between the expressions of a and b.

[*The partial differential equation of the second order.*]

[2.] Passing now to partial differential equations of the second order, let

(1) $$t=f(x, y, z, p, q, r, s)$$

be such an equation. We have now the three following total differential equations of the first order:

(2) $$0=px'+qy'-z',$$

(3) $$0=rx'+sy'-p',$$

(4) $$0=sx'+f(x, y, z, p, q, r, s)\,y'-q',$$

x', y', z', p', q' being here the total derivatives of x, y, z, p, q, considered as functions of any one independent variable of which r and s may also be conceived to be functions. And if we can discover any five relations between x, y, z, p, q, r and s, which, without any other relation being required, conduct to the system of the three total differential equations (2), (3), (4), we shall then be able to conclude that the partial differential coefficients of the first and second orders of z considered as a function of x and y (deduced from the five supposed relations by elimination of p, q, r, s) may be expressed as follows:

(5) $$z'(x)=p,\quad z'(y)=q,\quad z''(x)=r,\quad z'{\cdot}'(x,y)=s,\quad z''(y)=f(x, y, z, p, q, r, s);$$

and therefore that this function z of x and y will satisfy and be an integral of the proposed partial differential equation (1).

Since the three equations (2), (3), (4), even if freed from differentials, would not be in general sufficient to determine the five sought relations between x, y, z, p, q, r, s, we may conceive that $x+\epsilon\xi$, $y+\epsilon\eta$, $z+\epsilon\zeta$, $p+\epsilon\varpi$, $q+\epsilon\kappa$, $r+\epsilon\rho$, $s+\epsilon\sigma$ are also functions of some one independent variable, and satisfy as such three equations similar to (2), (3), (4), namely,

(6) $$0=(p+\epsilon\varpi)(x'+\epsilon\xi')+(q+\epsilon\kappa)(y'+\epsilon\eta')-(z'+\epsilon\zeta'),$$

(7) $$0=(r+\epsilon\rho)(x'+\epsilon\xi')+(s+\epsilon\sigma)(y'+\epsilon\eta')-(p'+\epsilon\varpi'),$$

(8) $$0=(s+\epsilon\sigma)(x'+\epsilon\xi')+(t+\epsilon\tau)(y'+\epsilon\eta')-(q'+\epsilon\kappa'),$$

in which we have put for abridgment

(9) $$t+\epsilon\tau=f(x+\epsilon\xi,\ y+\epsilon\eta,\ z+\epsilon\zeta,\ p+\epsilon\varpi,\ q+\epsilon\kappa,\ r+\epsilon\rho,\ s+\epsilon\sigma).$$

Developing according to powers of ϵ, reducing by the equations (2), (3), (4), dividing by ϵ, and passing to the limit at which ϵ vanishes, we find

(10) $$0=p\xi'+q\eta'-\zeta'+\varpi x'+\kappa y',$$

(11) $$0=r\xi'+s\eta'-\varpi'+\rho x'+\sigma y',$$

(12) $$0=s\xi'+t\eta'-\kappa'+\sigma x'+\tau y';$$

in which t is to be considered as representing the expression $f(x, y, z, p, q, r, s)$, and therefore by (9)

(13) $$\tau=\xi f'(x)+\eta f'(y)+\zeta f'(z)+\varpi f'(p)+\kappa f'(q)+\rho f'(r)+\sigma f'(s).$$

We may multiply the three differential equations (10), (11), (12) by λ, μ, ν respectively and add them, thus obtaining

(14) $$0=(\lambda p+\mu r+\nu s)\,\xi'+(\lambda q+\mu s+\nu t)\,\eta'-\lambda\zeta'-\mu\varpi'-\nu\kappa'$$
$$+\lambda x'\varpi+\lambda y'\kappa+\mu x'\rho+(\mu y'+\nu x')\,\sigma+\nu y'\tau.$$

This result may be put under the form:

$$(15)\qquad \begin{aligned} 0 = &\{(\lambda p + \mu r + \nu s)\,\xi + (\lambda q + \mu s + \nu t)\,\eta - \lambda\zeta - \mu\varpi - \nu\kappa\}' \\ &+ \xi\,\{\nu y' f'(x) - (\lambda p + \mu r + \nu s)'\} + \eta\,\{\nu y' f'(y) - (\lambda q + \mu s + \nu t)'\} \\ &+ \zeta\,\{\nu y' f'(z) + \lambda'\} + \varpi\,\{\nu y' f'(p) + \mu' + \lambda x'\} + \kappa\,\{\nu y' f'(q) + \nu' + \lambda y'\} \\ &+ \rho\,\{\mu x' + \nu y' f'(r)\} + \sigma\,\{\mu y' + \nu x' + \nu y' f'(s)\}. \end{aligned}$$

It will therefore give

$$(16)\qquad \begin{aligned} &(\lambda p + \mu r + \nu s)\,\xi + (\lambda q + \mu s + \nu t)\,\eta - \lambda\zeta - \mu\varpi - \nu\kappa \\ &\qquad = (\lambda_0 p_0 + \mu_0 r_0 + \nu_0 s_0)\,\xi_0 + (\lambda_0 q_0 + \mu_0 s_0 + \nu_0 t_0)\,\eta_0 - \mu_0\varpi_0 - \nu_0\kappa_0, \end{aligned}$$

in which $\lambda_0, \mu_0, \nu_0, \xi_0, \eta_0, \varpi_0, \kappa_0, p_0, q_0, r_0, s_0, t_0$ are the initial values of λ, μ, ν, ξ, η, ϖ, κ, p, q, r, s, t, corresponding to the assumed initial value 0 of z, if we suppose that the following differential equations are satisfied, as well as equations (2), (3), (4):

$$(17)\qquad 0 = \mu y' + \nu x' + \nu y' f'(s),$$

$$(18)\qquad 0 = \mu x' + \nu y' f'(r),$$

$$(19)\qquad 0 = \nu y' f'(q) + \nu' + \lambda y',$$

$$(20)\qquad 0 = \nu y' f'(p) + \mu' + \lambda x',$$

$$(21)\qquad 0 = \nu y' f'(z) + \lambda',$$

$$(22)\qquad 0 = \nu y' f'(y) - (\lambda q + \mu s + \nu t)',$$

$$(23)\qquad 0 = \nu y' f'(x) - (\lambda p + \mu r + \nu s)'.$$

Multiplying (23), (22), (21), (20), (19), (18) and (17) by x', y', z', p', q', r', s' respectively and adding the products, we find

$$(24)\qquad 0 = \lambda'(z' - px' - qy') + \mu'(p' - rx' - sy') + \nu'(q' - sx' - ty').$$

We may therefore on this account subtract one from the number of equations (2)–(4), (17)–(23), and consider these 10 equations as not equivalent to more, at most, than 9 distinct equations, adopted to determine the 9 variables x, y, p, q, r, s, λ, μ, ν as functions of z and of constants. We may also eliminate λ, λ', μ', ν' between the five equations (19)–(23) and so obtain a single equation giving the ratio of ν to μ; which, when compared with (17) and (18), will give two equations entirely freed from λ, μ, ν, but differential of the first order between x, y, z, p, q, r, s.

As an example, to illustrate these formulae, let the given partial differential equation be $t = 0$. Here the ten auxiliary total differential equations (equivalent only to nine distinct ones) are the following:

$$\begin{array}{lllll} z' = px' + qy', & p' = rx' + sy', & q' = sx', & 0 = (\lambda p + \mu r + \nu s)', & 0 = (\lambda q + \mu s)', \\ 0 = \lambda', & 0 = \mu' + \lambda x', & 0 = \nu' + \lambda y', & 0 = \mu x', & 0 = \mu y' + \nu x'. \end{array}$$

These equations give by integration x, y, z, p, q, r, s, λ, μ, ν each constant, or else x, y, z, p, q, λ each constant and $\mu = 0$, $\nu = 0$. But each alternative seems to render equation (16) identical.

As another example, let $t = r$. Here the ten total differential equations are:

$$\begin{array}{lllll} z' = px' + qy', & p' = rx' + sy', & q' = sx' + ry', & 0 = (\lambda p + \mu r + \nu s)', & 0 = (\lambda q + \mu s + \nu r)', \\ 0 = \lambda', & 0 = \mu' + \lambda x', & 0 = \nu' + \lambda y', & 0 = \mu x' + \nu y', & 0 = \mu y' + \nu x'; \end{array}$$

the ninth alone differing from the corresponding equation of the system of the last example. The two last of these ten equations give either

(I) $$\mu = \nu, \qquad x' + y' = 0,$$

or

(II) $$\mu = -\nu, \quad x' - y' = 0.$$

If we adopt the first alternative, then the 7th and 8th equations give $\lambda(x' - y') = 0$. Therefore either

(I, 1) $$x' = 0, \ y' = 0, \ \mu' = 0, \ \nu' = 0,$$

or else

(I, 2) $$\lambda = 0, \ \mu' = 0, \ \nu' = 0, \ \mu = \nu, \ x' + y' = 0.$$

In like manner we have

(II, 1) $$x' = 0, \ y' = 0, \ \mu' = 0, \ \nu' = 0,$$

or

(II, 2) $$\lambda = 0, \ \mu' = 0, \ \nu' = 0, \ \mu = -\nu, \ x' - y' = 0.$$

Thus we have, at all events, $\mu' = 0$, $\nu' = 0$, $\lambda' = 0$, and the only question is whether $x' = 0$, $y' = 0$ (a), or $\lambda = 0$, $x'^2 - y'^2 = 0$ (b). In the alternative (a) we have

$$z' = 0, \ p' = 0, \ q' = 0, \ 0 = \mu r' + \nu s', \ 0 = \mu s' + \nu r',$$

and in (b) we have

$$z' = px' + qy', \ p' = rx' + sy', \ q' = sx' + ry', \ 0 = \mu r' + \nu s', \ 0 = \mu s' + \nu r'.$$

If we adopt the supposition (I, 1), then $x, y, z, p, q, \lambda, \mu, \nu$ are each constant, $r + s$ is also constant, and $0 = \mu r' + \nu s'$, $0 = \mu s' + \nu r'$. Therefore one of the following alternatives is true:

(I, 1, 1) $$\mu = \nu, \qquad r + s = \text{const.},$$

(I, 1, 2) $$\mu = -\nu, \quad r - s = \text{const.},$$

(I, 1, 3) $$r = \text{const.}, \quad s = \text{const.},$$

(I, 1, 4) $$\mu = 0, \qquad \nu = 0.$$

In general, besides the two equations obtained by eliminating $\lambda, \mu, \nu, \lambda', \mu', \nu'$ from (17)–(23), which with the equations (2), (3), (4) make up a system of five differential equations equivalent only to four distinct equations of the first order between the variables x, y, z, p, q, r, s, we can deduce two other differential equations of the second order between the same 7 variables with the help of the derivatives of equations (17) and (18). And hence, in general, we can express x, y, p, q, r, s as functions of z and of eight arbitrary constants, such as $x_0, y_0, p_0, q_0, r_0, s_0, r_0', s_0'$, supposing $z_0 = 0$, $z' = 1$, $z_0' = 1$. We can also express $x_0', y_0', p_0', q_0', \dfrac{\mu_0}{\lambda_0}, \dfrac{\nu_0}{\lambda_0}, \dfrac{\lambda_0'}{\lambda_0}$ as functions of the same eight arbitrary constants, and $\dfrac{\lambda}{\lambda_0}, \dfrac{\mu}{\lambda_0}, \dfrac{\nu}{\lambda_0}$ as functions of them and of z. Hence, in (16),

$$\begin{aligned} \xi &= x'(z)\,\zeta + x'(x_0)\,\xi_0 + x'(y_0)\,\eta_0 + x'(p_0)\,\varpi_0 + x'(q_0)\,\kappa_0 + x'(r_0)\,\rho_0 + x'(s_0)\,\sigma_0 \\ &\qquad + x'(r_0')\,\rho_0' + x'(s_0')\,\sigma_0', \\ \eta &= y'(z)\,\zeta + \dots, \\ \varpi &= p'(z)\,\zeta + \dots, \\ \kappa &= q'(z)\,\zeta + \dots, \end{aligned} \tag{25}$$

and since $\zeta, \xi_0, \eta_0, \varpi_0, \kappa_0, \rho_0, \sigma_0, \rho_0', \sigma_0'$ may be treated as 9 independent and arbitrary data, we have the 9 equations:

$$(26)\quad \left\{\begin{array}{llllll}
(\lambda p+\mu r+\nu s)\,.\,x'(z) & +(\lambda q+\mu s+\nu t)\,.\,y'(z) & -\mu p'(z) & -\nu q'(z) & =\lambda, \\
(\qquad\qquad)\,.\,x'(x_0) & +(\qquad\qquad)\,.\,y'(x_0) & -\mu p'(x_0) & -\nu q'(x_0) & =\lambda_0 p_0+\mu_0 r_0+\nu_0 s_0, \\
(\qquad\qquad)\,.\,x'(y_0) & +(\qquad\qquad)\,.\,y'(y_0) & -\mu p'(y_0) & -\nu q'(y_0) & =\lambda_0 q_0+\mu_0 s_0+\nu_0 t_0, \\
(\qquad\qquad)\,.\,x'(p_0) & +(\qquad\qquad)\,.\,y'(p_0) & -\mu p'(p_0) & -\nu q'(p_0) & =-\mu_0, \\
(\qquad\qquad)\,.\,x'(q_0) & +(\qquad\qquad)\,.\,y'(q_0) & -\mu p'(q_0) & -\nu q'(q_0) & =-\nu_0, \\
(\qquad\qquad)\,.\,x'(r_0) & +(\qquad\qquad)\,.\,y'(r_0) & -\mu p'(r_0) & -\nu q'(r_0) & =0, \\
(\qquad\qquad)\,.\,x'(s_0) & +\quad\dots & -\quad\dots & -\quad\dots & =0, \\
(\qquad\qquad)\,.\,x'(r_0') & +\quad\dots & -\quad\dots & -\quad\dots & =0, \\
\qquad\qquad)\,.\,x'(s_0') & +\quad\dots & -\quad\dots & -\quad\dots & =0.
\end{array}\right.$$

Hence, treating all as variable,

$$(27)\quad (\lambda p+\mu r+\nu s)\,x'+(\lambda q+\mu s+\nu t)\,y'-\lambda z'-\mu p'-\nu q'$$
$$=(\lambda_0 p_0+\mu_0 r_0+\nu_0 s_0)\,.\,(x_0)'+(\lambda_0 q_0+\mu_0 s_0+\nu_0 t_0)\,.\,(y_0)'-\mu_0\,(p_0)'-\nu_0\,(q_0)'.$$

If then we establish the three equations (2), (3), (4), we shall have

$$(28)\quad 0=(\lambda_0 p_0+\mu_0 r_0+\nu_0 s_0)\,.\,(x_0)'+(\lambda_0 q_0+\mu_0 s_0+\nu_0 t_0)\,.\,(y_0)'-\mu_0\,(p_0)'-\nu_0\,(q_0)'.$$

In the case of the second example of page 398 we have the twelve differential equations:

$$\left\{\begin{array}{l}
z'=px'+qy',\quad p'=rx'+sy',\quad q'=sx'+ry',\quad \lambda'=0,\quad \mu'=-\lambda x',\quad \nu'=-\lambda y', \\
0=\lambda p'+\mu r'+\nu s',\quad 0=\lambda q'+\mu s'+\nu r',\quad 0=\mu x'+\nu y',\quad 0=\nu x'+\mu y', \\
0=\mu x''+\nu y''+\mu' x'+\nu' y',\quad 0=\nu x''+\mu y''+\nu' x'+\mu' y'.
\end{array}\right.$$

(April 25.)

We can satisfy the 9th and 10th of these equations by supposing $x+y=a$, $\mu=\nu$, a being an arbitrary constant. Then the two equations $\mu'=-\lambda x'$, $\nu'=-\lambda y'$ give $\mu'+\nu'=-\lambda(x'+y')$, that is, $\mu'=0$, $\nu'=0$, and so $\mu=\nu=$ const. They give also $\lambda(x'-y')=0$, a condition which can be satisfied by supposing $\lambda=0$ and which must be satisfied unless we treat x and y as each separately constant. If we suppose $\lambda=0$, $\mu=\nu=$ const., $x+y=a$, the 4th, 11th and 12th of the above equations will be satisfied of themselves, as well as the 5th, 6th, 9th and 10th. Also the 7th and 8th equations will agree in giving $r+s=$ const. $=c$; while the 2nd and 3rd will give $p+q=$ const. $=b$, $p'-q'=(r-s)(x'-y')$; and the 1st will give $z'=\frac{1}{2}(p-q)(x'-y')$. Thus the whole twelve equations will be satisfied, if we suppose

$$\lambda=0,\quad \mu=\nu=\text{const.},\quad x+y=a,\quad p+q=b,\quad r+s=c,$$
$$p'-q'=(r-s)(x'-y'),\quad z'=\tfrac{1}{2}(p-q)(x'-y').$$

The last two conditions may be satisfied by supposing

$$\tfrac{1}{2}(r-s)=\psi''(x-y)+\gamma,\quad \tfrac{1}{2}(p-q)=\psi'(x-y)+\gamma(x-y)+\beta,$$
$$z=\psi(x-y)+\tfrac{1}{2}\gamma(x-y)^2+\beta(x-y)+\alpha,$$

α, β, γ being any arbitrary constants and ψ being any arbitrary function. Now the six equations

$$\begin{cases} x+y=a, \quad p+q=b, \quad r+s=c, \quad z=\psi(x-y)+\tfrac{1}{2}\gamma(x-y)^2+\beta(x-y)+\alpha, \\ \tfrac{1}{2}(p-q)=\psi'(x-y)+\gamma(x-y)+\beta, \quad \tfrac{1}{2}(r-s)=\psi''(x-y)+\gamma \end{cases}$$

are sufficient to determine the six variables x, y, p, q, r, s as functions of z and of the six constants $a, b, c, \alpha, \beta, \gamma$; and we may substitute these six functions in the three differential equations

$$z'=px'+qy', \quad p'-q'=(r-s)(x'-y'), \quad p'+q'=(r+s)(x'+y'),$$

and so deduce three other differential equations between z and the six constants rendered variable. These three new equations will evidently not contain z' on account of the nature of the process of integration by which they were deduced, but will be of the form of linear relations between $a', b', c', \alpha', \beta', \gamma'$. They are the following:

$$\alpha'+(x-y)\beta'+\tfrac{1}{2}(x-y)^2\gamma'=\tfrac{1}{2}ba', \quad 0=\beta'+(x-y)\gamma', \quad b'=ca';$$

of which the third gives

$$b=\phi(a), \quad c=\phi'(a);$$

the second gives

$$\beta=\chi(\gamma), \quad (x-y)=-\chi'(\gamma);$$

and the first gives

$$\alpha'-\tfrac{1}{2}\{\chi'(\gamma)\}^2\gamma'=\tfrac{1}{2}\phi(a)a'.$$

Therefore

$$\alpha=\tfrac{1}{2}\int\{\chi'(\gamma)\}^2d\gamma+\tfrac{1}{2}\int\phi(a)\,da=\Xi(\gamma)+\Phi(a),$$

where

$$\Xi'(\gamma)=\tfrac{1}{2}\{\chi'(\gamma)\}^2, \quad \Phi'(a)=\tfrac{1}{2}\phi(a);$$

and hence

$$\alpha+\beta(x-y)+\tfrac{1}{2}\gamma(x-y)^2=\Xi(\gamma)+\Phi(a)-\chi(\gamma)\chi'(\gamma)+\gamma\Xi'(\gamma).$$

Consequently

$$\begin{cases} z=\psi\{-\chi'(\gamma)\}+\Xi(\gamma)+\tfrac{1}{2}\gamma\{\chi'(\gamma)\}^2-\chi(\gamma)\chi'(\gamma)+\Phi(a), \\ x=\tfrac{1}{2}a-\tfrac{1}{2}\chi'(\gamma), \quad y=\tfrac{1}{2}a+\tfrac{1}{2}\chi'(\gamma), \\ p=\Phi'(a)+\psi'\{-\chi'(\gamma)\}+\chi(\gamma)-\gamma\chi'(\gamma), \quad q=\Phi'(a)-\psi'\{-\chi'(\gamma)\}-\chi(\gamma)+\gamma\chi'(\gamma), \\ r=\Phi''(a)+\psi''\{-\chi'(\gamma)\}+\gamma, \quad s=\Phi''(a)-\psi''\{-\chi'(\gamma)\}-\gamma; \end{cases}$$

and it results from the nature of the foregoing investigation that this system of 7 relations between the 9 variables $x, y, z, p, q, r, s, a, \gamma$ may be considered as an integral of the proposed partial differential equation $t=r$, the functions Φ and ψ being altogether arbitrary and the functions Ξ and χ being subject only to the restriction $\Xi'(\gamma)=\tfrac{1}{2}\{\chi'(\gamma)\}^2$. Accordingly these 7 relations, considered with the derivatives of the first 5 of them, give

$$z'=px'+qy', \quad p'=rx'+sy', \quad q'=sx'+ry',$$

independent of any relation between x and y. But this integral system may be simplified; for, if we put for abridgement

$$\psi\{-\chi'(\gamma)\}+\Xi(\gamma)+\tfrac{1}{2}\gamma\{\chi'(\gamma)\}^2-\chi(\gamma)\chi'(\gamma)=\Psi\{-\chi'(\gamma)\},$$

and hence

$$\Psi'\{-\chi'(\gamma)\}=\psi'\{-\chi'(\gamma)\}-\gamma\chi'(\gamma)+\chi(\gamma), \quad \Psi''\{-\chi'(\gamma)\}=\psi''\{-\chi'(\gamma)\}+\gamma,$$

we can easily eliminate α and γ and deduce the 5 following relations between x, y, z, p, q, r, s:

$$z = \Psi(x-y) + \Phi(x+y),$$

$$p = \Psi'(x-y) + \Phi'(x+y), \quad q = -\Psi'(x-y) + \Phi'(x+y),$$

$$r = \Psi''(x-y) + \Phi''(x+y), \quad s = -\Psi''(x-y) + \Phi''(x+y);$$

of which the first is the sufficient and well-known integral of the equation $t = r$.

In general it seems likely enough that if we can integrate the system of the 3 total differential equations (2), (3), (4) and of their principal supplementaries (17)–(23), and thus, by elimination of λ, μ, ν, deduce x, y, p, q, r, s as functions of z and of constants; and if we then substitute these six functions instead of x, y, p, q, r, s in (2), (3), (4), treating now the constants as variable; the three resulting differential equations, which must evidently be linear relations between the differentials of the constants thus rendered variable, not involving the differential of z, will conduct by some new integration to the integral of the partial differential equation of the second order (1).

Now, if we substitute in (23) the values of λ', μ', ν' deduced from (21), (20), (19), we find

(29) $$0 = \nu y' \{f'(x) + pf'(z) + rf'(p) + sf'(q)\} - (\mu r' + \nu s');$$

and, similarly, from (22) we find

(30) $$0 = \nu y' \{f'(y) + qf'(z) + sf'(p) + tf'(q)\} - (\mu s' + \nu t'),$$

attending to (3) and (4). If we put

(31) $$r' = Rx' + Sy',$$

(32) $$s' = Sx' + Ty',$$

(33) $$t' = Tx' + Uy',$$

R, S, T, U being the partial differential coefficients of z of the third order, taken with respect to x and y, we shall have, by the proposed partial differential equation (1),

(34) $$T = f'(x) + pf'(z) + rf'(p) + sf'(q) + Rf'(r) + Sf'(s),$$

(35) $$U = f'(y) + qf'(z) + sf'(p) + tf'(q) + Sf'(r) + Tf'(s).$$

Hence (29) and (30) become

(36) $$0 = \nu y' \{T - Rf'(r) - Sf'(s)\} - \mu(Rx' + Sy') - \nu(Sx' + Ty'),$$

(37) $$0 = \nu y' \{U - Sf'(r) - Tf'(s)\} - \mu(Sx' + Ty') - \nu(Tx' + Uy');$$

and these are identically satisfied, by (17) and (18).

It seems then that we can in general dispense with the equations (22) and (23); or rather that we may substitute for them the equations (31)–(35), which by elimination of R, S, T, U will give in general a single differential equation of the first order between x, y, z, p, q, r, s, t.

The equations (17), (18) give

(38) $$\mu p' + \nu q' = \nu y' \{f - rf'(r) - sf'(s)\};$$

also by differentiation and by (19), (20)

(39) $$\mu y'' + \nu[x'' + y''f'(s) + y'\{f'(s)\}'] = y'\{\lambda x' + \nu y' f'(p)\} + \{x' + y'f'(s)\}\{\lambda y' + \nu y' f'(q)\},$$

(40) $$\mu x'' + \nu[y''f'(r) + y'\{f'(r)\}'] = x'\{\lambda x' + \nu y' f'(p)\} + y'f'(r)\{\lambda y' + \nu y' f'(q)\}.$$

We have also

$$(41)\qquad t' = x'\{f'(x)+pf'(z)+rf'(p)+sf'(q)\}+r'f'(r)+y'\{f'(y)+qf'(z)+sf'(p)+tf'(q)\} + s'f'(s),$$

and therefore may put equation (30) under the form

$$(42)\qquad 0=\{\mu+\nu f'(s)\}s'+\nu r'f'(r)+\nu x'\{f'(x)+pf'(z)+rf'(p)+sf'(q)\},$$

which gives, when combined with (29),

$$(43)\qquad 0=r'\{\nu y'f'(r)+\mu x'\}+s'\{\mu y'+\nu y'f'(s)+\nu x'\}.$$

But this is identically satisfied by (17) and (18). We may therefore consider the equation (30) as included in the three equations (17), (18), (29); which three last equations will give in general, by elimination of ν/μ, two differential equations of the first order between x, y, z, p, q, r, s, to be combined with the three equations (2), (3), (4). Another equation between the same variables, but of the second order, may be had by eliminating λ between (39) and (40) and then changing ν/μ to its value.

[*Examples.*]

[**3.**] In the example $t=r$, the equations (17), (18), (29) become

$$0=\mu y'+\nu x',\quad 0=\mu x'+\nu y',\quad 0=\mu r'+\nu s',$$

and give, by elimination of ν/μ, the two following:

$$x'^2=y'^2,\quad x's'=y'r'.$$

Also

$$x'r'=y's',$$

this latter being a consequence of the two former. In the same example, the equations (39) and (40) become

$$\mu y''+\nu x''=2\lambda x'y',\quad \mu x''+\nu y''=\lambda(x'^2+y'^2);$$

therefore

$$(\mu y''+\nu x'')(x'^2+y'^2)=2(\mu x''+\nu y'')x'y'.$$

Since

$$\frac{\nu}{\mu}=-\frac{y'}{x'}=-\frac{x'}{y'},$$

we have, by substitution of the first of these two equal values of ν/μ, the following differential equation of the second order:

$$\left(y''-\frac{y'x''}{x'}\right)(x'^2+y'^2)=2\left(x''-\frac{y'y''}{x'}\right)x'y';$$

which is identically satisfied, because the equation $x'^2=y'^2$ gives both $x'x''=y'y''$ and $x'y''=y'x''$.

In the less particular example

$$t=Ar+Bs,$$

the equations (17) and (18) become

$$0=\nu x'+(\mu+\nu B)y',\quad 0=\mu x'+\nu Ay';$$

equation (29) becomes $0=\mu r'+\nu s'$, and (30) becomes $0=\mu s'+\nu t'$. Thus, changing ν/μ to $-x'/Ay'$, we find

$$x'^2+Bx'y'=Ay'^2,\quad Ay'r'=x's',\quad Ay's'=x'(Ar'+Bs'),$$

of which the last is a consequence of the two first since those give

$$Ay's' = \frac{s'(x'^2 + Bx'y')}{y'} = x'(Ar' + Bs').$$

Also equations (39) and (40) become

$$\mu y'' + \nu(x'' + y''B) = \lambda(2x'y' + By'^2), \quad \mu x'' + \nu Ay'' = \lambda(x'^2 + Ay'^2),$$

which give, by elimination of λ,

$$\{\mu y'' + \nu(x'' + y''B)\}(x'^2 + Ay'^2) = (\mu x'' + \nu Ay'')(2x'y' + By'^2).$$

Therefore, eliminating ν/μ,

$$\{Ay'y'' - x'(x'' + y''B)\}(x'^2 + Ay'^2) = (Ay'x'' - Ax'y'')(2x'y' + By'^2);$$

but both sides of this equation vanish in consequence of the relation

$$x'^2 + Bx'y' = Ay'^2,$$

which gives

$$y'/x' = \text{const.}, \quad (y'/x')' = 0, \text{ and } Ay'y'' - x'x'' = \tfrac{1}{2}B(x'y')' = Bx'y''.$$

Still less particularly, let

$$t = Lr + Ms + N,$$

L, M, N being functions of x and y. Then

$$f'(x) = rL'(x) + sM'(x) + N'(x), \quad f'(y) = rL'(y) + sM'(y) + N'(y), \quad f'(z) = 0,$$
$$f'(p) = 0, \quad f'(q) = 0, \quad f'(r) = L, \quad f'(s) = M;$$

the equations (17) and (18) become

$$0 = \mu y' + \nu(x' + My'), \quad 0 = \mu x' + \nu Ly';$$

equation (29) becomes

$$0 = \nu y'\{rL'(x) + sM'(x) + N'(x)\} - (\mu r' + \nu s');$$

and equations (39), (40) become

$$\mu y'' + \nu(x'' + My'' + M'y') = \lambda(2x'y' + My'^2),$$
$$\mu x'' + \nu(Ly'' + L'y') = \lambda(x'^2 + Ly'^2).$$

Hence, by eliminating λ,

$$(x'^2 + Ly'^2)\{\mu y'' + \nu(x'' + My'' + M'y')\} = (2x'y' + My'^2)\{\mu x'' + \nu(Ly'' + L'y')\}.$$

Thus, by the elimination of ν/μ, we find the three following equations:

$$0 = Ly'^2 - x'^2 - Mx'y',$$
$$0 = Ly'r' - x's' + x'y'\{rL'(x) + sM'(x) + N'(x)\},$$
$$(x'^2 + Ly'^2)\{-Ly'y'' + x'x'' + x'My'' + x'M'y'\} = (2x'y' + My'^2)\{-Ly'x'' + Lx'y'' + L'x'y'\}.$$

The first of these equations gives

$$2Ly'y'' = 2x'x'' + M(x'y'' + y'x'') + M'x'y' - L'y'^2;$$

therefore

$$y'' = \frac{(2x' + My')x'' + (M'x' - L'y')y'}{2Ly' - Mx'} = y'\left\{\frac{x''}{x'} + \frac{M'x' - L'y'}{2Ly' - Mx'}\right\},$$

and

$$(x'^2 + Ly'^2)\{-x'^2(M'x' - L'y') + x'y'M'(2Ly' - Mx')\}$$
$$= (2x' + My')x'y'^2\{L(M'x' - L'y') + L'(2Ly' - Mx')\}.$$

Thus

$$\begin{aligned}0 = x'L'\{x'y'(x'^2+Ly'^2)+y'^2(2x'+My')(Mx'-Ly')\}\\ +x'M'\{(-x'^2+2Ly'^2-Mx'y')(x'^2+Ly'^2)-x'y'^2L(2x'+My')\}.\end{aligned}$$

Hence, setting aside the factor $x'y'^2$,

$$0 = L'\{(2Ly'-Mx')x'+(2x'+My')(Mx'-Ly')\}+M'\{L(x'^2+Ly'^2)-L(2x'^2+Mx'y')\},$$

in which the coefficients of L' and M' both vanish in consequence of the equation

$$0 = Ly'^2-x'^2-Mx'y'.$$

In the present example, $t = Lr+Ms+N$, the equation (38) gives

$$\mu p'+\nu q' = \nu N y';$$

and, since (18) gives $0 = \mu x'+\nu L y'$, we have, by elimination of ν/μ,

$$Ly'p'-x'q'+Nx'y' = 0.$$

This equation may take the place of equation (4), so that, upon the whole, we may establish the system of the five following equations:

$$\begin{cases} z' = px'+qy', \quad p' = rx'+sy', \quad 0 = Ly'^2-x'^2-Mx'y', \\ 0 = Ly'p'-x'q'+Nx'y', \quad 0 = Ly'r'-x's'+x'y'\{rL'(x)+sM'(x)+N'(x)\}, \end{cases}$$

which are, if possible, to be integrated all together by discovering functions of z and of constants which, when substituted for x, y, p, q, r, s, shall satisfy them. If we can discover such functions, we may then put $\nu/\mu = -x'/Ly'$, and we shall therefore have

$$\frac{}{\mu} = -\frac{y'}{x'+My'}.$$

Also (19) and (20) give

$$\left(\frac{\nu}{\mu}\right)' = \frac{\nu'}{\mu}-\frac{\nu\mu'}{\mu^2} = -\frac{\lambda}{\mu}y'+\frac{\nu\lambda}{\mu^2}x'.$$

Therefore

$$\frac{\lambda}{\mu} = \frac{-(\nu/\mu)'}{y'-(\nu/\mu)x'} = \frac{(x'/Ly')'}{y'+x'^2/Ly'} = \frac{x''-x'(Ly''+L'y')/Ly'}{Ly'^2+x'^2}.$$

The same equations (19), (20) give

$$\lambda = -\frac{\nu'}{y'} = -\frac{\mu'}{x'}, \quad \nu = -\frac{\mu x'}{Ly'};$$

therefore

$$\lambda = \frac{\mu'x'+\mu x''}{Ly'^2}-\frac{\mu x'(Ly''+L'y')}{L^2y'^3} = -\frac{\mu'}{x'}.$$

Thus

$$\lambda = -\frac{\mu'}{x'} = \frac{\mu}{Ly'^3}(y'x''-x'y'')+\frac{x'}{L^2y'^2}(L\mu'-L'\mu) = \frac{x'}{Ly'^2}\left\{\mu'-\frac{L'\mu}{L}-\mu\frac{M'x'-L'y'}{2Ly'-Mx'}\right\},$$

by the value of y'' deduced by differentiation from the equation

$$Ly'^2-x'^2-Mx'y' = 0.$$

Hence

$$\begin{aligned}0 &= (\lambda x'+\mu')L^2y'^2(2Ly'-Mx') = \mu'Ly'(2Ly'-Mx')^2-\mu x'^2\{L'(Ly'-Mx')+M'Lx'\}\\ &= x'^2\{\mu'Ly'(M^2+4L)-\mu LL'y'+\mu x'(ML'-LM')\},\end{aligned}$$

and so

$$\frac{\mu'}{\mu}=\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)y'}, \quad \frac{\lambda}{\mu}=-\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)x'y'}.$$

We had found before that

$$\frac{\nu}{\mu}=-\frac{x'}{Ly'}.$$

The equation (21) gives, in the present example, $\lambda'=0$; we ought therefore to have identically, or at least in virtue of the relation

$$0=Ly'^2-x'^2-Mx'y',$$

$$0=\frac{\{LL'y'+(LM'-ML')x'\}'}{LL'y'+(LM'-ML')x'}+\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)y'}-\frac{\{L(M^2+4L)x'y'\}'}{L(M^2+4L)x'y'}.$$

(If, to particularise a little further, we make

$$L=-\frac{x^2}{y^2}, \quad M=-\frac{2xy}{y^2}=-\frac{2x}{y},$$

we fall on the case of equal roots and have $M^2+4L=0$; to which case we shall afterwards return.)

If we suppose, at random, $L=x^2$, $M=0$, we have

$$\frac{\mu'}{\mu}=\frac{L'}{4L}=\frac{x'}{2x}, \quad \frac{\lambda}{\mu}=-\frac{1}{2x}, \quad \frac{\lambda'}{\lambda}=\frac{\mu'}{\mu}-\frac{x'}{x}=-\frac{x'}{2x}.$$

It certainly seems that we have not in general $\lambda'=0$, without a new relation giving the ratio y'/x', which may not coincide with the relation $0=Ly'^2-x'^2-Mx'y'$; and perhaps there may thus arise an *equation of condition* restricting the application of the method.

In the case of the equation

$$t=-\frac{x^2}{y^2}r-\frac{2x}{y}$$

the equation $Ly'^2=x'^2+Mx'y'$ becomes $0=(yx'-xy')^2$, and gives

$$\frac{y'}{x'}=\frac{y}{x}=\text{const.}=a.$$

Also the equation $0=Ly'p'-x'q'+Nx'y'$ becomes $y'p'/x'q'=-y^2/x^2$. Therefore

$$\frac{p'}{q'}=-\frac{y}{x}=-a,$$

and hence

$$p+aq=\text{const.}=b.$$

The 5th of the five equations in the middle of page 405 becomes

$$0=-\frac{x^2}{y^2}y'r'-x's'-\frac{2x'y'}{y^2}(xr+ys),$$

that is,

$$0=xr'+ys'+2\frac{x}{x}(xr+ys),$$

and so

$$0=\frac{r'+as'}{r+as}+2\frac{x'}{x},$$

or

$$(r+as)\,x^2=\text{const.}=c.$$

The first of the same equations gives

$$z'=(p+aq)\,x'=(b+2aq)\,x',$$

and hence

$$z-bx-\psi(x,\ a,\ b,\ c)=\text{const.}=\alpha,\quad \psi'(x)=2aq.$$

The second of the same equations gives

$$p'=(r+as)\,x'=\frac{cx'}{x^2},$$

and hence

$$p+\frac{c}{x}=\text{const.}=\beta.$$

Reciprocally if we differentiate these five integrals, treating a, b, c, α, β as variables, we find

$$y'=ax'+xa',\quad p'=-aq'-qa'+b',\quad (r'+as')\,x^2+2\,(r+as)\,xx'=c',$$

$$z'=bx'+xb'+\psi'(x)\,x'+\alpha',\quad p'=\frac{cx'}{x^2}-\frac{c'}{x}+\beta',$$

to which we are to join these five integral equations themselves and the equation $\psi'(x)=2aq$. Thus

$$\begin{aligned}0=px'+qy'-z'=bx'+q\,(ax'+y')-z'&=(b+2aq)\,x'+qxa'-z'\\&=qxa'-xb'-\alpha';\end{aligned}$$

$$0=rx'+sy'-p'=(r+as)\,x'+sxa'-\frac{cx'}{x^2}+\frac{c'}{x}-\beta'=sxa'+\frac{c'}{x}-\beta';$$

$$\begin{aligned}0=sx'+ty'-q'&=(s+at)\,x'+txa'-q'\\&=\frac{1}{a}\left\{-(r+as)\,x'+txaa'+qa'-b'+\frac{cx'}{x^2}-\frac{c'}{x}+\beta'\right\}\\&=\frac{1}{a}\left\{a'\,(q+txa)-b'-\frac{c'}{x}+\beta'\right\}=\frac{1}{a}\{(q+txa+sx)\,a'-b'\}.\end{aligned}$$

Therefore

$$\frac{b'}{a'}=q+x\,(s+ta)=q-\frac{x}{a}(r+as)=q-\frac{c}{ax}=q-\frac{c}{y},\quad \alpha'=x\,(qa'-b')=\frac{ca'}{a},$$

and so

$$\alpha=\chi(a),\quad c=a\chi'(a).$$

XVI.

CALCULUS OF PRINCIPAL RELATIONS

Read August, 1836.

[*British Association Report*, 1836, Part II, pp. 41–44.]

The method of principal relations is an extension of that mode of analysis which Sir William Hamilton has applied before to the sciences of optics and dynamics; its nature and spirit may be understood from the following sketch.

Let $x_1, x_2, \dots x_n$ be any number n of functions of any one independent variable s, with which they are connected by any one given differential equation of the first order, but not of the first degree,

$$0 = f(s, x_1, \dots x_n, ds, dx_1, \dots dx_n), \tag{1}$$

and also by $n-1$ other differential equations of the second order, to which the calculus of variations conducts, as supplementary to the given equation (1), and which may be thus denoted:

$$\frac{f'(x_1) - df'(dx_1)}{f'(dx_1)} = \dots = \frac{f'(x_n) - df'(dx_n)}{f'(dx_n)}. \tag{2}$$

Let also $a_1, \dots a_n$ be the n initial values of the n functions $x_1, \dots x_n$ and let $a'_1, \dots a'_n$ be the n initial values of their n derived functions or differential coefficients

$$x'_1 = \frac{dx_1}{ds}, \dots x'_n = \frac{dx_n}{ds},$$

corresponding to any assumed initial value a of the independent variable s. If we could integrate the system of the n differential equations (1) and (2), we should thereby obtain n expressions for the n functions $x_1, \dots x_n$ of the forms

$$\begin{aligned} x_1 &= \phi_1(s, a, a_1, \dots a_n, a'_1, \dots a'_n), \\ x_2 &= \phi_2(s, a, a_1, \dots a_n, a'_1, \dots a'_n), \\ & \\ x_n &= \phi_n(s, a, a_1, \dots a_n, a'_1, \dots a'_n); \end{aligned} \tag{3}$$

and, by the help of the initial equation analogous to (1), might then eliminate $a'_1, \dots a'_n$ and deduce a relation of the form

$$0 = \psi(s, x_1, \dots x_n, a, a_1, \dots a_n); \tag{4}$$

that is, a relation between the initial and final values of the $n+1$ connected variables $s, x_1, \dots x_n$. Reciprocally, the author has found that if this one relation (4) were known, it would be possible thence to deduce expressions for the n sought integrals (3) of the system of the n differential equations (1) and (2), or for the n sought relations between $s, x_1, \dots x_n$ and $a, a_1, \dots a_n, a'_1, \dots a'_n$, however large the number n may be; in such manner that all these many relations (3) are implicitly contained in the one relation (4), which latter relation the author proposes to call on this account the *principal integral relation*, or simply, the PRINCIPAL RELATION of the problem.

For he has found that the n following equations hold good,

$$\frac{f'(ds)}{\psi'(s)} = \frac{f'(dx_1)}{\psi'(x_1)} = \ldots = \frac{f'(dx_n)}{\psi'(x_n)}; \tag{5}$$

which may be put under the forms

$$\left.\begin{array}{l} a_1 = \phi_1(a,\ s,\ x_1,\ \ldots\ x_n,\ x_1',\ \ldots\ x_n'), \\ \\ a_n = \phi_n(a,\ s,\ x_1,\ \ldots\ x_n,\ x_1',\ \ldots\ x_n'), \end{array}\right\} \tag{6}$$

and are evidently transformations of the n sought integrals (3). And with respect to the mode in which, without previously effecting the integrations (3), it is possible to determine the *principal relation* (4), or the *principal function* which it introduces, when it is conceived to be resolved, as follows, for the originally independent variable s,

$$s = \phi(x_1,\ \ldots\ x_n,\ a,\ a_1,\ \ldots\ a_n), \tag{7}$$

the author remarks that a partial differential equation of the first order may be assigned, which this principal function ϕ must satisfy, and also an initial condition adapted to remove the arbitrariness which otherwise would remain. In fact the equations (5) may be thus written,

$$\frac{\delta(ds)}{\delta(dx_1)} = \frac{\delta s}{\delta x_1},\ \ldots\ \frac{\delta(ds)}{\delta(dx_n)} = \frac{\delta s}{\delta x_n}, \tag{8}$$

in which

$$\frac{\delta(ds)}{\delta(dx_i)} = -\frac{f'(dx_i)}{f'(ds)}, \quad \text{and} \quad \frac{\delta s}{\delta x_i} = \phi'(x_i), \tag{9}$$

and since, by (1), there subsists a known relation of the form

$$0 = F\left(s,\ x_1,\ \ldots\ x_n,\ \frac{\delta(ds)}{\delta(dx_1)},\ \ldots\ \frac{\delta(ds)}{\delta(dx_n)}\right), \tag{10}$$

the following relation also must hold good,

$$0 = F\left(s,\ x_1,\ \ldots\ x_n,\ \frac{\delta s}{\delta x_1},\ \ldots\ \frac{\delta s}{\delta x_n}\right), \tag{11}$$

that is, the principal function ϕ must satisfy the following partial differential equation of the first order,

$$0 = F(\phi,\ x_1,\ \ldots\ x_n,\ \phi'(x_1),\ \ldots\ \phi'(x_n)); \tag{12}$$

it must also satisfy the following initial condition,

$$0 = \lim_{s=a} f(a,\ a_1,\ \ldots\ a_n,\ \phi - a,\ x_1 - a_1,\ \ldots\ x_n - a_n). \tag{13}$$

Such are the most essential principles of the new method in analysis which Sir William Hamilton has proposed to designate by the name of the *Method of Principal Relations*, and of which perhaps the simplest *type* is the formula

$$\frac{\delta(ds)}{\delta(dx)} = \frac{\delta s}{\delta x}, \tag{14}$$

to be interpreted like the equations (8).

The simplest *example* which can be given to illustrate the meaning and application of these principles is perhaps that in which the differential equations are

$$0 = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 - 1 \tag{1'}$$

and
$$\frac{d\,(dx_1)}{dx_1}=\frac{d\,(dx_2)}{dx_2}. \tag{2'}$$

Here ordinary integration gives
$$x_1=a_1+a_1'(s-a), \quad x_2=a_2+a_2'(s-a); \tag{3'}$$
and consequently conducts to the following relation, (in this case the *principal* one,)
$$0=(x_1-a_1)^2+(x_2-a_2)^2-(s-a)^2, \tag{4'}$$
or
$$s=a+\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}, \tag{7'}$$
because by (1′) we have
$$a_1'^2+a_2'^2=1;$$
it enables us therefore to verify the relations (8) or (14), for it gives
$$\frac{\delta s}{\delta x_1}=\frac{x_1-a_1}{s-a}=\frac{dx_1}{ds}=\frac{\delta\,(ds)}{\delta\,(dx_1)},$$
and, in like manner,
$$\frac{\delta s}{\delta x_2}=\frac{\delta\,(ds)}{\delta\,(dx_2)}.$$

Reciprocally, in this example, the following known relation, deduced from (1′),
$$0=\left(\frac{\delta\,(ds)}{\delta\,(dx_1)}\right)^2+\left(\frac{\delta\,(ds)}{\delta\,(dx_2)}\right)^2-1, \tag{10'}$$
would have given, by the principles of the new method, this partial differential equation of the first order,
$$0=\left(\frac{\delta s}{\delta x_1}\right)^2+\left(\frac{\delta s}{\delta x_2}\right)^2-1, \tag{11'}$$
which might have been used, in conjunction with the initial condition
$$0=\lim_{s=a}\left\{\left(\frac{x_1-a_1}{s-a}\right)^2+\left(\frac{x_2-a_2}{s-a}\right)^2-1\right\}, \tag{13'}$$
to determine the form (7′) of the principal function s; and thence might have been deduced, by the same new principles, the ordinary integrals (3′) under the forms
$$a_1=x_1+a_1'(a-s), \quad a_2=x_2+a_2'(a-s). \tag{6'}$$

In so simple an instance as this there would be no advantage in using the new method; but in a great variety of questions, including all those of mathematical optics and mathematical dynamics, (at least, as those sciences have been treated by the author of this communication,) and in general all the problems in which it is required to integrate those systems of ordinary differential equations (whether of the second or of a higher order) to which the calculus of variations conducts, the method of principal relations assigns immediately a system of finite expressions for the integrals of the proposed equations, an object which can only very rarely be attained by any of the methods known before.

It seems, for example, to be impossible by any other method to express rigorously, in finite terms, the integrals of the differential equations of motion of a system of many points attracting or repelling one another; which yet was easily accomplished by a particular application of the general principles that have been here explained.* The author hopes to present these principles in a still more general form hereafter.

---

* See *Philosophical Transactions* for 1834 and 1835; also, Report of Edinburgh Meeting of the British Association. [Pages 103–216 of this volume.]

Part III

DYNAMICS OF LIGHT

XVII.

[ON THE PROPAGATION OF LIGHT IN CRYSTALS*]

[1835–1838.]

[Note Book 40.]

[CONTENTS

	PAGE
Equations of motion of an attracting or repelling system in vibration about a state of rest or motion.	413
Conditions for plane waves.	416
Values of constants for an infinitesimally strained cubic lattice.	418
Wave velocities and directions of vibrations.	423
Fresnel's wave surface.	432
Reflexion and refraction.	435
Extension of above to dispersion in a biaxal crystal.]	437

[*Equations of motion of an attracting or repelling system in vibration about a state of rest or motion.*]

1. Let x, y, z be the rectangular coordinates of any one point A of an attracting or repelling system at the time t; let $x + r\cos\alpha$, $y + r\cos\beta$, $z + r\cos\gamma$ be the coordinates of any other point B of the same system, at the same moment; and let the components of the accelerating or retarding action of B on A be denoted by $m\cos\alpha f(r)$, $m\cos\beta f(r)$, $m\cos\gamma f(r)$, m being a constant factor which may be called the *mass* of the point B, & the function $f(r)$ being positive for the case of attraction and negative for the case of repulsion; while the distance r between the two points is considered as always positive: then the differential equations of the motion of the point A, if we attend only to the action of B upon it, are

$$\frac{d^2x}{dt^2} = m\cos\alpha f(r),$$

$$\frac{d^2y}{dt^2} = m\cos\beta f(r), \qquad (1.)$$

$$\frac{d^2z}{dt^2} = m\cos\gamma f(r);$$

and if we take account of the actions of all the points B, C, &c., on the point A, the differential equations of the motion of A may be then denoted as follows,

$$\frac{d^2x}{dt^2} = S.m\cos\alpha f(r),$$

$$\frac{d^2y}{dt^2} = S.m\cos\beta f(r), \qquad (2.)$$

$$d^2z \quad = S.m\cos\gamma f(r),$$

---

* [See Cauchy, *Œuvres*, II^{e} Série, Tome IX, pp. 390–450; I^{e} Série, Tome II, pp. 111–177. Also Appendix, Note 11, p. 638. An abstract of this manuscript appears on page 450.]

the sign of summation S extending to all these other points B, C, &c., each of which has its own set of values of m, r, α, β, γ.

If the total number of attracting or repelling points A, B, C, &c. be n & if we attend only to their actions upon each other, there will be then $3n$ simultaneous differential equations of the 2nd order including and analogous to the 3 equations (2.); & the integrals of these $3n$ equations (if known) would express the $3n$ coordinates of the n points of the system as functions of the time t, involving also, besides the n masses, $6n$ arbitrary constants, which might be determined by the initial positions of the points and their initial components of velocities. And for every different supposition which we might make respecting these arbitrary initial circumstances, we should have (in general) a different result respecting the manner of motion of the system, though still consistent with the laws of mutual attraction or repulsion, as expressed by the differential equations.

2. Imagine then that while x, y, z and $x+r\cos\alpha$, $y+r\cos\beta$, $z+r\cos\gamma$ denote as before the coordinates of the points A & B at the time t in one possible state of motion (or rest) of the system, the coordinates of the same two points at the same moment in another possible state of motion of the same system are denoted by $x+\xi$, $y+\eta$, $z+\zeta$ and

$$x+r\cos\alpha+\xi+\Delta\xi, \quad y+r\cos\beta+\eta+\Delta\eta, \quad z+r\cos\gamma+\zeta+\Delta\zeta,$$

respectively; and let $r+\epsilon r$ denote the mutual distance of these two points in this new manner of motion, so that

$$r+\epsilon r=\sqrt{(r\cos\alpha+\Delta\xi)^2+(r\cos\beta+\Delta\eta)^2+(r\cos\gamma+\Delta\zeta)^2}. \qquad (3.)$$

Then

$$\frac{r\cos\alpha+\Delta\xi}{r+\epsilon r}, \quad \frac{r\cos\beta+\Delta\eta}{r+\epsilon r}, \quad \frac{r\cos\gamma+\Delta\zeta}{r+\epsilon r}, \text{ that is, } \frac{\cos\alpha+\frac{\Delta\xi}{r}}{1+\epsilon}, \quad \frac{\cos\beta+\frac{\Delta\eta}{r}}{1+\epsilon}, \quad \frac{\cos\gamma+\frac{\Delta\zeta}{r}}{1+\epsilon},$$

will denote the 3 new cosines by which the new quantity $mf(r+\epsilon r)$ is to be multiplied, in order to get the three new components of accelerating or retarding action of B on A; & the differential equations (2.) of the old motion (or rest) of the system will be changed for the new motion, to the following

$$\left.\begin{aligned}
\frac{d^2x}{dt^2}+\frac{d^2\xi}{dt^2}&=S.m\left(\cos\alpha+\frac{\Delta\xi}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon},\\
\frac{d^2y}{dt^2}+\frac{d^2\eta}{dt^2}&=S.m\left(\cos\beta+\frac{\Delta\eta}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon},\\
\frac{d^2z}{dt^2}+\frac{d^2\zeta}{dt^2}&=S.m\left(\cos\gamma+\frac{\Delta\zeta}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon}.
\end{aligned}\right\} \qquad (4.)$$

And subtracting the equations (2.) of the old motion, from the equations (4.) of the new, we find the following equations:

$$\left.\begin{aligned}
\frac{d^2\xi}{dt^2}&=S.m\cos\alpha\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\xi}{r}\frac{f(r+\epsilon r)}{1+\epsilon},\\
\frac{d^2\eta}{dt^2}&=S.m\cos\beta\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\eta}{r}\frac{f(r+\epsilon r)}{1+\epsilon},\\
\frac{d^2\zeta}{dt^2}&=S.m\cos\gamma\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\zeta}{r}\frac{f(r+\epsilon r)}{1+\epsilon},
\end{aligned}\right\} \qquad (5.)$$

which express, generally and rigorously, the laws of the differences between any two possible motions of any one system of attracting or repelling points: or, in other words, the laws of the dynamically possible deviations from any one dynamically possible manner of motion of an attracting or repelling system.

3. Let us now modify these general and rigorous expressions by introducing the particular & approximate supposition that the *length and direction of the varying line which connects any one moving point of the system with any other differ extremely little at any arbitrary moment t in the second manner of motion of the system from the length and direction of the same line at the same moment in the first manner of motion.* On this hypothesis the quantities $\frac{\Delta\xi}{r}$, $\frac{\Delta\eta}{r}$, $\frac{\Delta\zeta}{r}$ and ϵ will be extremely small, whether r be large or small, & we may neglect their squares and products; so that if we denote by $f'(r)$ the first differential coefficient of the function $f(r)$, the equations (5.) will take the simplified forms,

$$\begin{aligned}
\frac{d^2\xi}{dt^2} &= S.m\cos\alpha\{rf'(r)-f(r)\}\epsilon + S.mf(r)\frac{\Delta\xi}{r},\\
\frac{d^2\eta}{dt^2} &= S.m\cos\beta\{rf'(r)-f(r)\}\epsilon + S.mf(r)\frac{\Delta\eta}{r},\\
\frac{d^2\zeta}{dt^2} &= S.m\cos\gamma\{rf'(r)-f(r)\}\epsilon + S.mf(r)\frac{\Delta\zeta}{r},
\end{aligned} \qquad (6.)$$

in which, by (3.),

$$\epsilon = \cos\alpha\frac{\Delta\xi}{r} + \cos\beta\frac{\Delta\eta}{r} + \cos\gamma\frac{\Delta\zeta}{r}. \qquad (7.)$$

Such are the approximate laws of the dynamically possible deviations from one standard manner of motion of an attracting or repelling system, under the condition that the mutual distances of the several points of that system are altered extremely little by those deviations in length and in direction.

4. To illustrate by an example the foregoing general results, we may remark that they extend to the case of the solar system, considered as a system of points of which each attracts every other inversely as the square of the distance. In this case $f(r) = r^{-2}$, and the equations (2.) for the standard manner of motion, which may be supposed to be the actual motion of the system, become $\frac{d^2x}{dt^2} = S.\frac{m\cos\alpha}{r^2}$, $\frac{d^2y}{dt^2} = S.\frac{m\cos\beta}{r^2}$, $\frac{d^2z}{dt^2} = S.\frac{m\cos\gamma}{r^2}$. And if with this standard or actual motion of the solar system we compare any other motion which the same system might possibly have, (consistently with the present values of the masses & with the present law of attraction,) the deviations of the latter motion from the former are connected rigorously by equations of the form (5.)

$$\frac{d^2\xi}{dt^2} = S.m\cos\alpha\left(\frac{(r+\epsilon r)^{-2}}{1+\epsilon} - r^{-2}\right) + S.m\frac{\Delta\xi}{r}\cdot\frac{(r+\epsilon r)^{-2}}{1+\epsilon}, \text{ \&c.};$$

& if these deviations be imagined such as to alter very little the length and the direction of the distance of any one planet from any other or from the Sun, they may then be approximately expressed by equations of the form (6.), namely

$$\frac{d^2\xi}{dt^2} = -3S.m\cos\alpha\frac{\epsilon}{r^2} + S.\frac{m\Delta\xi}{r^3}, \text{ \&c.},$$

in which ϵ has the value (7.).

5. Let it next be supposed that *the law of mutual action is such that the system is capable of equilibrium* either exactly, or at least so very nearly that the difference shall produce no sensible effect in the final conclusions from this supposition: and let this possible state of equilibrium be taken as the *standard state*, to which the coordinates x, y, z refer & from which the deviations or displacements ξ, η, ζ are to be measured. Then the equations (6.) will determine approximately the motions of a displaced point of the system; and α, β, γ, r will become independent of the time and will satisfy the following equations of equilibrium,

$$0 = S.m\cos\alpha f(r), \quad 0 = S.m\cos\beta f(r), \quad 0 = S.m\cos\gamma f(r). \tag{8.}$$

More generally, we shall suppose that *the action of any one point of the system on any other is so small and extends sensibly to so small a distance, and that the distribution of these points in space is so symmetric*, as to satisfy, without sensible error, the conditions

$$\left.\begin{aligned} S.m\cos\alpha^{i}\cos\beta^{i'}\cos\gamma^{i''}r^{h}f(r) &= 0,\\ S.m\cos\alpha^{i}\cos\beta^{i'}\cos\gamma^{i''}r^{h}f'(r) &= 0, \end{aligned}\right\} \tag{9.}$$

in which h, i, i', i'' may denote any positive integers, or zero, *provided that the sum of the exponents* i, i', i'' *of the cosines is an odd number*. That is, we shall suppose that in the state of equilibrium, for every point B with mass m & with coordinates $x + r\cos\alpha$, $y + r\cos\beta$, $z + r\cos\gamma$ which is near enough to influence sensibly the point A with coordinates x, y, z, there is another near point C with an equal mass m and having for coordinates

$$x - r\cos\alpha, \quad y - r\cos\beta, \quad z - r\cos\gamma;$$

or at least that the results deduced from this supposition are not liable to sensible error. It is worth while to observe that when the extent of sensible action is supposed small, the approximate equations of motion of a displaced point (6.) may be obtained from the rigorous equations (5.) by supposing only that *within this small extent of sensible action the mutual distance of any two points changes very little in length and in direction.*

[*Conditions for plane waves.*]

6. The foregoing suppositions being admitted, we shall next inquire *whether under any, and under what conditions the approximate equations of motion of a displaced point* (6.) *are compatible with the uniform propagation of an indefinite series of plane waves, or of displacements which succeed each other periodically in space and time & are the same for all the points situated at any one moment on any one plane parallel to a given plane*; in other words, we shall inquire whether & under what conditions the variable displacements ξ, η, ζ can be supposed to be periodical functions of any one linear function, such as $ux + vy + wz + st + \theta$, of the coordinates of equilibrium x, y, z & the time t, consistently with the approximate equations of motion (6.). But to simplify this inquiry, *which is not into the most general motion possible but only into the conditions of the possibility of a particular motion proposed*, we shall consider at present only the simplest kind of periodical motion; & shall therefore assume, for the laws of displacement to be examined, the following expressions,

$$\left.\begin{aligned} \xi &= \xi^{\backprime}\sin(ux + vy + wz + st + \theta),\\ \eta &= \eta^{\backprime}\sin(ux + vy + wz + st + \theta),\\ \zeta &= \zeta^{\backprime}\sin(ux + vy + wz + st + \theta). \end{aligned}\right\} \tag{10.}$$

So that the question now is, whether under *any*, & if so then under *what* conditions, the constants

ξ', η', ζ', u, v, w, s, θ can be so chosen as to satisfy the differential equations of motion (6.) by the expressions of displacement (10.). If we can discover such conditions, we can then determine the *thickness* (λ) of the wave, & the *slowness* (μ) of its propagation; since these will be respectively equal to the square roots of the expressions $\frac{4\pi^2}{u^2+v^2+w^2}$ and $\frac{u^2+v^2+w^2}{}$

7. In this investigation, the operation Δ of differencing, by which we pass from the displacement of one to the displacement of another point of the system, affects only the coordinates of equilibrium x, y, z in the expressions of displacement (10.) & not the time t nor the constants ξ', η', ζ', u, v, w, s, θ, which are supposed to be the same for all the points of the system. We are therefore to substitute for $\Delta\xi$ in the equations (7.) and (6.) the value

$$\begin{aligned}\Delta\xi = {} & \xi' \sin(ux+vy+wz+u\,\Delta x+v\,\Delta y+w\,\Delta z+st+\theta)\\ & -\xi' \sin(ux+vy+wz+st+\theta)\\ = {} & -2\xi' \sin(ux+vy+wz+st+\theta)\left(\sin\frac{u\,\Delta x+v\,\Delta y+w\,\Delta z}{2}\right)^2\\ & +\xi' \cos(ux+vy+wz+st+\theta)\sin(u\,\Delta x+v\,\Delta y+w\,\Delta z)\\ = {} & -2\xi\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2\\ & +\xi' \cos(ux+vy+wz+st+\theta)\sin(ur\cos\alpha+vr\cos\beta+wr\cos\gamma),\end{aligned} \tag{11.}$$

because

$$\xi = \xi' \sin(ux+vy+wz+st+\theta)$$

and

$$\Delta x = r\cos\alpha, \quad \Delta y = r\cos\beta, \quad \Delta z = r\cos\gamma; \tag{12.}$$

and similar substitutions are to be made for $\Delta\eta$ and $\Delta\zeta$. The sign of summation S refers to the various values of m, r, α, β, γ corresponding to the various other points B, C, &c. of the system which attract or repel the point A, whose coordinates when displaced are $x+\xi$, $y+\eta$, $z+\zeta$; so that if we attend to the suppositions made in the 5th paragraph and put, for abridgment,

$$\begin{aligned}L &= S.\,2m\{\cos\alpha^2 f'(r)+\sin\alpha^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\\ M &= S.\,2m\{\cos\beta^2 f'(r)+\sin\beta^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\\ N &= S.\,2m\{\cos\gamma^2 f'(r)+\sin\gamma^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\end{aligned} \tag{13.}$$

and

$$\begin{aligned}P &= S.\,2m\cos\beta\cos\gamma\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\\ Q &= S.\,2m\cos\gamma\cos\alpha\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\\ R &= S.\,2m\cos\alpha\cos\beta\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\end{aligned} \tag{14.}$$

we shall have the three following conditions to be satisfied:

$$\begin{aligned}\frac{d^2\xi}{dt^2}&=-(L\xi+R\eta+Q\zeta),\\ \frac{d^2\eta}{dt^2}&=-(M\eta+P\zeta+R\xi),\\ \frac{d^2\zeta}{dt^2}&=-(N\zeta+Q\xi+P\eta),\end{aligned}\tag{15.}$$

or finally, by the supposed expressions of displacement,

$$\left.\begin{aligned}(L-s^2)\,\xi'+R\eta'+Q\zeta'=0,\\ (M-s^2)\,\eta'+P\zeta'+R\xi'=0,\\ (N-s^2)\,\zeta'+Q\xi'+P\eta'=0.\end{aligned}\right\}\tag{16.}$$

And these are the sought conditions for the possibility of the uniform propagation of a succession of plane waves with displacements of the forms (10.), in a system of attracting or repelling points distributed as supposed above. They do not at all restrict the constant θ but they give three relations between the constants u, v, w, s and those two other constants which express the ratios of ξ', η', ζ'.

[*Values of constants for an infinitesimally strained cubic lattice.*]

8. To simplify these conditions, we shall now introduce the additional supposition of an *equality between the quantities m, and nearly cubical arrangement of the attracting or repelling points* in their position of equilibrium, which differs from a cubical arrangement only by *three small and (in general) unequal dilatations or contractions in three rectangular directions*: in such a manner that in the expressions (13.), (14.) we may put

$$\left.\begin{aligned}r\cos\alpha=(1+a_,)\,r_,\cos\alpha_,\,,\\ r\cos\beta=(1+b_,)\,r_,\cos\beta_,\,,\\ r\cos\gamma=(1+c_,)\,r_,\cos\gamma_,\,,\end{aligned}\right\}\tag{17.}$$

$a_,$, $b_,$, $c_,$ being three small constant coefficients of dilatation or contraction & $r_,$, $\alpha_,$, $\beta_,$, $\gamma_,$ being the quantities analogous to r, α, β, γ in that exactly cubical arrangement of points from which the arrangement of the system, in the state of equilibrium considered lately, is now supposed to differ little. On this hypothesis, if we neglect all terms above the second dimension with respect to the small dilatations or contractions $a_,$, $b_,$, $c_,$, the accurate expression

$$r=r_,\sqrt{(1+a_,)^2\cos\alpha_,^2+(1+b_,)^2\cos\beta_,^2+(1+c_,)^2\cos\gamma_,^2}\tag{18.}$$

deduced from the equations (17.) will give approximately

$$\begin{aligned}r=r_,\{1+a_,\cos\alpha_,^2+b_,\cos\beta_,^2+c_,\cos\gamma_,^2+\tfrac{1}{2}(a_,-b_,)^2\cos\alpha_,^2\cos\beta_,^2+\tfrac{1}{2}(b_,-c_,)^2\cos\beta_,^2\cos\gamma_,^2\\ +\tfrac{1}{2}(c_,-a_,)^2\cos\gamma_,^2\cos\alpha_,^2\};\end{aligned}\tag{19.}$$

or simply,

$$r=r_,(1+a_,\cos\alpha_,^2+b_,\cos\beta_,^2+c_,\cos\gamma_,^2),\tag{20.}$$

if we neglect even the squares and products of $a_,$, $b_,$, $c_,$: and in this latter order of approximation, the equations (17.) will also give

$$\left.\begin{aligned}\cos\alpha=\cos\alpha_,(1+a_,\sin\alpha_,^2-b_,\cos\beta_,^2-c_,\cos\gamma_,^2),\\ \cos\beta=\cos\beta_,(1+b_,\sin\beta_,^2-c_,\cos\gamma_,^2-a_,\cos\alpha_,^2),\\ \cos\gamma=\cos\gamma_,(1+c_,\sin\gamma_,^2-a_,\cos\alpha_,^2-b_,\cos\beta_,^2).\end{aligned}\right\}\tag{21.}$$

On the same hypothesis, we may express the rectangular coordinates $r_{,}\cos\alpha_{,}$, $r_{,}\cos\beta_{,}$, $r_{,}\cos\gamma_{,}$ of the points of the cubical arrangement as follows:

$$r_{,}\cos\alpha_{,} = i_{,}h, \quad r_{,}\cos\beta_{,} = k_{,}h, \quad r_{,}\cos\gamma_{,} = l_{,}h, \tag{22.}$$

h denoting the *mean interval* of two adjacent points on the length of the side of any one of the elementary cubical spaces, and $i_{,}$, $k_{,}$, $l_{,}$ denoting any integer numbers positive or negative or null, which are however to be supposed subject to the conditions

$$i_{,}^2 + k_{,}^2 + l_{,}^2 > 0, \tag{23.}$$

in order not to include in the summation of the attractive or repulsive actions of the several points of the system on any one particular point the action of that point itself. And when, by substitution of the values (20.) and (21.), we have changed the 6 expressions (13.) & (14.) for the quantities L, M, N, P, Q, R to other expressions of the form

$$S.\,F(r_{,}\cos\alpha_{,},\ r_{,}\cos\beta_{,},\ r_{,}\cos\gamma_{,},\ r_{,}) \tag{24.}$$

as it is possible in every case to do, we may then transform these latter expressions to others of the form

$$\Sigma_{(i_{,},\,k_{,},\,l_{,})\,-\infty,\,-\infty,\,-\infty}^{\ \ \ \ \ \ \ \ \ \ \infty;\ \ \infty;\ \ \infty} F\left(i_{,}h,\ k_{,}h,\ l_{,}h,\ \sqrt{i_{,}^2+k_{,}^2+l_{,}^2}\,h\right) - F(0,0,0,0), \tag{25.}$$

the sign of summation $\Sigma_{(i_{,},\,k_{,},\,l_{,})\,-\infty,\,-\infty,\,-\infty}^{\ \ \ \ \ \ \ \ \ \ \infty;\ \ \infty;\ \ \infty}$ indicating a summation with respect to each of the three independent integers $i_{,}$, $k_{,}$, $l_{,}$ from $-\infty$ to ∞, and the term $-F(0,0,0,0)$ being added in order to allow for the condition (23.). Besides, by the supposed smallness & equality of the mass constants m, & by the supposed uniformity of distribution of the attracting or repelling points & the supposed smallness of extent of their sensible influence on each other, the expression (25.) may be transformed without sensible error into the following triple sum

$$\Sigma_{(i_{,},\,k_{,},\,l_{,})\,-\infty,\,-\infty,\,-\infty}^{\ \ \ \ \ \ \ \ \ \ \infty;\ \ \infty;\ \ \infty} \Phi\left(h\sqrt{i_{,}^2+k_{,}^2+l_{,}^2}\right) - \Phi(0) = S.\,\Phi(r_{,}), \tag{26.}$$

if we put for abridgment

$$\Phi(r_{,}) = \int_{-1}^{+1}\left\{\frac{1}{4\pi}\int_0^{2\pi} F\left(r_{,}\kappa_{,},\ r_{,}\sqrt{1-\kappa_{,}^2}\cos\omega_{,},\ r_{,}\sqrt{1-\kappa_{,}^2}\sin\omega_{,},\ r_{,}\right)d\omega_{,}\right\}d\kappa_{,}. \tag{27.}$$

It appears also to admit of proof that in virtue of the foregoing suppositions the triple sum (26.) may in general be very approximately expressed by the following single sum,

$$S.\,\Phi(r_{,}) = \Sigma_{(n)\,1}^{\infty}\,\Phi\left(h\sqrt[3]{\frac{3n}{4\pi}}\right): \tag{28.}$$

but the investigations of Poisson* respecting similar cases of molecular action seem to show that we must distinguish this last sum Σ from the definite integral $\int_0^\infty \frac{4\pi}{h^3}\Phi(r_{,})\,r_{,}^2 dr_{,}$, with which we might otherwise be tempted to confound it. Perhaps even the reduction of the triple sum (26.) to the single sum (28.) may in some cases sensibly err. So that it might be necessary, for some laws of molecular action, to employ the less simple expression

$$\begin{aligned} &S.\,F(r_{,}\cos\alpha_{,},\ r_{,}\cos\beta_{,},\ r_{,}\cos\gamma_{,},\ r_{,}) \\ &\qquad = \Sigma_{(i_{,},\,k_{,},\,l_{,})\,-\infty,\,-\infty,\,-\infty}^{\ \ \ \ \ \ \ \ \ \ \infty;\ \ \infty;\ \ \infty} \Phi\left(h\sqrt{i_{,}^2+k_{,}^2+l_{,}^2}\right) - \Phi(0) = S.\,\Phi(r_{,}) \end{aligned} \tag{29.}$$

and not to admit the transformation

$$S.\,F(r_{,}\cos\alpha_{,},\ r_{,}\cos\beta_{,},\ r_{,}\cos\gamma_{,},\ r_{,}) = \Sigma_{(n)\,1}^{\infty}\,\Phi\left(h\sqrt[3]{\frac{3n}{4\pi}}\right): \tag{30.}$$

---

* [Poisson, *Nouveaux Mémoires de l'Académie des Sciences*, Tome VI.]

but it seems probable that such cases are rare, & that the simpler expression (30.) will usually be very exact.

9. The expressions $r^{-2}f'(r) - r^{-3}f(r)$ and $r^{-1}f(r)$ become, by the substitution (20.),

$$r_{,}^{-2}f'(r_{,}) - r_{,}^{-3}f(r_{,}) + \{r_{,}^{-1}f''(r_{,}) - 3r_{,}^{-2}f'(r_{,}) + 3r_{,}^{-3}f(r_{,})\}(a_{,}\cos\alpha_{,}^{2} + b_{,}\cos\beta_{,}^{2} + c_{,}\cos\gamma_{,}^{2}) \qquad (31.)$$

and

$$r_{,}^{-1}f(r_{,}) + \{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}(a_{,}\cos\alpha_{,}^{2} + b_{,}\cos\beta_{,}^{2} + c_{,}\cos\gamma_{,}^{2}); \qquad (32.)$$

$f''(r_{,})$ denoting the second differential coefficient of the function $f(r_{,})$. We have also, by (17.),

$$\left.\begin{aligned} r^2\cos\alpha^2 &= r_{,}^2\cos\alpha_{,}^2(1+2a_{,}),\\ r^2\cos\beta^2 &= r_{,}^2\cos\beta_{,}^2(1+2b_{,}),\\ r^2\cos\gamma^2 &= r_{,}^2\cos\gamma_{,}^2(1+2c_{,}),\end{aligned}\right\} \qquad (33.)$$

and

$$\left.\begin{aligned} r^2\cos\beta\cos\gamma &= r_{,}^2\cos\beta_{,}\cos\gamma_{,}(1+b_{,}+c_{,}),\\ r^2\cos\gamma\cos\alpha &= r_{,}^2\cos\gamma_{,}\cos\alpha_{,}(1+c_{,}+a_{,}),\\ r^2\cos\alpha\cos\beta &= r_{,}^2\cos\alpha_{,}\cos\beta_{,}(1+a_{,}+b_{,});\end{aligned}\right\} \qquad (34.)$$

and, by the same substitutions,

$$\left(\sin\frac{ur\cos\alpha + vr\cos\beta + wr\cos\gamma}{2}\right)^2 = \left(\sin\frac{ur_{,}\cos\alpha_{,} + vr_{,}\cos\beta_{,} + wr_{,}\cos\gamma_{,}}{2}\right)^2$$
$$+\left(\frac{ur_{,}a_{,}\cos\alpha_{,} + vr_{,}b_{,}\cos\beta_{,} + wr_{,}c_{,}\cos\gamma_{,}}{2}\right)\sin(ur_{,}\cos\alpha_{,} + vr_{,}\cos\beta_{,} + wr_{,}\cos\gamma_{,}). \qquad (35.)$$

If then we denote by $L_{,}$, $M_{,}$, $N_{,}$, $P_{,}$, $Q_{,}$, $R_{,}$ the values of L, M, N, P, Q, R in the cubical arrangement of the points, and by

$$\frac{\delta L}{\delta a_{,}},\quad \frac{\delta L}{\delta b_{,}},\quad \frac{\delta L}{\delta c_{,}},\quad \&\text{c.}$$

the coefficients of the first powers of the small dilatations or contractions $a_{,}$, $b_{,}$, $c_{,}$ in the general expressions of these six quantities L, &c., and if we put for abridgment

$$\iota_{,} = \frac{r_{,}}{2}(u\cos\alpha_{,} + v\cos\beta_{,} + w\cos\gamma_{,}), \qquad (36.)$$

we shall have these 6 equations

$$\left.\begin{aligned} L_{,} &= S.\,2m\{\cos\alpha_{,}^2 f'(r_{,}) + \sin\alpha_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ M_{,} &= S.\,2m\{\cos\beta_{,}^2 f'(r_{,}) + \sin\beta_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ N_{,} &= S.\,2m\{\cos\gamma_{,}^2 f'(r_{,}) + \sin\gamma_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\end{aligned}\right\} \qquad (37.)$$

$$\left.\begin{aligned} P_{,} &= S.\,2m\cos\beta_{,}\cos\gamma_{,}\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ Q_{,} &= S.\,2m\cos\gamma_{,}\cos\alpha_{,}\{\qquad\qquad\qquad\}\sin\iota_{,}^2,\\ R_{,} &= S.\,2m\cos\alpha_{,}\cos\beta_{,}\{\qquad\qquad\qquad\}\sin\iota_{,}^2,\end{aligned}\right\} \qquad (38.)$$

together with 18 others of which it is sufficient to write these 4,

$$\begin{aligned}\frac{\delta L}{\delta a_{,}} = {} & S.\,6m\cos\alpha_{,}^2\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2\\ & + S.\,2m\cos\alpha_{,}^4\{r_{,}f''(r_{,}) - 3f'(r_{,}) + 3r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2\\ & + S.\,mur_{,}\cos\alpha_{,}\{\cos\alpha_{,}^2 f'(r_{,}) + \sin\alpha_{,}^2 r_{,}^{-1}f(r_{,})\}\sin 2\iota_{,}; \end{aligned} \qquad (39.)$$

$$\frac{\delta L}{\delta b_{,}} = S.\,2m\cos\beta_{,}^{2}\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^{2}$$
$$+ S.\,2m\cos\alpha_{,}^{2}\cos\beta_{,}^{2}\{r_{,}f''(r_{,}) - 3f'(r_{,}) + 3r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^{2}$$
$$+ S.\,mvr_{,}\cos\beta_{,}\{\cos\alpha_{,}^{2}f'(r_{,}) + \sin\alpha_{,}^{2}r_{,}^{-1}f(r_{,})\}\sin 2\iota_{,}; \qquad (40.)$$

$$\frac{\delta P}{\delta a_{,}} = S.\,2m\cos\alpha_{,}^{2}\cos\beta_{,}\cos\gamma_{,}\{r_{,}f''(r_{,}) - 3f'(r_{,}) + 3r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^{2} \qquad (41.)$$
$$+ S.\,mur_{,}\cos\alpha_{,}\cos\beta_{,}\cos\gamma_{,}\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin 2\iota_{,};$$

$$\frac{\delta P}{\delta b_{,}} = P_{,} + S.\,2m\cos\beta_{,}^{3}\cos\gamma_{,}\{r_{,}f''(r_{,}) - 3f'(r_{,}) + 3r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^{2}$$
$$+ S.\,mvr_{,}\cos\beta_{,}^{2}\cos\gamma_{,}\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin 2\iota_{,}: \qquad (42.)$$

and the 6 quantities L, M, N, P, Q, R, in the formula of the 7th paragraph, will now be represented as follows,

$$L = L_{,} + a_{,}\frac{\delta L}{\delta a_{,}} \cdot - b_{,}\frac{\delta L}{\delta b_{,}} + c_{,}\frac{\delta L}{\delta c_{,}}, \ \&\text{c.} \qquad (43.)$$

10. To attain a still greater simplicity, we shall suppose further that *the extent of sensible action of one attracting or repelling point upon another is small, yet not successively small, with respect to the thickness of a wave*, that is, with respect to the interval in space between two successive planes on which the attracting or repelling points of the system are similarly displaced from their positions of equilibrium, at any one moment of time. This interval or thickness λ is easily seen to be expressible by the formula

$$\lambda = \frac{2\pi}{\;} \qquad (44.)$$

if we put for abridgment

$$k = \sqrt{u^2 + v^2 + w^2}, \qquad (45.)$$

employing the symbols u, v, w to denote the same three constants of a wave as in the equations (10.). We shall, therefore, *suppose that the product*

$$kr = 2\pi\,.\,\frac{r}{\lambda}, \quad \text{or} \quad kr_{,} = 2\pi\,.\,\frac{r_{,}}{\lambda}, \qquad (46.)$$

is small, for all those distances r, *or* r *for which the function* $f(r)$, *or* $f(r_{,})$, *or even any one of its differential coefficients, has any sensible value; yet not so very small but that its square and perhaps higher powers may be sensible.* And then, if we put

$$u = ka, \quad v = kb, \quad w = kc, \qquad (47.)$$

(so that a, b, c shall denote the cosines of the angles which one of the two outer seminormals to the surfaces of the wave makes with the three positive semiaxes of coordinates,) we shall be able to develope the square of the sine of the trinomial

$$\iota_{,} = \tfrac{1}{2}(ur_{,}\cos\alpha_{,} + vr_{,}\cos\beta_{,} + wr_{,}\cos\gamma_{,})$$
$$= \tfrac{1}{2}kr_{,}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,}), \qquad (48.)$$

according to the ascending powers of this small product $kr_{,}$, in a converging series of which the three first terms are

$$\sin\iota_{,}^{2} = \frac{k^2r_{,}^2}{4}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,})^2$$
$$- \frac{k^4r_{,}^4}{48}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,})^4 \qquad (49.)$$
$$+ \frac{k^6r_{,}^6}{1440}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,})^6:$$

and in like manner we may develope $\sin 2\iota_{,}$ in a series of which the first terms are

$$\sin 2\iota_{,} = kr_{,}\,(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,}) - \frac{k^3 r_{,}^3}{6}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,})^3. \qquad (50.)$$

11. We shall now neglect the terms which are of the fourth or higher dimensions with respect to $kr_{,}$ in the 6 expressions $L_{,}, M_{,}, N_{,}, P_{,}, Q_{,}, R_{,}$; and also the terms which are of the third or higher dimensions with respect to the same small quantity $kr_{,}$ in the 6 small remaining parts

$$a_{,}\frac{\delta L}{\delta a_{,}} + b_{,}\frac{\delta L}{\delta b_{,}} + c_{,}\frac{\delta L}{\delta c_{,}}, \text{ \&c.},$$

of the 6 expressions (43.). And thus we shall have

$$\left.\begin{aligned} \bar{L} &= \{(L+G)\,a^2 + (R+H)\,b^2 + (Q+I)\,c^2\}\,k^2, \\ \bar{M} &= \{(M+H)\,b^2 + (P+I)\,c^2 + (R+G)\,a^2\}\,k^2, \\ \bar{N} &= \{(N+I)\,c^2 + (Q+G)\,a^2 + (P+H)\,b^2\}\,k^2, \end{aligned}\right\} \qquad (51.)$$

and

$$\bar{P} = 2Pbck^2, \quad \bar{Q} = 2Qcak^2, \quad \bar{R} = 2Rabk^2, \qquad (52.)$$

if we put for abridgment

$$G = G_{,} + a_{,}\frac{\delta G}{\delta a_{,}} + b_{,}\frac{\delta G}{\delta b_{,}} + c_{,}\frac{\delta G}{\delta c_{,}}, \qquad (53.)$$

with 8 other equations for H, I, L, M, N, P, Q, R, and if we also put

$$\begin{aligned} G_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\alpha_{,}^2, \\ H_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\beta_{,}^2, \\ I_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\gamma_{,}^2, \end{aligned} \qquad (54.)$$

$$\begin{aligned} L_{,} &= S.\frac{mr_{,}}{2}\{r_{,}f'(r_{,}) - f(r_{,})\}\cos\alpha_{,}^4, \\ M_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\qquad\}\cos\beta_{,}^4, \\ N_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\qquad\}\cos\gamma_{,}^4, \end{aligned} \qquad (55.)$$

$$\begin{aligned} P_{,} &= S.\frac{mr_{,}}{2}\{r_{,}f'(r_{,}) - f(r_{,})\}\cos\beta_{,}^2\cos\gamma_{,}^2, \\ Q_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\qquad\}\cos\gamma_{,}^2\cos\alpha_{,}^2, \\ R_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\qquad\}\cos\alpha_{,}^2\cos\beta_{,}^2, \end{aligned} \qquad (56.)$$

$$\left.\begin{aligned}
&\frac{\delta G}{\delta a_,}=2G_,+L_,\,, && \frac{\delta G}{\delta b_,}=R_,\,, && \frac{\delta G}{\delta c_,}=Q_,\,,\\
&\frac{\delta H}{\delta b_,}=2H_,+M_,\,, && \frac{\delta H}{\delta c_,}=P_,\,, && \frac{\delta H}{\delta a_,}=R_,\,,\\
&\frac{\delta I}{\delta c_,}=2I_,+N_,\,, && \frac{\delta I}{\delta a_,}=Q_,\,, && \frac{\delta I}{\delta b_,}=P_,\,,
\end{aligned}\right\} \tag{57.}$$[*]

$$\left.\begin{aligned}
&\frac{\delta L}{\delta a_,}=4L_,+S.\phi(r_,)\cos\alpha_,^6, && \frac{\delta L}{\delta b_,}=S.\phi(r_,)\cos\alpha_,^4\cos\beta_,^2, && \frac{\delta L}{\delta c_,}=S.\phi(r_,)\cos\alpha_,^4\cos\gamma_,^2,\\
&\frac{\delta M}{\delta b}=4M_,+S.\phi(r_,)\cos\beta_,^6, && \frac{\delta M}{\delta c_,}=S.\phi(r_,)\cos\beta_,^4\cos\gamma_,^2, && \frac{\delta M}{\delta a_,}=S.\phi(r_,)\cos\beta_,^4\cos\alpha_,^2,\\
&\frac{\delta N}{\delta c_,}=4N_,+S.\phi(r_,)\cos\gamma_,^6, && \frac{\delta N}{\delta a_,}=S.\phi(r_,)\cos\gamma_,^4\cos\alpha_,^2, && \frac{\delta N}{\delta b_,}=S.\phi(r_,)\cos\gamma_,^4\cos\beta_,^2,
\end{aligned}\right\} \tag{58.}$$

and finally

$$\left.\begin{aligned}
&\frac{\delta P}{\delta a_,}=\frac{\delta Q}{\delta b_,}=\frac{\delta R}{\delta c_,}=S.\phi(r_,)\cos\alpha_,^2\cos\beta_,^2\cos\gamma_,^2,\\
&\frac{\delta P}{\delta b_,}=2P_,+\frac{\delta M}{\delta c_,}, \quad \frac{\delta Q}{\delta c_,}=2Q_,+\frac{\delta N}{\delta a_,}, \quad \frac{\delta R}{\delta a_,}=2R_,+\frac{\delta L}{\delta b_,},\\
&\frac{\delta P}{\delta c_,}=2P_,+\frac{\delta N}{\delta b_,}, \quad \frac{\delta Q}{\delta a_,}=2Q_,+\frac{\delta L}{\delta c_,}, \quad \frac{\delta R}{\delta b_,}=2R_,+\frac{\delta M}{\delta a},
\end{aligned}\right\} \tag{59.}$$

in which

$$\phi(r_,)=\frac{m}{2}r_,^5\frac{d}{dr_,}\left(\frac{1}{r_,}\frac{d}{dr_,}\frac{f(r_,)}{r_,}\right)=\frac{m}{2}\{r_,^3f''(r_,)-3r_,^2f'(r_,)+3r_,f(r_,)\}. \tag{60.}$$

[*Wave velocities and directions of vibrations.*]

12. Again, by the 8th paragraph, we may transform every sum of the form

$$S.\psi(r_,)\cos\alpha_,^{2i_{,,}}\cos\beta_,^{2k_{,,}}\cos\gamma_,^{2l_{,,}} \tag{61.}$$

into an equivalent sum of the form

$$(i_{,,},k_{,,},l_{,,})\,S\psi(r_,)=S.\psi(r_,)\cos\alpha_,^{2i_{,,}}\cos\beta_,^{2k_{,,}}\cos\gamma_,^{2l_{,,}}, \tag{62.}$$

in which $(i_{,,}, k_{,,}, l_{,,})$ is a fraction depending on the integer exponents $i_{,,}, k_{,,}, l_{,,}$ by the following rule

$$(i_{,,},k_{,,},l_{,,})=\frac{1}{\pi}\int_0^\pi\cos\omega_,^{2k_{,,}}\sin\omega_,^{2l_{,,}}d\omega_,\,.\int_0^1\kappa_,^{2i_{,,}}(1-\kappa_,^2)^{k_{,,}+l_{,,}}d\kappa_,\,. \tag{63.}$$

It is easily found to be a consequence of this rule that we may change the order of the 3 exponents $i_{,,}, k_{,,}, l_{,,}$ in any manner, without altering the value of the fraction $(i_{,,}, k_{,,}, l_{,,})$;[†] and therefore that it is sufficient to calculate the values of these 6 different combinations,

$$(1,0,0),\quad(2,0,0),\quad(1,1,0),\quad(3,0,0),\quad(2,1,0),\quad(1,1,1), \tag{64.}$$

---

[*] [Equations (57.)–(60.) follow easily from results of the type

$$\frac{\partial}{\partial a_,}(r\cos\alpha)=r_,\cos\alpha_,,\quad \frac{\partial r}{\partial a_,}=r_,\cos^2\alpha_,,\quad \frac{\partial}{\partial b_,}(r\cos\beta)=0,\ \text{etc.},$$

which are a direct consequence of (17.) and (19.).]

[†] [$(i_{,,}, k_{,,}, l_{,,})=\Gamma(i_{,,}+\frac{1}{2})\,\Gamma(k_{,,}+\frac{1}{2})\,\Gamma(l_{,,}+\frac{1}{2})/\{2\pi\Gamma(i_{,,}+k_{,,}+l_{,,}+\frac{3}{2})\}$.]

in order to be able to apply the transformation (62.) to all those expressions of the last paragraph which are of the form (61.). The 6 values (64.) are easily found to be:

$$\begin{array}{lll} (1,0,0)=\tfrac{1}{3}; & (2,0,0)=\tfrac{1}{5}; & (1,1,0)=\tfrac{1}{15}; \\ (3,0,0)=\tfrac{1}{7}; & (2,1,0)=\tfrac{1}{35}; & (1,1,1)=\tfrac{1}{105}. \end{array} \qquad (65.)$$

If then we put for abridgment

$$R_0 = S.\frac{m}{6} r_{,} f(r_{,}),$$

$$R_1 = S.\frac{m}{30}\{r_{,}^2 f'(r_{,}) - r_{,} f(r_{,})\}, \qquad (66.)$$

$$R_2 = S.\frac{m}{210}\{r_{,}^3 f''(r_{,}) - 3r_{,}^2 f'(r_{,}) + 3r_{,} f(r_{,})\},$$

we shall have the following results:

$$G_{,} = H_{,} = I_{,} = R_0, \qquad (67.)$$

$$L_{,} = M_{,} = N_{,} = 3R_1, \qquad (68.)$$

$$P_{,} = Q_{,} = R_{,} = R_1, \qquad (69.)$$

$$\frac{\delta G}{\delta a_{,}} = \frac{\delta H}{\delta b_{,}} = \frac{\delta I}{\delta c_{,}} = 2R_0 + 3R_1, \qquad (70.)$$

$$\frac{\delta G}{\delta b_{,}} = \frac{\delta H}{\delta c_{,}} = \frac{\delta I}{\delta a_{,}} = R_1, \qquad (71.)$$

$$\frac{\delta G}{\delta c_{,}} = \frac{\delta H}{\delta a_{,}} = \frac{\delta I}{\delta b_{,}} = R_1, \qquad (72.)$$

$$\frac{\delta L}{\delta a_{,}} = \frac{\delta M}{\delta b_{,}} = \frac{\delta N}{\delta c_{,}} = 12R_1 + 15R_2, \qquad (73.)$$

$$\frac{\delta L}{\delta b_{,}} = \frac{\delta M}{\delta c_{,}} = \frac{\delta N}{\delta a_{,}} = 3R_2, \qquad (74.)$$

$$\frac{\delta L}{\delta c_{,}} = \frac{\delta M}{\delta a_{,}} = \frac{\delta N}{\delta b_{,}} = 3R_2, \qquad (75.)$$

$$\frac{\delta P}{\delta a_{,}} = \frac{\delta Q}{\delta b_{,}} = \frac{\delta R}{\delta c_{,}} = R_2, \qquad (76.)$$

$$\frac{\delta P}{\delta b_{,}} = \frac{\delta Q}{\delta c_{,}} = \frac{\delta R}{\delta a_{,}} = 2R_1 + 3R_2, \qquad (77.)$$

$$\frac{\delta P}{\delta c_{,}} = \frac{\delta Q}{\delta a_{,}} = \frac{\delta R}{\delta b_{,}} = 2R_1 + 3R_2. \qquad (78.)$$

And substituting these values, in the expressions of the form (53.), we find

$$\left.\begin{aligned} G &= R_0(1 + 2a_{,}) + R_1(3a_{,} + b_{,} + c_{,}), \\ H &= R_0(1 + 2b_{,}) + R_1(3b_{,} + c_{,} + a_{,}), \\ I &= R_0(1 + 2c_{,}) + R_1(3c_{,} + a_{,} + b_{,}), \end{aligned}\right\} \qquad (79.)$$

$$\left.\begin{aligned} L &= 3R_1(1+4a,)+3R_2(5a,+b,+c,),\\ M &= 3R_1(1+4b,)+3R_2(5b,+c,+a,),\\ N &= 3R_1(1+4c,)+3R_2(5c,+a,+b,),\end{aligned}\right\}\quad (80.)$$

$$\left.\begin{aligned} P &= R_1(1+2b,+2c,)+R_2(a,+3b,+3c,),\\ Q &= R_1(1+2c,+2a,)+R_2(b,+3c,+3a,),\\ R &= R_1(1+2a,+2b,)+R_2(c,+3a,+3b,);\end{aligned}\right\}\quad (81.)$$

and therefore

$$\left.\begin{aligned} L+G &= R_0(1+2a,)+R_1(3+15a,+b,+c,)+3R_2(5a,+b,+c,),\\ M+H &= R_0(1+2b,)+R_1(3+15b,+c,+a,)+3R_2(5b,+c,+a,),\\ N+I &= R_0(1+2c,)+R_1(3+15c,+a,+b,)+3R_2(5c,+a,+b,),\end{aligned}\right\}\quad (82.)$$

$$\left.\begin{aligned} R+H &= R_0(1+2b,)+R_1(1+3a,+5b,+c,)+R_2(3a,+3b,+c,),\\ P+I &= R_0(1+2c,)+R_1(1+3b,+5c,+a,)+R_2(3b,+3c,+a,),\\ Q+G &= R_0(1+2a,)+R_1(1+3c,+5a,+b,)+R_2(3c,+3a,+b,),\end{aligned}\right\}\quad (83.)$$

$$\left.\begin{aligned} Q+I &= R_0(1+2c,)+R_1(1+3a,+b,+5c,)+R_2(3a,+b,+3c,),\\ R+G &= R_0(1+2a,)+R_1(1+3b,+c,+5a,)+R_2(3b,+c,+3a,),\\ P+H &= R_0(1+2b,)+R_1(1+3c,+a,+5b,)+R_2(3c,+a,+3b,).\end{aligned}\right\}\quad (84.)$$

Hence, by (51.) and (52.),

$$\begin{aligned} k^{-2}\overline{L} = R_0+R_1+2R_1a^2+2R_0(a,a^2+b,b^2+c,c^2)+R_1(3a,+b,+c,)+3R_2(a,+b,+c,)&\\ +4R_1(3a,a^2+b,b^2+c,c^2)+2R_2(6a,a^2-c,b^2-b,c^2);&\quad (85.)\end{aligned}$$

$$\begin{aligned} k^{-2}\overline{M} = R_0+R_1+2R_1b^2+2R_0(b,b^2+c,c^2+a,a^2)+R_1(3b,+c,+a,)+3R_2(a,+b,+c,)&\\ +4R_1(3b,b^2+c,c^2+a,a^2)+2R_2(6b,b^2-a,c^2-c,a^2);&\quad (86.)\end{aligned}$$

$$\begin{aligned} k^{-2}\overline{N} = R_0+R_1+2R_1c^2+2R_0\,(c,c^2+a,a^2+b,b^2)+R_1(3c,+a,+b,)+3R_2(a,+b,+c,)&\\ +4R_1(3c,c^2+a,a^2+b,b^2)+2R_2(6c,c^2-b,a^2-a,b^2);&\quad (87.)\end{aligned}$$

$$\left.\begin{aligned} k^{-2}\overline{P} &= 2R_1(1+2b,+2c,)\,bc+2R_2(a,+3b,+3c,)\,bc,\\ k^{-2}\overline{Q} &= 2R_1(1+2c,+2a,)\,ca+2R_2(b,+3c,+3a,)\,ca,\\ k^{-2}\overline{R} &= 2R_1(1+2a,+2b,)\,ab+2R_2(c,+3a,+3b,)\,ab.\end{aligned}\right\}\quad (88.)$$

And hence, finally, if we can put for abridgment

$$-s^2k^{-2}+R_0+R_1+(2R_0+4R_1)(a,a^2+b,b^2+c,c^2)+(R_1+3R_2)(a,+b,+c,)=2\sigma,\quad (89.)$$

the equations (16.) become

$$\left.\begin{aligned} &\{\sigma+R_1(a^2+a,+4a,a^2)+R_2(6a,a^2-c,b^2-b,c^2)\}\,\xi'\\ &\quad+\{R_1(1+2a,+2b,)+R_2(3a,+3b,+c,)\}\,ab\eta'\\ &\quad+\{R_1(1+2a,+2c,)+R_2(3a,+3c,+b,)\}\,ac\zeta'=0,\end{aligned}\right\}\quad (90.)$$

$$\left.\begin{aligned} &\{\sigma+R_1(b^2+b,+4b,b^2)+R_2(6b,b^2-a,c^2-c,a^2)\}\,\eta'\\ &\quad+\{R_1(1+2b,+2c,)+R_2(3b,+3c,+a,)\}\,bc\zeta'\\ &\quad+\{R_1(1+2b,+2a,)+R_2(3b,+3a,+c,)\}\,ba\xi'=0,\end{aligned}\right\}\quad (91.)$$

$$\left.\begin{aligned} &\{\sigma+R_1(c^2+c,+4c,c^2)+R_2(6c,c^2-b,a^2-a,b^2)\}\,\zeta'\\ &\quad+\{R_1(1+2c,+2a,)+R_2(3c,+3a,+b,)\}\,ca\xi'\\ &\quad+\{R_1(1+2c,+2b,)+R_2(3c,+3b,+a,)\}\,cb\eta'=0.\end{aligned}\right\}\quad (92.)$$

It is to be remembered that in these 4 last equations the quantities R_0, R_1, R_2 and $a_{,}, b_{,}, c_{,}$ are constant for any one medium depending only on the distribution of the attracting or repelling points of the system in their positions of equilibrium and on their law of action; $\xi^{\backprime}, \eta^{\backprime}, \zeta^{\backprime}$ are also constants for any one succession of plane waves and are proportional to the three coordinate displacements of any one vibrating point from its position of equilibrium; a, b, c are the cosines of the angles of direction of a normal to such a wave; and s^2k^{-2}, with which the auxiliary quantity σ is connected by the equation (89.), is the square of the velocity wherewith that wave is propagated.

13. In the particular case when

$$b_{,} = a_{,}, \tag{93.}$$

that is, when the system of points is supposed to be, in its position of equilibrium, dilated equally or contracted equally in the 2 rectangular directions of x and y from the standard cubical arrangement, the three equations (90.), (91.), (92.) become

$$\begin{aligned}
&\{\sigma + R_1(a^2 + a_{,} + 4a_{,}a^2) + R_2(6a_{,}a^2 - a_{,}c^2 - c_{,}b^2)\}\xi^{\backprime}\\
&\quad + \{R_1(1 + 4a_{,}) + R_2(6a_{,} + c_{,})\}ab\eta^{\backprime}\\
&\quad + \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}ac\zeta^{\backprime} = 0;
\end{aligned} \tag{94.}$$

$$\begin{aligned}
&\{\sigma + R_1(b^2 + a_{,} + 4a_{,}b^2) + R_2(6a_{,}b^2 - a_{,}c^2 - c_{,}a^2)\}\eta^{\backprime}\\
&\quad + \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}bc\zeta^{\backprime}\\
&\quad + \{R_1(1 + 4a_{,}) + R_2(6a_{,} + c_{,})\}ba\xi^{\backprime} = 0;
\end{aligned} \tag{95.}$$

$$\begin{aligned}
&\{\sigma + R_1(c^2 + c_{,} + 4c_{,}c^2) + R_2(6c_{,}c^2 - a_{,}a^2 - a_{,}b^2)\}\zeta^{\backprime}\\
&\quad + \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}ca\xi^{\backprime}\\
&\quad + \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}cb\eta^{\backprime} = 0.
\end{aligned} \tag{96.}$$

And if we further suppose that these two equal dilatations or contractions in the directions of x & y to vanish, so that

$$a_{,} = 0, \quad b_{,} = 0, \tag{97.}$$

(which comes to taking for the standard cubical arrangement $x_{,}, y_{,}, z_{,}$ that which differs from the position of equilibrium x, y, z only by a dilatation or contraction in the direction of the axis of z,) we shall have, more simply, the 3 equations following:

$$\left.\begin{aligned}
&(\sigma + R_1a^2 - R_2c_{,}b^2)\xi^{\backprime} + (R_1 + R_2c_{,})ab\eta^{\backprime} + (R_1 + 2R_1c_{,} + 3R_2c_{,})ac\zeta^{\backprime} = 0,\\
&(\sigma + R_1b^2 - R_2c_{,}a^2)\eta^{\backprime} + (R_1 + R_2c_{,})ab\xi^{\backprime} + (R_1 + 2R_1c_{,} + 3R_2c_{,})bc\zeta^{\backprime} = 0,\\
&\{\sigma + R_1c^2 + R_1c_{,}(1 + 4c^2) + 6R_2c_{,}c^2\}\zeta^{\backprime} + (R_1 + 2R_1c_{,} + 3R_2c_{,})c(a\xi^{\backprime} + b\eta^{\backprime}) = 0;
\end{aligned}\right\} \tag{98.}$$

which give, by elimination of the ratios of $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$, the following cubic equation

$$\begin{aligned}
0 &= \sigma^3 + \sigma^2R_1 + \sigma^2c_{,}\{R_1 - R_2 + (4R_1 + 7R_2)c^2\}\\
&\quad + \sigma c_{,}(1 - c^2)R_1(R_1 - R_2) - \sigma c_{,}^2(1 - c^2)\{4R_1^2c^2 + R_1R_2(1 + 16c^2) + 15R_2^2c^2\}\\
&\quad - c_{,}^2(1 - c^2)^2R_1^2R_2 + c_{,}^3(1 - c^2)^2c^2(2R_1 + 3R_2)^2R_2\\
&= \{\sigma - c_{,}(1 - c^2)R_2\}\{\sigma + c_{,}(1 - c^2)R_1\}\{\sigma + R_1 + c_{,}c^2(5R_1 + 6R_2)\}\\
&\quad - 9\{\sigma - c_{,}(1 - c^2)R_2\}c_{,}^2(1 - c^2)c^2(R_1 + R_2)^2:
\end{aligned} \tag{99.}$$

and the three roots of this cubic equation will give three corresponding values of the square s^2k^{-2} of the velocity of propagation of a plane wave by the formula

$$s^2k^{-2} = R_0 + R_1 + c_{,}(R_1 + 3R_2) + 2c_{,}c^2(R_0 + 2R_1) - 2\sigma, \tag{100.}$$

to which the formula (89.) reduces itself by the supposition (97.). With respect to the three directions of displacement of a vibrating point corresponding to these three velocities of propagation of a wave, they may be deduced from the equations (98.) by determining, for each value of the velocity or of the auxiliary quantity σ, the ratios of $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$.

14. One root of the cubic equation (99.) is evidently

$$\sigma_1 = c_{,}(1 - c^2)R_2; \tag{101.}$$

which is easily seen to correspond, by the equations (98.), to the following relations between $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$,

$$a\xi^{\backprime}_1 + b\eta^{\backprime}_1 = 0, \quad \zeta^{\backprime}_1 = 0, \tag{102.}$$

that is, to a direction of displacement or vibration which is contained upon the surface of the wave & is perpendicular to the axis of z, that is, to the direction of dilatation or contraction of the system of points when in equilibrium. And the square of the velocity of a wave when propagated with such vibrations is, by (100.) and (101.),

$$s_1^2k^{-2} = R_0 + R_1 + c_{,}(R_1 + R_2) + 2c_{,}(R_0 + 2R_1 + R_2)c^2; \tag{103.}$$

so that it consists of a constant part, namely,

$$R_0 + R_1 + c_{,}(R_1 + R_2), \tag{104.}$$

which is independent of the direction of the wave, and a variable part, namely,

$$2c_{,}(R_0 + 2R_1 + R_2)c^2, \tag{105.}$$

which is proportional to the square c^2 of the cosine of the angle between the normal to the surface of the wave and the direction of dilatation or contraction.

Another root of the same cubic (99.) is, very nearly,

$$\sigma_2 = -c_{,}(1 - c^2)R_1, \tag{106.}$$

the error being of the order of the square of the dilatation or contraction, which we have already supposed that we may neglect. This root corresponds to the following value of the square of the velocity of the wave,

$$s_2^2k^{-2} = R_0 + R_1 + 3c_{,}(R_1 + R_2) + 2c_{,}(R_0 + R_1)c^2; \tag{107.}$$

which consists like the former value (103.) of a constant part, in this case

$$R_0 + R_1 + 3c_{,}(R_1 + R_2), \tag{108.}$$

and a part proportional to the square of the cosine of the angle between the direction of dilatation or contraction & the normal to the surface of the wave, namely the part

$$2c_{,}(R_0 + R_1)c^2. \tag{109.}$$

The vibrations, for this velocity of propagation of the wave, have their directions determined very nearly by the following equations, deduced from (98.) and (106.),

$$\frac{\xi^{\backprime}_2}{a} = \frac{\eta^{\backprime}_2}{b} = \frac{\zeta^{\backprime}_2}{c - \frac{1}{c}}\left\{1 + \frac{3c_{,}}{R_1}(R_1 + R_2)\right. \tag{110.}$$

They are therefore very nearly tangential to the surface of the wave, & are perpendicular to the vibrations which correspond to the former velocity.

Finally, the remaining root of the cubic equation (99.) is very nearly

$$\sigma_3 = -R_1 - c_{,}c^2(5R_1+6R_2); \qquad (111.)$$

it corresponds to vibrations which are perpendicular to both the two former directions of vibration and very nearly normal to the wave, being determined very approximately by the equations

$$\frac{\xi_3^{\backprime}}{a} = \frac{\eta_3^{\backprime}}{b} = \frac{\zeta_3^{\backprime}}{c}\left\{1 - \frac{3c_{,}}{R_1}(R_1+R_2)\right\}; \qquad (112.)$$

and for these last vibrations the square of the velocity of propagation of the wave is very nearly

$$s_3^2k^{-2} = R_0 + 3R_1 + c_{,}(R_1+3R_2) + 2c_{,}(R_0+7R_1+6R_2)c^2, \qquad (113.)$$

so that it is composed of a constant and a variable part like the squares of the two former velocities. With respect to the small angular deviation of the third set of vibrations from the normal, or of the second set of vibrations from the surface of the wave, it varies very nearly as the sine of twice the angle between the normal and the direction of dilatation or of contraction.

15. Results of the same kind would have followed if we had made in (89.), (90.), (91.) and (92.)

$$b_{,} = a_{,}, \quad c_{,} = 0; \qquad (114.)$$

that is, if we had supposed the arrangement of the attracting or repelling points of the system, when in equilibrium, to differ from the standard cubical arrangement by two equal dilatations or equal contractions in the two rectangular directions of x and y. We should then have had this formula for the square of the velocity of the wave,

$$s^2k^{-2} = R_0 + R_1 + 2a_{,}(R_1+3R_2) + 2a_{,}(R_0+2R_1)(1-c^2) - 2\sigma; \qquad (115.)$$

the auxiliary quantity σ, and the ratios of $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$, which determine the direction of vibration, being found from the following equations:

$$\left.\begin{aligned}
0 &= \{\sigma + R_1a^2 + a_{,}R_1(1+4a^2) + a_{,}R_2(6a^2-c^2)\}\xi^{\backprime} \\
&\quad + \{R_1 + 2a_{,}(2R_1+3R_2)\}ab\eta^{\backprime} + \{R_1+2a_{,}(R_1+2R_2)\}ac\zeta^{\backprime}, \\
0 &= \{\sigma + R_1b^2 + a_{,}R_1(1+4b^2) + a_{,}R_2(6b^2-c^2)\}\eta^{\backprime} \\
&\quad + \{R_1 + 2a_{,}(2R_1+3R_2)\}ab\xi^{\backprime} + \{R_1+2a_{,}(R_1+2R_2)\}bc\zeta^{\backprime}, \\
0 &= \{\sigma + R_1c^2 - a_{,}R_2(1-c^2)\}\zeta^{\backprime} + \{R_1+2a_{,}(R_1+2R_2)\}c(a\xi^{\backprime}+b\eta^{\backprime}),
\end{aligned}\right\} \qquad (116.)$$

which give, by elimination,

$$\begin{aligned}
0 = \{\sigma + a_{,}(R_1-R_2c^2)\}\{\sigma+a_{,}(R_1c^2-R_2)\}\{\sigma+R_1+a_{,}(1-c^2)(5R_1+6R_2)\} \\
-9\{\sigma+a_{,}(R_1-R_2c^2)\}a_{,}^2(1-c^2)c^2(R_1+R_2)^2. \qquad (117.)
\end{aligned}$$

One root of this cubic is

$$\sigma_1 = -a_{,}(R_1-R_2c^2) \qquad (118.)$$

& corresponds to the following expressions for the velocity & the vibrations,

$$\left.\begin{aligned}
s_1^2k^{-2} = R_0 + R_1 + 2a_{,}(R_0+4R_1+3R_2) - 2a_{,}(R_0+2R_1+R_2)c^2, \\
a\xi_1^{\backprime} + b\eta_1^{\backprime} = 0, \quad \zeta_1^{\backprime} = 0;
\end{aligned}\right. \qquad (119.)$$

another root is, nearly,

$$\sigma_2 = -a_{,}(R_1c^2-R_2) \qquad (120.)$$

& corresponds to the expressions

$$\begin{aligned}
s_2^2k^{-2} = R_0 + R_1 + 2a_{,}(R_0+3R_1+2R_2) - 2a_{,}(R_0+R_1)c^2, \\
\frac{\xi_2^{\backprime}}{a} = \frac{\eta_2^{\backprime}}{b} = \frac{\zeta_2^{\backprime}}{c-\frac{1}{}}\left\{1 - 3a_{,}\frac{R_1+R_2}{R_1}\right\}; \qquad (121.)
\end{aligned}$$

and the third root is, nearly,

$$\sigma_3 = -R_1 - a_,(1-c^2)(5R_1+6R_2), \tag{122.}$$

which gives the expressions

$$\left.\begin{aligned} s_3^2 k^{-2} &= R_0 + 3R_1 + 2a_,(R_0+8R_1+9R_2) - 2a_,(R_0+7R_1+6R_2)c^2, \\ \frac{\xi_3^{\backprime}}{a} &= \frac{\eta_3^{\backprime}}{b} = \frac{\zeta_3^{\backprime}}{c}\left\{1+3a_,\frac{R_1+R_2}{R_1}\right\}. \end{aligned}\right\} \tag{123.}$$

And all these results are evidently similar in kind to those obtained in the foregoing paragraph and lead to analogous conclusions.

16. In general, if we put for abridgment

$$\left.\begin{aligned} c\xi^{\backprime} - a\zeta^{\backprime} &= \xi^{\backprime\backprime}, \\ c\eta^{\backprime} - b\zeta^{\backprime} &= \eta^{\backprime\backprime}, \\ a\xi^{\backprime} + b\eta^{\backprime} + c\zeta^{\backprime} &= \zeta^{\backprime\backprime}, \end{aligned}\right\} \tag{124.}$$

we shall get the expressions

$$\left.\begin{aligned} \xi^{\backprime} &= \frac{b^2+c^2}{c}\xi^{\backprime\backprime} - \frac{ab}{c}\eta^{\backprime\backprime} + a\zeta^{\backprime\backprime}, \\ \eta^{\backprime} &= -\frac{ab}{c}\xi^{\backprime\backprime} + \frac{a^2+c^2}{c}\eta^{\backprime\backprime} + b\zeta^{\backprime\backprime}, \\ \zeta^{\backprime} &= -a\xi^{\backprime\backprime} - b\eta^{\backprime\backprime} + c\zeta^{\backprime\backprime}, \end{aligned}\right\} \tag{125.}$$

by means of which the system of the 3 equations (90.), (91.), (92.) may easily be transformed into the following:

$$\begin{aligned} 0 = \xi^{\backprime\backprime}\{\sigma + a_,R_1(b^2+c^2) - b_,R_2(a^2+c^2) + c_,(R_1a^2 - R_2b^2)\} \\ + a(c_, - a_,)(R_1+R_2)(b\eta^{\backprime\backprime} - 3c\zeta^{\backprime\backprime}); \end{aligned} \tag{126.}$$

$$\begin{aligned} 0 = \eta^{\backprime\backprime}\{\sigma + b_,R_1(a^2+c^2) - a_,R_2(b^2+c^2) + c_,(R_1b^2 - R_2a^2)\} \\ + b(c_, - b_,)(R_1+R_2)(a\xi^{\backprime\backprime} - 3c\zeta^{\backprime\backprime}); \end{aligned} \tag{127.}$$

$$\begin{aligned} 0 = c\zeta^{\backprime\backprime}\{\sigma + R_1 + (5R_1+6R_2)(a_,a^2 + b_,b^2 + c_,c^2)\} \\ + 3(R_1+R_2)a\xi^{\backprime\backprime}\{a_, - (a_,a^2+b_,b^2+c_,c^2)\} \\ + 3(R_1+R_2)b\eta^{\backprime\backprime}\{b_, - (a_,a^2+b_,b^2+c_,c^2)\}. \end{aligned} \tag{128.)*}$$

It is easy hence to perceive that either the two quantities σ and $\zeta^{\backprime\backprime}$, or else the three quantities $\sigma + R_1$, $\xi^{\backprime\backprime}$ and $\eta^{\backprime\backprime}$, must be considered as small of the order of the dilatations or contractions $a_,, b_,, c_,$; of which two cases, the first corresponds to nearly tangential & the second case to nearly normal vibrations. In the first case we may suppress the products of $\zeta^{\backprime\backprime}$, σ, $a_,$, $b_,$, $c_,$ and then the equations (126.), (127.), (128.) reduce themselves to the three following:

$$0 = \xi^{\backprime\backprime}\{\sigma + a_,R_1(b^2+c^2) - b_,R_2(a^2+c^2) + c_,(R_1a^2 - R_2b^2)\} + \eta^{\backprime\backprime}(c_, - a_,)(R_1+R_2)ab; \tag{129.}$$

$$0 = \eta^{\backprime\backprime}\{\sigma + b_,R_1(a^2+c^2) - a_,R_2(b^2+c^2) + c_,(R_1b^2 - R_2a^2)\} + \xi^{\backprime\backprime}(c_, - b_,)(R_1+R_2)ab; \tag{130.}$$

$$0 = c\zeta^{\backprime\backprime}R_1 + 3(R_1+R_2)\{aa_,\xi^{\backprime\backprime} + bb_,\eta^{\backprime\backprime} - (a\xi^{\backprime\backprime} + b\eta^{\backprime\backprime})(a_,a^2+b_,b^2+c_,c^2)\}: \tag{131.}$$

---

* [The terms in (90.), (91.), (92.) not involving $a_,, b_,, c_,$ relate to a system of revolution, the axis of revolution being the direction a, b, c. This is the reason for adopting the transformation (124.), which is the simplest of its kind. (126.), (127.), (128.) do not follow directly from (90.), (91.), (92.). A transformation of the type of (124.) is first employed, e.g. (90.) multiplied by c, (91.) multiplied by a, etc.]

and of these the third determines a plane of vibration coinciding nearly with the surface of the wave, while the two first determine in general two distinct directions of vibration upon that plane, to each of which corresponds a certain velocity of propagation determined by the same two equations through the help of the two values assigned by them for the auxiliary quantity σ. In the second case the equation (128.) reduces itself to the form

$$\sigma = -R_1 - (5R_1 + 6R_2)(a, a^2 + b, b^2 + c, c^2), \qquad (132.)$$

which determines this auxiliary quantity σ, and therefore also, by (89.), the velocity of propagation of the wave when the vibrations are nearly normal to its surface, having the direction determined by the equations (126.), (127.), which now reduce themselves to the following:

$$\left.\begin{aligned} \xi^{\backprime\backprime} &= 3ac\,\frac{R_1 + R_2}{R_1}(a, -c,)\,\zeta^{\backprime\backprime}, \\ \eta^{\backprime\backprime} &= 2bc\,\frac{R_1 + R_2}{R_1}(b, -c,)\,\zeta^{\backprime\backprime}. \end{aligned}\right\} \qquad (133.)$$

It is easy to prove that this line of nearly normal vibration is perpendicular to the plane (131.) of the two nearly tangential vibrations, of which the equation is, when referred to the rectangular coordinates $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$,

$$\begin{aligned} 0 = R_1(a\xi^{\backprime} + b\eta^{\backprime} + c\zeta^{\backprime}) + 3(R_1 + R_2)\{ab(a, -b,)(b\xi^{\backprime} - a\eta^{\backprime}) \\ + bc(b, -c,)(c\eta^{\backprime} - b\zeta^{\backprime}) + ca(c, -a,)(a\zeta^{\backprime} - c\xi^{\backprime})\}. \end{aligned} \qquad (134.)$$

And in this plane the two nearly tangential vibrations themselves, of which the directions are determined by the following quadratic equation, deduced from (129.) & (130.) by elimination of the auxiliary quantity σ,

$$0 = \eta^{\backprime\backprime 2}(c, -a,)\,ab - \xi^{\backprime\backprime 2}(c, -b,)\,ab + \xi^{\backprime\backprime}\eta^{\backprime\backprime}\{(c, -b,)(a^2 + c^2) - (c, -a,)(b^2 + c^2)\}, \qquad (135.)$$

are easily seen to be perpendicular to each other as well as to the direction (133.) of nearly normal vibration;* for if we consider them as represented separately by the 2 equations

$$\eta_1^{\backprime\backprime} = g_{,}\,\xi_1^{\backprime\backprime}, \quad \eta_2^{\backprime\backprime} = g_2\,\xi_2^{\backprime\backprime}, \qquad (136.)$$

in which, by (135.),

$$g_1 + g_2 = \frac{(c, -a,)(b^2 + c^2) - (c, -b,)(a^2 + c^2)}{(c, -a,)\,ab}, \quad g_1 g_2 = -\frac{c, -b.}{c. -a.} \qquad (137.)$$

then the two normal planes to the wave, containing the two directions (135.) or (136.) & making with each other an angle which in the present order of approximation must be considered as equal to the angle between those two directions themselves, have for their two equations in the rectangular coordinates $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$,

$$\left.\begin{aligned} c\eta_1^{\backprime} - b\zeta_1^{\backprime} &= g_1(c\xi_1^{\backprime} - a\zeta_1^{\backprime}), \\ c\eta_2^{\backprime} - b\zeta_2^{\backprime} &= g_2(c\eta_2^{\backprime} - a\zeta_2^{\backprime}), \end{aligned}\right\} \qquad (138.)$$

and are therefore perpendicular to each other, because, by (137.),

$$c^2(1 + g_1 g_2) + (b - ag_1)(b - ag_2) = 0. \qquad (139.)$$

With respect to the 2 velocities of propagation of the wave corresponding to these two rectangular & nearly tangential vibrations (135.) or (136.), they may be determined (as was above observed) by the formula (89.) through the help of the two values of the auxiliary quantity σ by these 2 linear equations

---

* [Three directions perpendicular to one another are given by (90.), (91.), (92.), if the cubic in σ has unequal roots.]

$$\left.\begin{aligned}\sigma_1 &= -a_, R_1 (b^2+c^2) + b_, R_2 (a^2+c^2) - c_, (R_1 a^2 - R_2 b^2) + g_1 (a_, - c_,)(R_1 + R_2) ab,\\ \sigma_2 &= -a_, R_1 (b^2+c^2) + b_, R_2 (a^2+c^2) - c_, (R_1 a^2 - R_2 b^2) + g_2 (a_, - c_,)(R_1 + R_2) ab;\end{aligned}\right\} \quad (140.)$$

or, jointly, by this one quadratic,

$$\begin{aligned}0 = \sigma^2 &+ \sigma (R_1 - R_2)\{a_, (b^2+c^2) + b_, (c^2+a^2) + c_, (a^2+b^2)\}\\ &+ (R_1^2 + R_2^2)(a_, b_, c^2 + b_, c_, a^2 + c_, a_, b^2)\\ &- R_1 R_2 \{a_,^2 (b^2+c^2)^2 + b_,^2 (c^2+a^2)^2 + c_,^2 (a^2+b^2)^2 + 2a_, b_, a^2 b^2 + 2b_, c_, b^2 c^2 + 2c_, a_, c^2 a^2\},\end{aligned} \quad (141.)$$

which may be put under the form

$$\begin{aligned}&[2\sigma + (R_1 - R_2)\{a_, (b^2+c^2) + b_, (c^2+a^2) + c_, (a^2+b^2)\}]^2\\ &\quad = (R_1 + R_2)^2 \{a_,^2 (b^2+c^2)^2 + b_,^2 (c^2+a^2)^2 + c_,^2 (a^2+b^2)^2 + 2a_, b_, (a^2 b^2 - c^2)\\ &\qquad + 2b_, c_, (b^2 c^2 - a^2) + 2c_, a_, (c^2 a^2 - b^2)\}\\ &\quad = (R_1 + R_2)^2 \{(c_, - a_,)^2 (b^2+c^2)^2 + (c_, - b_,)^2 (a^2+c^2)^2 + 2 (c_, - a_,)(c_, - b_,)(a^2 b^2 - c^2)\}\\ &\quad = (R_1 + R_2)^2 \{(a_, - b_,)^2 (c^2+a^2)^2 + (a_, - c_,)^2 (b^2+a^2)^2 + 2 (a_, - b_,)(a_, - c_,)(b^2 c^2 - a^2)\}.\end{aligned} \quad (142.)$$

17. When any two of the three quantities $a_,, b_,, c_,$ are equal to each other, the last expression (142.) becomes an exact square, & the quadratic equation (141.) resolves itself into two linear factors. For example, in the case (93.) when

$$b_, = a_,,$$

the roots of the quadratic (141.) may be thus separately and rationally expressed,

$$\left.\begin{aligned}\sigma_1 &= -a_, (R_1 - R_2) + (c_, - a_,) R_2 (1 - c^2),\\ \sigma_2 &= -a_, (R_1 - R_2) + (a_, - c_,) R_1 (1 - c^2),\end{aligned}\right\} \quad (143.)$$

and are connected by (140.) with the following separate & rational values of g_1, g_2,

$$g_1 = -\frac{\text{[illegible]}}{b}, \quad g_2 = \frac{\text{[illegible]}}{a}, \quad (144.)$$

which are to be substituted in the equations (136.), while the equation (131.) of the plane of nearly tangential vibrations becomes

$$0 = R_1 \zeta^{\backprime\backprime} + 3 (R_1 + R_2)(a_, - c_,) c (a\xi^{\backprime\backprime} + b\eta^{\backprime\backprime}), \quad (145.)$$

& the formula (89.) for the velocity of propagation of the wave becomes

$$s^2 k^{-2} = R_0 + R_1 + 2a_, (R_0 + 3R_1 + 3R_2) + c_, (R_1 + 3R_2) + 2 (c_, - a_,) c^2 (R_0 + 2R_1) - 2\sigma; \quad (146.)$$

so that in this case the two directions of nearly tangential vibration, with the two corresponding velocities of propagation of the wave, are expressed in the following manner,*

$$\left.\begin{aligned}&a\xi_1^{\backprime} + b\eta_1^{\backprime} = 0, \quad \zeta_1^{\backprime} = 0,\\ &s_1^2 k^{-2} = R_0 + R_1 + 2a_, (R_0 + 4R_1 + 3R_2) + c_, (R_1 + R_2) + 2 (c_, - a_,)(R_0 + 2R_1 + R_2) c^2,\end{aligned}\right\} \quad (147.)$$

and

$$\left.\begin{aligned}&\frac{\xi_2^{\backprime}}{a} = \frac{\eta_2^{\backprime}}{b} = \frac{\zeta_2^{\backprime}}{c - \frac{1}{c}} \left\{1 + 3 (c_, - a_,) \frac{R_1 + R_2}{R_1}\right\},\\ &s_2^2 k^{-2} = R_0 + R_1 + 2a_, (R_0 + 3R_1 + 2R_2) + 3c_, (R_1 + R_2) + 2 (c_, - a_,)(R_0 + R_1) c^2.\end{aligned}\right\} \quad (148.)$$

---

* [In the equations determining the directions of the vibrations the square of $(c_, - a_,)$ is neglected.]

In the same case (93.) the direction of nearly normal vibration & the corresponding velocity of the wave are determined by the following formula, deduced from equations (133.) & (132.) combined with (124.) & (89.):

$$\left.\begin{aligned} &\frac{\xi_3'}{a}=\frac{\eta_3'}{b}=\frac{\zeta_3'}{c}\left\{1+3\,(a,-c,)\,\frac{R_1+R_2}{R_1}\right\} \\ &s_3^2k^{-2}=R_0+3R_1+2a,(R_0+8R_1+9R_2)+c,(R_1+3R_2) \\ &\qquad\qquad +2\,(c,-a,)\,(R_0+7R_1+6R_2)\,c^2. \end{aligned}\right\} \tag{149.}$$

These results agree with and include those of the 14th & 15th paragraphs; and give the following expression for the difference of the squares of the two velocities of propagation of a wave, corresponding to the two directions of tangential or nearly tangential vibration,

$$s_1^2k^{-2}-s_2^2k^{-2}=2\,(a,-c,)\,(R_1+R_2)\,(1-c^2). \tag{150.}$$

[*Fresnel's wave surface.*]

[**18.**] $$\begin{aligned} 0=\sigma^2&+\sigma\,(R_1-R_2)\,\{a,(b^2+c^2)+b,(c^2+a^2)+c,(a^2+b^2)\} \\ &+(R_1^2+R_2^2)\,(a,b,c^2+b,c,a^2+c,a,b^2)\,(a^2+b^2+c^2) \\ &-R_1R_2\,\{a_{,}^2\,(b^2+c^2)^2+b_{,}^2\,(c^2+a^2)^2+2a,b,a^2b^2+2b,c,b^2c^2+2c,a,c^2a^2\}; \end{aligned}$$

if $c,=0$, then

$$\begin{aligned} 0=\sigma^2&+\sigma\,(R_1-R_2)\,\{a,(b^2+c^2)+b,(c^2+a^2)\} \\ &+(R_1^2+R_2^2)\,a,b,c^2\,(a^2+b^2+c^2)-R_1R_2\,\{a_{,}^2\,(b^2+c^2)^2+2a,b,a^2b^2+b_{,}^2\,(c^2+a^2)^2\}; \end{aligned}$$

also

$$2\sigma=\{-s^2k^{-2}+R_0+R_1+(R_1+3R_2)\,(a,+b,)\}\,(a^2+b^2+c^2)+2\,(R_0+2R_1)\,(a,a^2+b,b^2);$$

when $a=0,\ b=0,\ c=1,$ then $\sigma=R_2a,-R_1b,$ or $\sigma=R_2b,-R_1a,$;

when $b=0,\ c=0,\ a=1,$ then $\sigma=-R_1b,$ or $\sigma=R_2b,$;

when $c=0,\ a=0,\ b=1,$ then $\sigma=-R_1a,$ or $\sigma=R_2a,$.

In the 1st case

$$s^2k^{-2}-(R_0+R_1)=(a,+3b,)\,(R_1+R_2) \quad\text{or}\quad =(3a,+b,)\,(R_1+R_2);$$

in the 2nd case

$$=2R_0a,+R_1\,(5a,+3b,)+3R_2\,(a,+b,) \quad\text{or}\quad =2R_0a,+R_1\,(5a,+b,)+R_2\,(3a,+b,);$$

in the 3rd case

$$=2R_0b,+R_1\,(3a,+5b,)+3R_2\,(a,+b,) \quad\text{or}\quad =2R_0b,+R_1\,(a,+5b,)+R_2\,(a,+3b,).$$

If these 6 values of s^2k^{-2} be called (1), (2), (3), (4), (5), (6), then

$$(3)-(1)=2a,(R_0+2R_1+R_2);\quad (5)-(2)=2b,(R_0+2R_1+R_2);$$
$$(6)-(4)=2\,(b,-a,)\,(R_0+2R_1+R_2).$$

If, then, we suppose $R_0+2R_1+R_2=0$, we shall have

$$\begin{aligned} (1)=(3)&=(R_0+R_1)\,(1-a,-3b,), \\ (2)=(5)&=(R_0+R_1)\,(1-3a,-b,), \\ (4)=(6)&=(R_0+R_1)\,(1-a,-b,). \end{aligned}$$

Is then

$$\frac{a^2}{s^2k^{-2}-(2)}+\frac{b^2}{s^2k^{-2}-(1)}+\frac{c^2}{s^2k^{-2}-(4)}=0,$$

that is,

$$0=s^4k^{-4}-s^2k^{-2}\{a^2\,\overline{(1)+(4)}+b^2\,\overline{(2)+(4)}+c^2\,\overline{(1)+(2)}\}+a^2(1)(4)+b^2(2)(4)+c^2(1)(2)?$$

$$\frac{a^2(1)(4)}{(R_1+R_2)^2}=a^2(1-2a_{,}-4b_{,}+a_{,}^2+4a_{,}b_{,}+3b_{,}^2),$$

$$\frac{b^2(2)(4)}{(R_1+R_2)^2}=b^2(1-4a_{,}-2b_{,}+3a_{,}^2+4a_{,}b_{,}+b_{,}^2),$$

$$\frac{c^2(1)(2)}{(R_1+R_2)^2}=c^2(1-4a_{,}-4b_{,}+3a_{,}^2+10a_{,}b_{,}+3b_{,}^2);$$

$$\text{Sum}=1-4a_{,}-4b_{,}+3a_{,}^2+4a_{,}b_{,}+3b_{,}^2+2(a_{,}a^2+b_{,}b^2)-2(a_{,}^2a^2-3a_{,}b_{,}c^2+b_{,}^2b^2);$$

$$-a^2\frac{(1)+(4)}{R_1+R_2}=2(1-a_{,}-2b_{,})a^2,$$

$$-b^2\frac{(2)+(4)}{R_1+R_2}=2(1-2a_{,}-b_{,})b^2,$$

$$-c^2\frac{(1)+(2)}{R_1+R_2}=2(1-2a_{,}-2b_{,})c^2;$$

$$\text{Sum}=2(1-2a_{,}-2b_{,})+2(a_{,}a^2+b_{,}b^2).$$

Putting

$$R_1=-R_2+(R_1+R_2),\quad R_2=\rho^{\backprime}(R_1+R_2),\quad \sigma=\sigma^{\backprime}(R_1+R_2),$$

then

$$\frac{s^2k^{-2}}{R_1+R_2}=-1+(2\rho^{\backprime}+1)(a_{,}+b_{,})-2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-2\sigma^{\backprime}.$$

$$\begin{aligned}\because\quad 0=&\,1-4a_{,}-4b_{,}+3a_{,}^2+4a_{,}b_{,}+3b_{,}^2+2(a_{,}a^2+b_{,}b^2)-2(a_{,}^2a^2-3a_{,}b_{,}c^2+b_{,}^2b^2)\\&-2(1-2a_{,}-2b_{,}+a_{,}a^2+b_{,}b^2)\{1+2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-(2\rho^{\backprime}+1)(a_{,}+b_{,})+2\sigma^{\backprime}\}\\&+\{1+2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-(2\rho^{\backprime}+1)(a_{,}+b_{,})+2\sigma^{\backprime}\}^2=\Sigma_0^{\backprime}+\Sigma_1^{\backprime}+\Sigma_2^{\backprime};^{*}\end{aligned}$$

where

$$\Sigma_0^{\backprime}=1-2+1=0,$$

$$\begin{aligned}\Sigma_1^{\backprime}=&-4a_{,}-4b_{,}+2(a_{,}a^2+b_{,}b^2)+4(a_{,}+b_{,})-2(a_{,}a^2+b_{,}b^2)-4\rho^{\backprime}(a_{,}a^2+b_{,}b^2)\\&+2(2\rho^{\backprime}+1)(a_{,}+b_{,})-4\sigma^{\backprime}+4\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-2(2\rho^{\backprime}+1)(a_{,}+b_{,})+4\sigma^{\backprime}=0,\end{aligned}$$

$$\tfrac{1}{4}\Sigma_2^{\backprime}=\sigma^{\backprime 2}+\sigma^{\backprime}(1-2\rho^{\backprime})(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)+a_{,}b_{,}c^2-\rho^{\backprime}(1-\rho^{\backprime})(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)^2;$$

$$\begin{aligned}\because\quad 0=&\,\sigma^2+\sigma(R_1-R_2)(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)+a_{,}b_{,}c^2(R_1+R_2)^2\\&-R_1R_2(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)^2;\end{aligned}$$

$$\begin{aligned}0=&\,\sigma^2+\sigma(R_1-R_2)\{a_{,}(b^2+c^2)+b_{,}(c^2+a^2)\}+(R_1^2+R_2^2)a_{,}b_{,}c^2(a^2+b^2+c^2)\\&-R_1R_2\{a_{,}^2(b^2+c^2)^2+2a_{,}b_{,}a^2b^2+b_{,}^2(a^2+c^2)\};\end{aligned}$$

& this quadratic equation does in fact agree with that lately found for σ. *Thus Fresnel's law for the square of the normal velocity of a wave in a biaxal crystal would in fact result from the supposition of a rectangular and nearly cubical arrangement of attracting and repelling points,*

---

* [I.e. absolute term, terms of 1st degree, terms of 2nd degree in $a_{,}$, $b_{,}$, $\sigma^{\backprime}$.]

if we suppose the law of force & the mean interval to be such that $R_0+2R_1+R_2=0$*: & the vibrations would then be perpendicular to the plane of polarisation, at least for uniaxal and ∵ probably for biaxal crystals.*

But if, instead, we suppose R_0 and R_1 to be so small with respect to R_2 that when multiplied by $a_{,}$ or $b_{,}$ they may be neglected, then the quadratic in σ becomes

$$0=\sigma^2-\sigma R_2(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)+R_2^2a_{,}b_{,}c^2;$$

& the question now is, whether this quadratic can be deduced from the developement of the formula

$$0=\frac{}{\sigma-R_2a_{,}}+\frac{b^2}{\sigma-R_2b_{,}}+\frac{c^2}{\sigma};$$

which gives, accordingly,

$$0=\sigma^2-\sigma R_2\{a_{,}(b^2+c^2)+b_{,}(c^2+a^2)\}+R_2^2a_{,}b_{,}c^2.$$

Thus we should equally deduce Fresnel's law of velocity of a wave in a biaxal crystal, by supposing R_0 & R_1 very small with respect to R_2; but we should now have the vibrations *parallel* to the plane of polarisation, at least for uniaxal crystals and probably for biaxal also.

Supposing $R_0=-2R_1-R_2$, we shall deduce, very nearly, Fresnel's law of polarisation in a biaxal crystal if we can prove that ξ', η', ζ' are very nearly proportional to

$$\frac{a}{\sigma+R_2(a_{,}a^2+b_{,}b^2)+R_1a_{,}-R_2b_{,}},\quad \frac{b}{\sigma+R_2(a_{,}a^2+b_{,}b^2)+R_1b_{,}-R_2a_{,}},$$
$$\frac{c}{\sigma+R_2(a_{,}a^2+b_{,}b^2)-R_2(a_{,}+b_{,})}.$$

That is,

$$0=\{\sigma+R_2(a_{,}a^2+b_{,}b^2-b_{,})\}(c\xi'-a\zeta')+a_{,}(R_1c\xi'+R_2a\zeta'),$$
$$0=\{\sigma+R_2(a_{,}a^2+b_{,}b^2-a_{,})\}(c\eta'-a\zeta')+b_{,}(R_1c\eta'+R_2b\zeta');$$

that is [(125.)], since $a\xi'+b\eta'+c\zeta'=0$ very nearly,

$$0=\xi''\{\sigma+R_2(b_{,}b^2-b_{,})+a_{,}R_1(b^2+c^2)\}-ab\eta''(R_1+R_2)a_{,},$$
$$0=\eta''\{\sigma+R_2(a_{,}a^2-a_{,})+b_{,}R_1(a^2+c^2)\}-ab\xi''(R_1+R_2)b_{,};$$

& these do in fact agree with the equations (129.), (130.).

But if R_0 and R_1 be very small compared with R_2, these last equations give

$$\frac{\xi''}{\eta''}=\frac{aba_{,}R_2}{\sigma+R_2(b^2-1)b_{,}}=\frac{\sigma+R_2(a^2-1)a_{,}}{abb_{,}R_2};$$

also

$$\frac{\xi'}{\eta'}=\frac{(a^2-1)\xi''+ab\eta''}{(b^2-1)\eta''+ab\xi''},\quad \frac{\zeta'}{\eta'}=\frac{ac\xi''+bc\eta''}{(b^2-1)\eta''+ab\xi''};$$

$$\text{is}\quad \frac{a\xi'}{\sigma-R_2a_{,}}+\frac{b\eta'}{\sigma-R_2b_{,}}+\frac{c\zeta'}{\sigma}=0?\quad \text{is}\quad \frac{a(c\xi'-a\zeta')}{\sigma-R_2a_{,}}+\frac{b(c\eta'-b\zeta')}{\sigma-R_2b_{,}}=0?$$

$$\text{is}\quad \frac{a\xi''}{\sigma-R_2a_{,}}+\frac{b\eta''}{\sigma-R_2b_{,}}=0?$$

$$\text{is}\quad (\sigma-R_2b_{,})(\sigma-R_2a_{,}+R_2a^2a_{,})+b^2b_{,}R_2(\sigma-R_2a_{,})=0?$$

$$\text{is}\quad \sigma^2-\sigma R_2\{a_{,}(b^2+c^2)+b_{,}(a^2+c^2)\}+R_2^2a_{,}b_{,}c^2=0?\quad \text{Yes.}$$

Reflexion & Refraction.

(June 27, 1838.) (See the series of sheets begun at Kilmore near Nenagh, Sept. 27th, 1836.)*

[**19.**] Components of attractive action of $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ on x, y, z ... $\frac{\Delta x}{r} f(r)$, $\frac{\Delta y}{r} f(r)$, $\frac{\Delta z}{r} f(r)$. Suppose $S.f(r) = 0$ and $S.rf'(r) = 0$, in order to account for the equilibrium of a point placed at the separating surface of two ordinary media.

To account for the change of velocity which corresponds to ordinary refraction & yet to leave the density almost unaltered, suppose

$$\frac{S.r\{r^2 f'(r) + 4rf(r)\}'}{S.\{r^2 f'(r) + 4rf(r)\}} \text{ very great;}$$

but

$$\frac{S.r[r\{r^2 f'(r) + 4rf(r)\}']'}{S.r\{r^2 f'(r) + 4rf(r)\}'}$$

not very great, at least for the gases; because taking the velocity in vacuo for unity, we are to have

$$\frac{30}{\mu^2} = S.\{(r + \alpha r)^2 f'(r + \alpha r) + 4(r + \alpha r) f(r + \alpha r)\}, \quad 30 = S.\{r^2 f'(r) + 4rf(r)\},$$

$$\frac{\mu^2 - 1}{\mu^2} = A_1 \alpha + A_2 \alpha^2 + \ldots, \quad A_1 = \frac{S.\{r^3 f''(r) + 6r^2 f'(r) + 4rf(r)\}}{S.\{r^2 f'(r) + 4rf(r)\}}, \quad A_2 = \ldots.$$

However, for the gases, the extreme smallness of dilatation of the ether may be sufficient to account for the (probably) near proportion of this to the change of velocity.

To account for the independence of velocity on colour in vacuo, suppose

$$S.\{r^4 f'(r) + 6r^3 f(r)\} = 0;$$

or at least

$$\frac{S.r\{r^4 f'(r) + 6r^3 f(r)\}'}{S.\{r^4 f'(r) + 6r^3 f(r)\}} \text{ very great.}$$

We are then to suppose in like manner

$$\frac{S.r\{r^6 f'(r) + 8r^5 f(r)\}'}{S.\{r^6 f'(r) + 8r^5 f(r)\}} = 0, \text{ \&c.}$$

We have $\because$ been led by *both refraction and dispersion* to suppose the quotient

$$\frac{S.r\{r^{2i} f'(r) + \overline{2i+2}\, r^{2i-1} f(r)\}'}{S.\{r^{2i} f'(r) + \overline{2i+2}\, r^{2i-1} f(r)\}} \text{ great;}$$

---

* [These sheets seem to have been lost. If we take equations (13.), p. 417, and replace $mf(r)$ by $\frac{f(r)}{r}$ and expand, we get, on taking average values for Δx^2, Δx^4, etc., i.e. $\frac{1}{3}r^2$, $\frac{1}{5}r^4$, etc.,

$$(\text{velocity})^2 = \frac{1}{30} S.\{r^2 f'(r) + 4rf(r)\} - \frac{\pi^2}{105\lambda^2} S.\{r^4 f'(r) + 6r^3 f(r)\} + \text{etc.}$$

In the other medium every distance r is supposed to be increased to $r + \alpha r$.]

& it is remarkable that this quotient becomes, when the sums for all space are changed to sums along one radius vector & these again to single definite integrals,

$$\frac{\int_0^\infty r^3\{r^{2i}f'(r)+\overline{2i+2}\,r^{2i-1}f(r)\}'\,dr}{\int_0^\infty \{r^{2i+2}f'(r)+\overline{2i+2}\,r^{2i+1}f(r)\}\,dr},$$

in which denominator $=\Delta_0^\infty\{r^{2i+2}f(r)\}$ & numerator $=\Delta_0^\infty\{r^{2i+3}f'(r)+(2i-1)\,r^{2i+2}f(r)\}$; so that $\Delta_0^\infty.\,r^3\{r^{2i}f(r)\}'$ ought on this hypothesis to be greater than $\Delta_0^\infty.\,r^2\{r^{2i}f(r)\}$.

(June 28th.)

Let refracting surface be plane of xy, z being negligible for incident and reflected rays, but positive for refracted. Also let all the vibrations be parallel to y, so that $\xi=0$, $\zeta=0$. Let incident vibration be

$$\eta_1=\grave{\eta}_1\cos\left(\frac{2\pi}{\lambda}\overline{z-t}\right),\quad \text{refracted}\ \ \eta_2=\grave{\eta}_2\cos\left(\frac{2\pi}{\lambda}\overline{\mu z-t}\right),\quad \text{reflected}\ \ \eta_3=\grave{\eta}_3\cos\left(\frac{2\pi}{\lambda}\overline{z+t}\right),$$

the incidence being perpendicular. When $z<0$, $\eta=\eta_1+\eta_3$; when $z>0$, $\eta=\eta_2$; when $z=0$, both expressions hold, $\grave{\eta}_2=\grave{\eta}_1+\grave{\eta}_3$. When $z<0$, x, y, z are integers i, k, l, l being negative; when $z>0$, x, y, z are integers i, k, l multiplied by $1+\alpha$, α being a *very* small fraction positive or negative & l being positive. The components of attraction on that point, which when at rest was 0, 0, 0 & when in motion 0, $\grave{\eta}_2\cos\frac{2\pi t}{\lambda}$, 0, are, in the directions $+x$, $+y$, $+z$,

$$S.\frac{x}{r}\frac{f(r+\epsilon r)}{1+\epsilon}=S.\frac{x\epsilon}{r}\{rf'(r)-f(r)\}=S.\ \frac{xy}{r^3}\{rf'(r)-f(r)\}\left(\eta-\grave{\eta}_2\cos\frac{2\pi t}{\lambda}\right),^*$$

$$S.\frac{y}{r}\frac{f(r+\epsilon r)}{1+\epsilon}+S.\left(\eta-\grave{\eta}_2\cos\frac{2\pi t}{\lambda}\right)\frac{f(r)}{r}=S.\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\}\left(\eta-\grave{\eta}_2\cos\frac{2\pi t}{\lambda}\right),$$

$$S.\frac{z}{r}\frac{f(r+\epsilon r)}{1+\epsilon}=S.\frac{z\epsilon}{r}\{rf'(r)-f(r)\}=S.\ \frac{zy}{r^3}\{rf'(r)-f(r)\}\left(\eta-\grave{\eta}_2\cos\frac{2\pi t}{\lambda}\right).$$

The 1st and 3rd vanish; the 2nd ought to be $=-\left(\frac{2\pi}{\lambda}\right)^2\grave{\eta}_2\cos\frac{2\pi t}{\lambda}$. We ought ∴ to have

$$\left(\frac{2\pi}{\lambda}\right)^2=\underset{z>0}{S}.\operatorname{vers}\frac{2\pi\mu z}{\lambda}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\}+\underset{z<0}{S}.\operatorname{vers}\frac{2\pi z}{\lambda}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\},$$

$$0=(\grave{\eta}_1+\grave{\eta}_3)\underset{z>0}{S}.\sin\frac{2\pi\mu z}{\lambda}\left\{\frac{y^2}{r^3}\{\qquad\}+\frac{f(r)}{r}\right\}+(\grave{\eta}_1-\grave{\eta}_3)\underset{z<0}{S}.\sin\frac{2\pi z}{\lambda}\left\{\frac{y^2}{r^3}\{\qquad\}+\frac{f(r)}{r}\right\};$$

that is,

$$1=\left(\mu^2\underset{z>0}{S}+\underset{z<0}{S}\right).\frac{z^2}{2}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\};$$

$$0=\left\{(\grave{\eta}_1+\grave{\eta}_3)\,\mu\underset{z>0}{S}+(\grave{\eta}_1-\grave{\eta}_3)\underset{z<0}{S}\right\}\left\{\frac{zy^2}{r^3}\{rf'(r)-f(r)\}+\frac{zf(r)}{r}\right\}.$$

---

* [ϵr is the increase in r when y is changed to $y+\eta-\eta_2\cos\frac{2\pi t}{\lambda}$.]

The 1st is satisfied because[*]

$$\mu^2 \underset{z>0}{S}\,(\&\text{c.}) = \underset{z<0}{S}\,(\&\text{c.}) = \tfrac{1}{2};$$

the 2nd becomes

$$0 = \left(\mu^2 \underset{z>0}{S} + \underset{z<0}{S}\right)\left\{\frac{zy^2}{r^3}\{rf'(r) - f(r)\} + \frac{zf(r)}{r}\right.$$

when we introduce the lately established laws of reflection and refraction [i.e. $\eta_1' + \eta_3' = \mu(\eta_1' - \eta_3')$]. Now

$$\underset{z>0}{S}\, F(x, y, z, r) = \underset{l>0}{S}\, F(i + \alpha i,\ k + \alpha k,\ l + \alpha l,\ n + \alpha n);$$

$$\underset{z>0}{S}\left\{\frac{zy^2}{r^3}\{rf'(r) - f(r)\} + \frac{zf(r)}{r}\right\} = \underset{l>0}{S}\left\{\frac{lk^2}{n}\left(\frac{f(n)}{n}\right)' + \frac{lf(n)}{n}\right\}$$
$$+ \alpha \underset{l>0}{S}\left\{lk^2\left(\frac{f(n)}{n}\right)'' + \frac{lf(n)}{n} + \frac{l}{n}(2k^2 + n^2)\left(\frac{f(n)}{n}\right)'\right\} = \alpha \underset{l>0}{S}.\, lk^2 \frac{f''(n)}{n},$$

because $Sf(n) = 0$, $S.\,nf'(n) = 0$; & the same reason which led us to establish these 2 last equations seems sufficient to lead us to establish also the equation $S.\,n^2 f''(n) = 0$; whereby the condition ending the last sentence is satisfied, even without supposing, according to the laws of reflected and refracted vibrations $\eta_1' + \eta_3' = \mu(\eta_1' - \eta_3')$. Thus the case of *perpendicular incidence* appears to be satisfactorily explained even without introducing the law of the *vis viva*, provided that we do employ the principle of *equivalent vibrations*.

[*Extension of above to dispersion in a biaxal crystal.*]

(July 9th, 1838.)

[20.]

$$\frac{d^2x}{dt^2} = S.\,\Delta x \frac{f(r)}{r}; \quad \frac{d^2\delta x}{dt^2} = S.\left\{\Delta\delta x \frac{f(r)}{r} + \Delta x \left(\frac{f(r)}{r}\right)' \delta r\right\},$$

$$\delta r = \frac{\Delta x}{r}\Delta\delta x + \frac{\Delta y}{r}\Delta\delta y + \frac{\Delta z}{r}\Delta\delta z; \quad \delta x = \text{const.} \times \cos\frac{2\pi}{\lambda}(t - V), \quad V = \sigma x + \tau y + \upsilon z;$$

$$\Delta\delta x = -2\delta x\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2 + \text{useless terms [i.e. terms which vanish on summation]};$$

$$\Delta V = \sigma\Delta x + \tau\Delta y + \upsilon\Delta z; \quad \sigma^2 + \tau^2 + \upsilon^2 = \omega^{-2};$$

let

$$L = S.\left\{\frac{f(r)}{r} + \frac{\Delta x^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega^2}{2\pi^2}\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2, \quad M = \quad , \quad N = \quad ,$$

$$P = S.\,\frac{\Delta y\Delta z}{r}\left(\frac{f(r)}{r}\right)'\frac{\lambda^2\omega^2}{2\pi^2}\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2, \quad Q = \quad , \quad R = \quad ,$$

then

$$0 = (L - \omega^2)\,\delta x + R\,\delta y + Q\,\delta z = (M - \omega^2)\,\delta y + P\,\delta z + R\,\delta x = (N - \omega^2)\,\delta z + Q\,\delta x + P\,\delta y;$$

let plane of polarisation be xy, then we have 2 different velocities ω_1, ω_2, determined as follows:

Ist... $\Delta V = \sigma_1\Delta x = \omega_1^{-1}\Delta x, \quad \delta x = 0, \quad \delta z = 0, \quad P = 0, \quad Q = 0, \quad R = 0,$

$$\omega_1^2 = M = S.\left\{\frac{f(r)}{r} + \frac{\Delta y^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega_1^2}{2\pi^2}\left(\sin\frac{\pi\Delta x}{\lambda\omega_1}\right)^2$$
$$= \tfrac{1}{2}S\left\{\frac{f(r)}{r}\Delta x^2 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\} - \frac{\pi^2}{6\lambda^2\omega_1^2}S\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\} + \&\text{c.};$$

---

[*] [This result follows from the equations in the footnote of page 435 when we use the fact that the average values over a hemisphere of y^2z^2 and z^2 are $r^4/30$ and $r^2/6$ respectively.]

$$\text{II}^{\text{nd}}\ldots\ \omega_2^2=\tfrac{1}{2}S\left\{\frac{f(r)}{r}\Delta y^2+\left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\}-\frac{\pi^2}{6\lambda^2\omega_2^2}S\left\{\frac{f(r)}{r}\Delta y^4+\left(\frac{f(r)}{r}\right)'\frac{\Delta y^4\Delta x^2}{r}\right\}+\&\text{c.}$$

Is $\omega_1^2=\omega_2^2$ sensibly?

$$\Delta x=\Delta x_{,}(1+a_{,}),\quad \Delta y=\Delta y_{,}(1+b_{,}),\quad \Delta z=\Delta z_{,}(1+c_{,}),\quad r=r_{,}+\frac{a_{,}\Delta x_{,}^2+b_{,}\Delta y_{,}^2+c_{,}\Delta z_{,}^2}{r_{,}};$$

$a_{,}$, $b_{,}$, $c_{,}$ are small dilatations or contractions & the lower accents refer to a cubical system;

$$\tfrac{1}{2}S.\frac{f(r)}{r}\Delta x^2=\frac{1+2a_{,}}{2}S.\frac{f(r_{,})}{r_{,}}\Delta x_{,}^2+\tfrac{1}{2}S.\left(\frac{f(r_{,})}{r_{,}}\right)'\frac{\Delta x_{,}^2}{r_{,}}(a_{,}\Delta x_{,}^2+b_{,}\Delta y_{,}^2+c_{,}\Delta z_{,}^2)$$
$$=\frac{1+2a_{,}}{6}S.r_{,}f(r_{,})+\tfrac{1}{30}(3a_{,}+b_{,}+c_{,})S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)';$$

$$\tfrac{1}{2}S.\left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}=\frac{1+2a_{,}+2b_{,}}{2}S.\left(\frac{f(r_{,})}{r_{,}}\right)'\frac{\Delta x_{,}^2\Delta y_{,}^2}{r_{,}}+\tfrac{1}{2}S.\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)'\frac{\Delta x_{,}^2\Delta y_{,}^2}{r_{,}}(a_{,}\Delta x_{,}^2+\ +\ \ )$$
$$=\frac{1+2a_{,}+2b_{,}}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'+\frac{3a_{,}+3b_{,}+c_{,}}{210}S.r_{,}^5\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)';$$

$$\because\ (\text{if }\lambda=\infty)\quad \omega_1^2=(1+2a_{,})\left\{\tfrac{1}{6}S.r_{,}f(r_{,})+\tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'\right\}$$
$$+(3a_{,}+3b_{,}+c_{,})\left\{\tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'+\tfrac{1}{210}S.r_{,}^5\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)'\right\};$$

$$\because\qquad \omega_1^2-\omega_2^2=2(a_{,}-b_{,})\left\{\tfrac{1}{6}S.r_{,}f(r_{,})+\tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'\right\}=2(a_{,}-b_{,})$$

& $\because$ insensible if the value of ω_1 for the cubical system be unity.

But we must show also that

$$S.\left\{\frac{f(r)}{r}(\Delta x^2+\Delta y^2)+\left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\}(\Delta x^2-\Delta y^2)$$

is much less than $$S.\left\{\frac{f(r)}{r}\Delta x^4+\left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\},$$

in order to account for the existence of Fresnel's law of propagation in dispersive crystals.

$$S.\frac{f(r)}{r}\Delta x^4=\frac{1+4a_{,}}{5}S_{,}r^3f(r)+\frac{5a_{,}+b_{,}+c_{,}}{35}S_{,}r^5\left(\frac{f(r)}{r}\right)',^{*}$$

$$S.\left(\frac{f(r)}{r}\right)'\Delta x^4\Delta y^2=\frac{1+4a_{,}+2b_{,}}{35}S_{,}r^5\left(\frac{f(r)}{r}\right)'+\frac{5a_{,}+3b_{,}+c_{,}}{315}S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)',$$

$$\cdot\ \frac{\text{diff}^{\text{ce}}}{\text{sum}}\text{ of }S\left\{\frac{f(r)}{r}\Delta x^4+\left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\}\quad\text{and}\quad S\left\{\frac{f(r)}{r}\Delta y^4+\left(\frac{f(r)}{r}\right)'\frac{\Delta y^4\Delta x^2}{r}\right\}$$

$$=(a_{,}-b_{,})\frac{126S_{,}r^3f(r)+27S_{,}r^5\left(\frac{f(r)}{r}\right)'+S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)'}{63(1+2a_{,}+2b_{,})S_{,}r^3f(r)+9(1+6a_{,}+6b_{,}+c_{,})S_{,}r^5\left(\frac{f(r)}{r}\right)'+(4a_{,}+4b_{,}+c_{,})S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)'}$$

$$=\frac{a_{,}-b_{,}}{9}\ \frac{18+S_{,}\left\{9r^5\left(\frac{f}{r}\right)'+r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}\div S_{,}\left\{7r^3f+r^3\left(\frac{f}{r}\right)'\right\}}{1+2a_{,}+2b_{,}+\tfrac{1}{9}(4a_{,}+4b_{,}+c_{,})S_{,}\div S_{,}\ (\text{as in num}^{\text{r}})}$$

$$=\frac{a_{,}-b_{,}}{4a_{,}+4b_{,}+c_{,}}\text{ nearly;}$$

---

* [$S_{,}r^3f(r)$ means $S.r_{,}^3f(r_{,})$, etc.]

which may easily be supposed to be very small. Yet this result seems to indicate that Fresnel's law of propagation will hold less perfectly for the more refrangible rays.

Consider III^rd^ that velocity ω_3 which corresponds to the same plane wave as in I^st^ case, but to vibrations parallel to z & ∵ perpendicular to axis of crystal; that is,

III^rd^... $$\Delta V = \sigma_3 \Delta x = \omega_3^{-1} \Delta x, \quad \delta x = 0, \quad \delta y = 0,$$

$$\omega_3^2 = N = S.\left\{\frac{f(r)}{r} + \frac{\Delta z^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega_3^2}{2\pi^2}\left(\sin\frac{\pi\Delta x}{\lambda\omega_3}\right)^2$$
$$= \tfrac{1}{2} S\left\{\frac{f(r)}{r}\Delta x^2 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2 \Delta z^2}{r}\right\} - \frac{\pi^2}{6\lambda^2\omega_3^2} S\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta z^2}{r}\right\} + \&c.$$
$$= (1+2a_{,})\, S_{,}\left\{\tfrac{1}{6} r f + \tfrac{1}{30} r^3\left(\frac{f}{r}\right)'\right\} + (3a_{,} + b_{,} + 3c_{,})\, S_{,}\left\{\tfrac{1}{30} r^3 \left(\frac{f}{r}\right)' + \tfrac{1}{210} r^5\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}$$
$$- \frac{\pi^2}{1890\lambda^2\omega_3^2}\left\{63(1+4a_{,})\, S_{,} r^3 f + 9(1+9a_{,} + b_{,} + 3c_{,})\, S_{,} r^5 \left(\frac{f}{r}\right)'\right.$$
$$\left. + (5a_{,} + b_{,} + 3c_{,})\, S_{,} r^7 \left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\};$$

$$\omega_1^2 - \omega_3^2 = 2(b_{,} - c_{,})\left[S_{,}\left\{\tfrac{1}{30} r^3\left(\frac{f}{r}\right)' + \tfrac{1}{210} r^5 \left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\} - \frac{\pi^2}{1890\lambda^2\omega_1^2} S_{,}\left\{9r^5\left(\frac{f}{r}\right)' + r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}\right];$$

which is the effect of a double refraction on the square of velocity of the wave and is sensible on account of the greatness of the factor

$$S_{,}\left\{\tfrac{1}{30} r^3 \left(\frac{f}{r}\right)' + \frac{r^5}{210}\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}.$$

(July 9th, 1838.)

I think that [the] ordinary [ray of] violet light should have a slightly but sensibly different velocity in Iceland Spar, according as its course is parallel or perpendicular to the axis & I will ask Prof. Lloyd to try this experimentally.*

(July 10th, 1838.)

$$\omega_1^2 = 1 + \frac{(3a_{,} + 3b_{,} + c_{,})}{210} S_{,}\left\{7r^3\left(\frac{f}{r}\right)' + r^5\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\} - \frac{\pi^2(5a_{,} + 3b_{,} + c_{,})}{1890\lambda^2\omega_1^2} S_{,}\left\{9r^5\left(\frac{f}{r}\right)' + r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\};$$
$$\omega_2^2 = 1 + \frac{(3a_{,} + 3b_{,} + c_{,})}{210} S_{,} \qquad - \frac{\pi^2(3a_{,} + 5b_{,} + c_{,})}{1890\lambda^2\omega_2^2} S_{,}$$
$$\omega_3^2 = 1 + \frac{(3a_{,} + b_{,} + 3c_{,})}{210} S_{,} \qquad - \frac{\pi^2(5a_{,} + b_{,} + 3c_{,})}{1890\lambda^2\omega_3^2} S_{,}\left\{\right.$$

Let

$$\omega_1^2 = \Omega_1^{(1)} - \frac{\Omega_1^{(2)}}{\lambda^2\omega_1^2}, \quad \omega_2^2 = \Omega_2^{(1)} - \frac{\Omega_2^{(2)}}{\lambda^2\omega_2^2}, \quad \omega_3^2 = \Omega_3^{(1)} - \frac{\Omega_3^{(2)}}{\lambda^2\omega_3^2};$$

hence

$$\Omega_2^{(2)} + \Omega_3^{(2)} = 2\Omega_1^{(2)}, \quad \text{if } c_{,} = a_{,}.$$

Using Rudberg's indices as given by Powell in *Phil. Trans.* 1836, Part I, page 18, I find for Iceland Spar by calculations of this date... $(\omega_H^{-1})_1 = 1{,}6806$; $(\omega_H^{-1})_2 = 1{,}6860$. These then I conjecture or rather conclude from the foregoing theory (if *it* be right) to be the two indices of refraction of an *ordinary* ray H according as it passes through Iceland Spar in one or other of two different directions, namely perpendicular or parallel to axis [the axis of y].

---

* I did ask him in presence of Poggendorf and Lord Adare in Observatory (S.E.L. room) on July 10th.

(July 11th, 1838.)

But I believe that it will be more accurate to consider the observed quantity 0,00014478... as being $\Omega_1^{(2)}$ and not as $\frac{\Omega_1^{(2)}+\Omega_2^{(2)}}{2}$. [Then] $(\omega_H^{-1})_2 = 1{,}69152$. I am led to think it likely that the value of the same index for light *parallel to axis* would be found to be 1,6915: the difference being about the 200th part of either. Malus's differences between observation and calculation seem to have been about as great as this, sometimes.

(July 12th, 1838.)

The comparison of those parts of the squares of the ordinary and extraordinary velocities which are *independent of dispersion* give $b_{,} = \frac{8}{5}a_{,}$ & the comparison of those parts which *depend on dispersion* give $b_{,} = 2a_{,}$; so that, by each method, the *ether in Iceland Spar* ought to be about twice as much dilated or compressed (I suspect dilated) in the direction of the axis of the crystal as in a direction perpendicular thereto.

Again, supposing $1-\Omega_1^{(1)} : 1-\Omega_3^{(1)} = 4a_{,} + 3b_{,} : 6a_{,} + b_{,}$, we see that the second changes faster than the first when $a_{,}$ recedes from 0 and $b_{,}$ approaches 0, if this recession and this approach be nearly equal. Suppose that $db_{,} = -\frac{4}{3}da_{,}$, $a_{,} > 0$, $da_{,} > 0$, then $d\Omega_1^{(1)} = 0$, $\frac{d\,(1-\Omega_3^{(1)})}{1-\Omega_3^{(1)}} = \frac{\frac{14}{3}da_{,}}{6a_{,}+b_{,}}$, $\because > 0$; & thus we shall explain the phenomenon observed by Mitscherlich of the extraordinary index changing much faster than the ordinary with *heat & increasing* while the crystal tended to become cubical....

[21.] How far would my theory agree with the experiments of Malus as cited by Biot, *Traité de Physique*, Vol. III, the plane of incidence containing the axis of the crystal?

Let
$$\Delta V = \sigma\Delta x + \tau\Delta y, \quad \upsilon = 0.$$

[Introducing new notation

$$\upsilon_y, \quad \omega_x^2 + \omega_y^2 = 1,$$
$$R_0' = \frac{\pi^2}{30} S_{,}\, r^3 f(r), \quad R_1' = \frac{\pi^2}{210} S_{,}\, r^5 \left(\frac{f}{r}\right)', \quad R_2' = \frac{\pi^2}{1890} S_{,}\, r^7 \left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)',$$

expressions for L, M, N, R are obtained: $P=0$, $Q=0$. It will be sufficient to quote L and R.]

$$\begin{aligned}
L = {} & \omega_x^2\{(1+2a_{,})\,R_0 + (3a_{,}+b_{,}+c_{,})\,R_1 + 3\,(1+4a_{,})\,R_1 + 3\,(5a_{,}+b_{,}+c_{,})\,R_2\} \\
& + \omega_y^2\{(1+2b_{,})\,R_0 + (a_{,}+3b_{,}+c_{,})\,R_1 + (1+2a_{,}+2b_{,})\,R_1 + (3a_{,}+3b_{,}+c_{,})\,R_2\} \\
& - \frac{\omega_x^4}{\lambda^2\omega^2}\{(1+4a_{,})\,R_0' + (5a_{,}+b_{,}+c_{,})\,R_1' + 5\,(1+6a_{,})\,R_1' + 5\,(7a_{,}+b_{,}+c_{,})\,R_2'\} \\
& - \frac{2\omega_x^2\omega_y^2}{\lambda^2\omega^2}\{(1+2a_{,}+2b_{,})\,R_0' + (3a_{,}+3b_{,}+c_{,})\,R_1' + 3\,(1+4a_{,}+2b_{,})\,R_1' + 3\,(5a_{,}+3b_{,}+c_{,})\,R_2'\} \\
& - \frac{\omega_y^4}{\lambda^2\omega^2}\{(1+4b_{,})\,R_0' + (a_{,}+5b_{,}+c_{,})\,R_1' + (1+2a_{,}+4b_{,})\,R_1' + (3a_{,}+5b_{,}+c_{,})\,R_2'\};
\end{aligned}$$

$$R = 2\omega_x\omega_y\{(1+2a_{,}+2b_{,})R_1+(3a_{,}+3b_{,}+c_{,})R_2\}$$
$$-\frac{4\omega_x^3\omega_y}{\lambda^2\omega^2}\{(1+4a_{,}+2b_{,})R_1'+(5a_{,}+3b_{,}+c_{,})R_2'\}-\frac{4\omega_x\omega_y^3}{\lambda^2\omega^2}\{(1+2a_{,}+4b_{,})R_1'$$
$$+(3a_{,}+5b_{,}+c_{,})R_2'\};$$
$$0=(L-\omega^2)\,\delta x+R\,\delta y=(M-\omega^2)\,\delta y+R\,\delta x=(N-\omega^2)\,\delta z;$$

for one set of vibrations $\delta x=0$, $\delta y=0$, $\omega^2=N$; these vibrations are perpendicular to the plane of xy & are tangential to wave & correspond to the extraordinary ray if the axis of the crystal be (as before) the axis of y; for the other set of vibrations, namely those which give the ordinary ray, we have

$$\delta z=0,\quad (\omega^2-L)(\omega^2-M)=R^2,\quad R(\delta y^2-\delta x^2)=(M-L)\,\delta x\,\delta y.$$

[ω^2 is then found, neglecting squares and higher powers of $\lambda^{-2}\omega^{-2}$.]

$$\omega^2=(1+a_{,}+b_{,})(R_0+R_1)+(3a_{,}+3b_{,}+c_{,})(R_1+R_2)$$
$$-\lambda^{-2}\omega^{-2}\{(1+2a_{,}+2b_{,})(R_0'+R_1')+(4a_{,}+4b_{,}+c_{,})(R_1'+R_2')\}$$
$$+(a_{,}-b_{,})(\omega_x^2-\omega_y^2)\{R_0+R_1-\lambda^{-2}\omega^{-2}(2R_0'+3R_1'+R_2')\};$$

and this expression may, with sufficient accuracy, be written

$$\omega^2=R_0+R_1+(3a_{,}+3b_{,}+c_{,})(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')\{(4a_{,}+4b_{,}+c_{,})+(a_{,}-b_{,})(\omega_x^2-\omega_y^2)\}.$$

This includes the 2 expressions

$$\omega_1^2=R_0+R_1+(3a_{,}+3b_{,}+c_{,})(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')(5a_{,}+3b_{,}+c_{,}),$$

and $$\omega_2^2=R_0+R_1+(3a_{,}+3b_{,}+c_{,})(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')(3a_{,}+5b_{,}+c_{,});$$

which correspond (as we have seen) to the cases of an ordinary ray perpendicular and parallel to the axis.

In general the square of the *ordinary* velocity ought, by this theory, for Iceland Spar, to *increase* as the ray recedes from the axis, the increase being proportional to the *square of the sine of the inclination* just as for the extraordinary ray, except that this increase is much less rapid for the ordinary ray. And the double wave surface for Iceland Spar, and for light of any one colour, ought to be *a system of 2 oblate spheroids of revolution touching at their common poles.*

For the extraordinary light

$$\omega^2=N=\omega_x^2\{(1+2a_{,})(R_0+R_1)+(3a_{,}+b_{,}+3c_{,})(R_1+R_2)\}$$
$$+\omega_y^2\{(1+2b_{,})(R_0+R_1)+(a_{,}+3b_{,}+3c_{,})(R_1+R_2)\}$$
$$-\omega_x^2\lambda^{-2}\omega^{-2}\{(1+4a_{,})(R_0'+R_1')+(5a_{,}+b_{,}+3c_{,})(R_1'+R_2')\}$$
$$-\omega_y^2\lambda^{-2}\omega^{-2}\{(1+4b_{,})(R_0'+R_1')+(a_{,}+5b_{,}+3c_{,})(R_1'+R_2')\}$$
$$=R_0+R_1+(2a_{,}+2b_{,}+3c_{,})(R_1+R_2)-3(a_{,}+b_{,}+c_{,})\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$+(a_{,}-b_{,})(\omega_x^2-\omega_y^2)\{R_1+R_2-2\lambda^{-2}\omega^{-2}(R_1'+R_2')\}.$$

Or, referring to axis of crystal, by making $\omega_x^2-\omega_y^2=2\omega_x^2-1$ & observing that $c_{,}=a_{,}$, we have for ordinary light ...

$$\omega_o^2=R_0+R_1+(4a_{,}+3b_{,})(R_1+R_2)-(4a_{,}+5b_{,})\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$-2(a_{,}-b_{,})\lambda^{-2}\omega^{-2}(R_1'+R_2')\,\omega_x^2;$$

for extraordinary ...

$$\omega_e^2=R_0+R_1+(4a_{,}+3b_{,})(R_1+R_2)-(4a_{,}+5b_{,})\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$+2(a_{,}-b_{,})\,\omega_x^2\{R_1+R_2-2\lambda^{-2}\omega^{-2}(R_1'+R_2')\}.$$

Hence $$\omega_e^2-\omega_o^2=2\,(a,-b,)\,\omega_x^2\{R_1+R_2-\lambda^{-2}\omega^{-2}\,(R_1'+R_2')\},$$
ω_x^2 being square of sine of inclination to axis.*

In the notation lately used
$$\Omega_1^{(2)}=(R_1'+R_2')\,(6a,+3b,);\quad \Omega_2^{(2)}=(R_1'+R_2')\,(4a,+5b,);\quad \Omega_3^{(2)}=(R_1'+R_2')\,(8a,+b,);$$
& $$\Omega_1^{(2)}=\frac{\Omega_2^{(2)}+\Omega_3^{(2)}}{2}.$$
The common difference of the increasing arithmetical progression $\Omega_3^{(2)}$, $\Omega_1^{(2)}$, $\Omega_2^{(2)}$ is
$$2\,(b,-a,)\,(R_1'+R_2').$$
Making $R_0+R_1=1$ & introducing a and b with the same meanings as Biot, we have
$$a^2=1+(6a,+b,)\,(R_1+R_2)-(8a,+b,)\,(R_1'+R_2')\,\lambda^{-2}\omega^{-2};$$
$$b^2=1+(4a,+3b,)\,(R_1+R_2)-(6a,+3b,)\,(R_1'+R_2')\,\lambda^{-2}\omega^{-2};$$
$$\omega_o^2=b^2-\frac{\Omega\omega_y^2}{\lambda^2\omega^2};\quad \omega_e^2=a^2-\left(\Psi+\frac{2\Omega}{\lambda^2\omega}\right. \quad ..2.$$
$$\Omega=2\,(b,-a,)\,(R_1'+R_2');\quad \Psi=2\,(a,-b,)\,(R_1+R_2);$$
$$a^2-b^2=\Psi+\frac{\Omega}{\lambda^2b^2};\quad \Psi=a^2-b^2-\frac{\Omega}{\lambda^2b^2};$$
$$\omega_e^2=a^2\omega_x^2+\left(b^2-\frac{\Omega}{\lambda^2b^2}\right)\omega_y^2.$$
Thus the received constructions and rules for determining the course of the ordinary and extraordinary rays through Iceland Spar are to be modified by supposing that a and b are the two equatorial semidiameters of my two oblate spheroids & that $\sqrt{b^2-\frac{\Omega^2}{\lambda^2b^2}}$ is the common polar semidiameter of both. [Hamilton finds from numerical calculations that "Malus' experiments as quoted by Biot are quite inadequate to decide a question which turns on such small quantities."]

(July 13th, 1838.)

The recent calculations, having been carried on nearly to the conclusion without supposing $c,=a,$, apply so far to a biaxal crystal also, and show that the section of the wave surface in such a crystal made by any one of the 3 principal planes is composed of 2 concentric and coaxal ellipses, though one of them is *very nearly circular*, its ellipticity being proportional to the product of the doubly refractive & dispersive powers or to the *difference of the 2 dispersions*.

To examine other sections of the biaxal wave surface & in particular to examine *whether it has really two optic axes* now supposed and indeed known to exist, let
$$L=L^{(1)}-\lambda^{-2}\omega^{-2}L^{(2)},\quad M=\quad,\quad N=\quad,\quad P=\quad,\quad Q=\quad,\quad R=\quad;$$
& let $$\Delta V=\sigma\Delta x+\tau\Delta y+v\Delta z;$$

---

* [Various terms in the expressions for ω_o^2 and ω_e^2 have been neglected. They would however have disappeared in the expression for $\omega_e^2-\omega_o^2$.]

then $L^{(1)}$, $M^{(1)}$, $N^{(1)}$ have all one common part, namely

$$\omega_x^2\{(1+2a_{,})R_0+(3a_{,}+b_{,}+c_{,})R_1\}+\omega_y^2\{(1+2b_{,})R_0+(a_{,}+3b_{,}+c_{,})R_1\}$$
$$+\omega_z^2\{(1+2c_{,})R_0+(a_{,}+b_{,}+3c_{,})R_1\}$$
$$=(R_0+R_1)\{1+2(a_{,}\omega_x^2+b_{,}\omega_y^2+c_{,}\omega_z^2)\}-R_1(1-a_{,}-b_{,}-c_{,});$$

while the other parts are

$$\omega_x^2\{3(1+4a_{,})R_1+3(5a_{,}+b_{,}+c_{,})R_2\}+\omega_y^2\{(1+2a_{,}+2b_{,})R_1+(3a_{,}+3b_{,}+c_{,})R_2\}$$
$$+\omega_z^2\{(1+2a_{,}+2c_{,})R_1+(3a_{,}+b_{,}+3c_{,})R_2\},$$
$$\omega_y^2\{3(1+4b_{,})R_1+3(a_{,}+5b_{,}+c_{,})R_2\}+\omega_z^2\{(1+2b_{,}+2c_{,})R_1+(a_{,}+3b_{,}+3c_{,})R_2\}$$
$$+\omega_x^2\{(1+2a_{,}+2b_{,})R_1+(3a_{,}+3b_{,}+c_{,})R_2\},$$
$$\omega_z^2\{3(1+4c_{,})R_1+3(a_{,}+b_{,}+5c_{,})R_2\}+\omega_x^2\{(1+2a_{,}+2c_{,})R_1+(3a_{,}+b_{,}+3c_{,})R_2\}$$
$$+\omega_y^2\{(1+2b_{,}+2c_{,})R_1+(a_{,}+3b_{,}+3c_{,})R_2\}.$$

Of these the first is

$$=(a_{,}+b_{,}+c_{,})(R_1+R_2)+2(R_1+R_2)(a_{,}\omega_x^2+b_{,}\omega_y^2+c_{,}\omega_z^2)+(1-a_{,}-b_{,}-c_{,})R_1$$
$$+2a_{,}(R_1+R_2)(1+4\omega_x^2)+2\{(a_{,}+b_{,}+c_{,})(R_1+R_2)+(1-a_{,}-b_{,}-c_{,})R_1\}\omega_x^2,$$

the 1st line being common to all 3, but the 2nd line being peculiar to $L^{(1)}$. As to $R^{(1)}$, it has been assigned.

Our fundamental *equations of vibration* are $0=(L-\omega^2)\delta x+R\,\delta y+Q\,\delta z \;=\; =$. These give

$$\omega^2\xi''=\omega_z(L\,\delta x+R\,\delta y+Q\,\delta z)-\omega_x(Q\,\delta x+P\,\delta y+N\,\delta z),$$
$$\omega^2\eta''=\omega_z(R\,\delta x+M\,\delta y+P\,\delta z)-\omega_y(Q\,\delta x+P\,\delta y+N\,\delta z),$$
$$\omega^2\zeta''=\omega_x(L\,\delta x+R\,\delta y+Q\,\delta z)+\omega_y(R\,\delta x+M\,\delta y+P\,\delta z)+\omega_z(Q\,\delta x+P\,\delta y+N\,\delta z),$$

if $\qquad \xi''=\omega_z\delta x-\omega_x\delta z, \quad \eta''=\omega_z\delta y-\omega_y\delta z, \quad \zeta''=\omega_x\delta x+\omega_y\delta y+\omega_z\delta z.$

These last expressions give

$$\omega_z\delta x=\xi''-\omega_x(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta''),$$
$$\omega_z\delta y=\eta''-\omega_y(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta''),$$
$$\omega_z\delta z=\quad -\omega_z(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta'').$$

$$\because \quad \omega^2\omega_z\xi''=\omega_z(L\xi''+R\eta'')-\omega_x(Q\xi''+P\eta'')+\{-\omega_z(\omega_xL+\omega_yR+\omega_zQ)$$
$$+\omega_x(\omega_xQ+\omega_yP+\omega_zN)\}(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta''),$$
$$\omega^2\omega_z\eta''=\omega_z(R\xi''+M\eta'')-\omega_y(Q\xi''+P\eta'')+\{-\omega_z(\omega_xR+\omega_yM+\omega_zP)$$
$$+\omega_y(\omega_xQ+\omega_yP+\omega_zN)\}(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta''),$$
$$\omega^2\omega_z\zeta''=\omega_x(L\xi''+R\eta'')+\omega_y(R\xi''+M\eta'')+\omega_z(Q\xi''+P\eta'')$$
$$-\{\omega_x(\omega_xL+\omega_yR+\omega_zQ)+\omega_y(\omega_xR+\omega_yM+\omega_zP)+\omega_z(\omega_xQ+\omega_yP+\omega_zN)\}$$
$$\times(\omega_x\xi''+\omega_y\eta''-\omega_z\zeta'').$$

Put $\qquad R=R'\omega_x\omega_y, \quad P=P'\omega_y\omega_z, \quad Q=Q'\omega_x\omega_z;$

then, omitting the small terms containing $\zeta^{\backprime\backprime}$ as a factor [i.e. the vibrations are assumed to be nearly tangential],

$$\begin{aligned} 0 = \xi^{\backprime\backprime}\{-\omega^2 + L - Q^{\backprime}\omega_x^2 + \omega_x^2\,(N - L + Q^{\backprime}\,(\omega_x^2 - \omega_z^2) + (P^{\backprime} - R^{\backprime})\,\omega_y^2)\} \\ + \omega_x\omega_y\eta^{\backprime\backprime}\{R^{\backprime} - P^{\backprime} + N - L + Q^{\backprime}\,(\omega_x^2 - \omega_z^2) + (P^{\backprime} - R^{\backprime})\,\omega_y^2\}, \\ 0 = \eta^{\backprime\backprime}\{-\omega^2 + M - P^{\backprime}\omega_y^2 + \omega_y^2\,(N - M + P^{\backprime}\,(\omega_y^2 - \omega_z^2) + (Q^{\backprime} - R^{\backprime})\,\omega_x^2)\} \\ + \omega_x\omega_y\xi^{\backprime\backprime}\{R^{\backprime} - Q^{\backprime} + N - M + P^{\backprime}\,(\omega_y^2 - \omega_z^2) + (Q^{\backprime} - R^{\backprime})\,\omega_x^2\}. \end{aligned}$$

But by an investigation made lately [pp. 432–434] these formulae will give *Fresnel's laws of velocity & polarisation* to semiaxes $a_{,,}$, $b_{,,}$, $c_{,,}$, provided we can have the 4 following relations:

$$\begin{aligned} c_{,,}^2 + (1 - \omega_y^2)\,(b_{,,}^2 - c_{,,}^2) &= (L - Q^{\backprime}\omega_x^2)\,(1 - \omega_x^2) + \omega_x^2\,(N - Q^{\backprime}\omega_z^2) + \omega_x^2\omega_y^2\,(P^{\backprime} - R^{\backprime}), \\ c_{,,}^2 + (1 - \omega_x^2)\,(a_{,,}^2 - c_{,,}^2) &= (M - P^{\backprime}\omega_y^2)\,(1 - \omega_y^2) + \omega_y^2\,(N - P^{\backprime}\omega_z^2) + \omega_x^2\omega_y^2\,(Q^{\backprime} - R^{\backprime}), \\ a_{,,}^2 - c_{,,}^2 &= R^{\backprime} - P^{\backprime} + N - Q^{\backprime}\omega_z^2 - (L - Q^{\backprime}\omega_x^2) + \omega_y^2\,(P^{\backprime} - R^{\backprime}), \\ b_{,,}^2 - c_{,,}^2 &= R^{\backprime} - Q^{\backprime} + N - P^{\backprime}\omega_z^2 - (M - P^{\backprime}\omega_y^2) + \omega_x^2\,(Q^{\backprime} - R^{\backprime}). \end{aligned}$$

Of these 4 equations, it is remarkable that *any one is a consequence of the 3 others*, independently of L, M, N, $P^{\backprime}$, $Q^{\backprime}$, $R^{\backprime}$, ω_x, ω_y and in virtue of the relation $\omega_z^2 = 1 - \omega_x^2 - \omega_y^2$. They give

$$\begin{aligned} c_{,,}^2 &= L + M + \omega_z^2\,(P^{\backprime} + Q^{\backprime} - R^{\backprime}) - (L\omega_x^2 + M\omega_y^2 + N\omega_z^2 + 2R^{\backprime}\omega_x^2\omega_y^2 + 2P^{\backprime}\omega_y^2\omega_z^2 + 2Q^{\backprime}\omega_z^2\omega_x^2), \\ a_{,,}^2 &= M + N + \omega_x^2\,(Q^{\backprime} + R^{\backprime} - P^{\backprime}) - (L\omega_x^2 + M\omega_y^2 + N\omega_z^2 + 2R^{\backprime}\omega_x^2\omega_y^2 + 2P^{\backprime}\omega_y^2\omega_z^2 + 2Q^{\backprime}\omega_z^2\omega_x^2), \\ b_{,,}^2 &= N + L + \omega_y^2\,(R^{\backprime} + P^{\backprime} - Q^{\backprime}) - (L\omega_x^2 + M\omega_y^2 + N\omega_z^2 + 2R^{\backprime}\omega_x^2\omega_y^2 + 2P^{\backprime}\omega_y^2\omega_z^2 + 2Q^{\backprime}\omega_z^2\omega_x^2); \end{aligned}$$

so that it only remains to examine *how nearly these 3 quantities are constant.* For this examination it will be convenient to calculate

$$L\omega_x + R\omega_y + Q\omega_z, \quad R\omega_x + M\omega_y + N\omega_z, \quad Q\omega_x + P\omega_y + N\omega_z;$$

or rather

$$L + R^{\backprime}\omega_y^2 + Q^{\backprime}\omega_z^2, \quad M + P^{\backprime}\omega_z^2 + R^{\backprime}\omega_x^2, \quad N + Q^{\backprime}\omega_x^2 + P^{\backprime}\omega_y^2.$$

Any one of these will give the 2 others by transformation of the letters in the cycles a, b, c and x, y, z.

Confining ourselves at first to the terms independent of dispersion, we have

$$c_{,,}^2 = R_0 + R_1 + (R_1 + R_2)\,(3a_{,} + 3b_{,} + c_{,}); \quad a_{,,}^2 = R_0 + R_1 + (R_1 + R_2)\,(a_{,} + 3b_{,} + 3c_{,});$$
$$b_{,,}^2 = R_0 + R_1 + (R_1 + R_2)\,(3a_{,} + b_{,} + 3c_{,}).$$

How nearly would this same constancy hold when we take account of dispersion?...

..."neglecting $R_0' + R_1'$ on account of there being no dispersion in a vacuum"....

We may then determine the velocity & polarisation of light in a biaxal crystal by Fresnel's rules, (the vibrations however being parallel to the plane of polarisation & at least *very nearly* tangential to the wave) if we employ *variable semi-axes* $a_{,,}$, $b_{,,}$, $c_{,,}$ to construct the auxiliary ellipsoid or the wave surface; these variable semi-axes depending on the colour & on the direction, according to the formulae

$$\begin{aligned} a_{,,}^2 &= R_0 + R_1 + \left(R_1 + R_2 - \frac{R_1' + R_2'}{\lambda^2\omega^2}\right)(a_{,} + 3b_{,} + 3c_{,}) - 2\,\frac{R_1' + R_2'}{\lambda^2\omega^2}\,(a_{,}\omega_x^2 + b_{,}\omega_y^2 + c_{,}\omega_z^2), \\ b_{,,}^2 &= R_0 + R_1 + \left(\qquad\qquad\qquad\qquad\right)(3a_{,} + b_{,} + 3c_{,}) - 2\,\frac{R_1' + R_2'}{\lambda^2\omega^2}\,(\qquad\qquad\qquad\qquad), \\ c_{,,}^2 &= R_0 + R_1 + \left(\qquad\qquad\qquad\qquad\right)(3a_{,} + 3b_{,} + c_{,}) - 2\,\frac{R_1' + R_2'}{\lambda^2\omega^2}\,(\qquad\qquad\qquad\qquad). \end{aligned}$$

It is remarkable that the two differences

$$a_{,,}^2 - b_{,,}^2 = 2\left(R_1 + R_2 - \frac{R_1' + R_2'}{\lambda^2\omega^2}\right)(b_, - a_,), \quad b_{,,}^2 - c_{,,}^2 = 2\left(R_1 + R_2 - \frac{R_1' + R_2'}{\lambda^2\omega^2}\right)(c_, - b_,),$$

are independent of the direction of the ray or wave....

But it is very important to observe that although we have only been led to add a common part to each of Fresnel's 2 values of the square of the velocity, so that *the difference of those 2 squares is still proportional to the product of the sines of the inclinations of the wave-normal to 2 fixed axes A & B* and thus the phenomena of the rings appears to be satisfied, yet on the other hand these axes seem in the present theory to be *independent of the colour*; which is *not true for light*. And this would seem to be a fatal objection, unless it can be removed by some modification of my hypothesis....

[Here follows an attempt at reconciling the values of $a_{,,}$, $b_{,,}$, $c_{,,}$ for topaz with the above results.]

XVIII.

ON THE PROPAGATION OF LIGHT *IN VACUO*

Read August, 1838.

[*British Association Report*, Newcastle 1838, pp. 2–6.*]

The object of this communication was to advance the state of our knowledge respecting the law which regulates the attractions or repulsions of the particles of the ether on each other. The general differential equations of motion of any system of attracting or repelling points being reducible to the form

$$\frac{d^2x}{dt^2} = S\,.\,m,\Delta x f(r), \qquad (1.)$$

the equations of minute vibration are of the form

$$\frac{d^2\delta x}{dt^2} = S\,.\,m,\{\Delta\delta x\,.\,f(r) + \Delta x\,.\,\delta f(r)\}, \qquad (2.)$$

in which

$$\delta f(r) = f'(r)\,\delta r, \qquad (3.)$$

and

$$\delta r = \frac{\Delta x}{r}\Delta\delta x + \frac{\Delta y}{r}\Delta\delta y + \frac{\Delta z}{r}\Delta\delta z. \qquad (4.)$$

A mode of satisfying the differential equations (2.), and at the same time of representing a large class of the phenomena of light, is to assume

$$\frac{\delta x}{\xi'} = \frac{\delta y}{\eta'} = \frac{\delta z}{\zeta'} = \text{const.} + \cos\frac{2\pi(vt - ax - by - cz)}{\lambda}; \qquad (5.)$$

in which ξ', η', ζ' are constants, depending on the extent and direction of vibration; a, b, c are the cosines of the inclinations of the direction of propagation of a plane wave to the positive semi-axes of x, y, z; v is the velocity of propagation of that wave, and λ is the length of an undulation; and π is the semi-circumference of a circle, of which the radius is unity. With this assumption (5.), and with a natural and obvious supposition respecting a certain symmetry of arrangement in the ether, causing the sums of odd powers [of Δx, Δy, Δz] to vanish, it is permitted to substitute in (2.) the expressions

$$\frac{d^2\delta x}{dt^2} = -\left(\frac{2\pi v}{\lambda}\right)^2\delta x, \qquad (6.)$$

$$\Delta\delta x = -\text{vers}\,\Delta\theta\,.\,\delta x, \qquad (7.)$$

in which

$$\Delta\theta = -\frac{2\pi}{\lambda}(a\Delta x + b\Delta y + c\Delta z); \qquad (8.)$$

[By a curious error Hamilton is given the distinction F.R.S. in the title to this paper.]

and thus arises a system of conditions of the form

$$\begin{aligned}\xi^{\backprime}\left(\frac{2\pi v}{\lambda}\right)^2 = \quad & \xi^{\backprime} mS\,.\left\{f(r)+\frac{\Delta x^2}{r}f'(r)\right\}\text{vers}\,\Delta\theta\\ & +\eta^{\backprime} mS\,.\frac{\Delta x\,\Delta y}{r}f'(r)\,\text{vers}\,\Delta\theta\\ & +\zeta^{\backprime} mS\,.\frac{\Delta x\,\Delta z}{r}f'(r)\,\text{vers}\,\Delta\theta, \qquad (9.)\end{aligned}$$

the masses $m_{,}$ of the etherial particles being supposed each $=m$. Three conditions of this form (9.) exist for every particle, and determine, in general, for any given values of a, b, c, λ, that is, for any given direction of propagation, and any given length of wave, the value of v, and the ratios of $\xi^{\backprime}$, $\eta^{\backprime}$, $\zeta^{\backprime}$, that is, the velocity of propagation of the wave, and the direction of vibration of the particle. Accordingly, with some slight differences of notation, they have been proposed for this purpose by Cauchy, and adopted by other mathematicians. Suppose now, for simplicity, that the plane wave is vertical, so that $c=0$; and let, at first, the direction of its propagation coincide with the positive semi-axis of x, so that b also vanishes and a is $=1$. Then, for transversal vibrations, the expression for the square of the velocity of propagation is

$$v^2=\left(\frac{\lambda}{2\pi}\right)^2 mS\,.\left\{f(r)+\frac{r^2-\Delta x^2}{2r}f'(r)\right\}\text{vers}\,\frac{2\pi\,\Delta x}{\lambda}; \qquad (10.)$$

which appears to extend not only to the interplanetary spaces, but also to all ordinary transparent media, and contains, for them, the theoretical law of dispersion, which was first discovered by Cauchy, namely, the expression

$$v^2=A_0-A_1\lambda^{-2}+A_2\lambda^{-4}\ \&\text{c.}, \qquad (11.)$$

in which

$$A_i=\frac{(2\pi)^{2i}m}{1\,.\,2\,.\,3\,.\,4\ldots(2i+2)}S\left\{f(r)+\frac{r^2-\Delta x^2}{2r}f'(r)\right\}\Delta x^{2i+2}. \qquad (12.)$$

But, in order that this law may agree with the phenomena, it is essential that the series (11.) should be convergent, even in its earliest terms; and this consideration enables us to exclude the supposition which has occurred to some mathematicians, that the particles of the ether attract each other with forces which are inversely as the squares of the distances between them. For, if we suppose $rf(r)=r^{-2}$, and therefore $f(r)=r^{-3}$, $f'(r)=-3r^{-4}$, we shall have

$$A_i=\frac{1}{2}\,\frac{(2\pi)^{2i}m}{1\,.\,2\,.\,3\,.\,4\ldots(2i+2)}S\{-r^{-3}+3r^{-5}\Delta x^2\}\,\Delta x^{2i+2}; \qquad (13.)$$

and by extending the summation to particles, distant by several times the length of an undulation from the particle which they are supposed to attract, these sums (13.) become extremely large, and the terms of the series (11.) diverge very rapidly at first, though they always finish by converging. In fact, if we conceive a sphere, whose radius $=n\lambda=n$ times the length of an undulation (n being a large multiplier), and whose centre is at the attracted particle; and if we consider only the combined effect of the actions of all the particles within this sphere, we may, as a good approximation, convert each sum (13.) into a triple definite integral, and thus obtain, for the general term of the series (11.), the expression*

$$(-1)^i A_i \lambda^{-2i} = \frac{(-1)^i 4\pi m n^2 \lambda^2}{(2i+5)\epsilon^3} \cdot \frac{(2\pi n)^{2i}}{1 \,.\, 2 \,.\, 3 \ldots (2i+3)}, \tag{14.}$$

ϵ being the mean interval between any two adjacent particles of the ether, so that the number of such particles contained in any sphere of radius r is nearly $= \frac{4\pi r^3}{3\epsilon^3}$, if r be a large multiple of ϵ. And hence we find, by taking the sum of all these terms (14.), the expression

$$v^2 = \frac{\lambda^2 m}{\pi \epsilon^3} \left\{ \frac{1}{3} + \frac{\cos 2\pi n}{(2\pi n)^2} - \frac{\sin 2\pi n}{(2\pi n)^3} \right\}; \tag{15.}$$

so that, by taking the limit to which v^2 tends, when n is taken greater and greater, we get at last as a near approximation

$$v^2 = \frac{\lambda^2 m}{3\pi\epsilon^3}, \tag{16.}$$

and

$$\frac{\lambda}{v} = \sqrt{\frac{3\pi\epsilon^3}{m}}. \tag{17.}$$

But $\frac{\lambda}{v}$ expresses the time of oscillation of any one vibrating particle; this time would therefore be nearly constant, if the particles attracted each other according to the law of the inverse square of the distance; and consequently this law is inadmissible, as being incompatible with the law of dispersion. It had appeared to Sir William Hamilton important to reproduce these results, though he remarked that they agree substantially with those of Cauchy, because the law of the inverse square was one which naturally offered itself to the mind, and had, in fact, been proposed by at least one mathematician of high talent. There was, however, another law which had great claims on the attention of mathematicians, as having been proposed by Cauchy to represent the phenomena of the propagation of light *in vacuo*, namely, the law of a repulsive action, proportional inversely to the fourth power, or to the square of the square of the distance. M. Cauchy had, indeed, supposed that this law might hold good only for small distances, but in examining into its admissibility, it appeared fair to treat it as extending to all the neighbouring particles which act on any one. But against this law also, Sir William Hamilton brought forward objections, which were founded partly on algebraical, and partly on numerical calculations, and which appeared to him decisive.

The spirit of these objections consisted in showing that the law in question would give too great a preponderance to the effect of the immediately adjacent particles, and would thereby produce irregularities which are not observed to exist. In particular, if it be supposed that

$$S \,.\, r^i \Delta x^2 = S \,.\, r^i \Delta y^2 = S \,.\, r^i \Delta z^2,$$

$$S \,.\, r^i \Delta x^4 = S \,.\, r^i \Delta y^4 = S \,.\, r^i \Delta z^4,$$

$$S \,.\, r^i \Delta x^2 \Delta y^2 = S \,.\, r^i \Delta y^2 \Delta z^2 = S \,.\, r^i \Delta z^2 \Delta x^2,$$

$$\left[S = \frac{2\pi}{\epsilon^3} \int_{-1}^{+1} d\mu \int_0^{n\lambda} dr \, (3\mu^2 - 1) \, \mu^{2i+2} \, r^{2i+1}. \right]$$

and also, in (5.), that $c=0$, $a=b$, and that λ is much greater than ϵ, it is found that the two values v^2 and $v_{,}^2$ of the square of the velocity v, corresponding to vertical and to horizontal but transversal vibrations, are connected by the relation

$$v_{,}^2 = -\tfrac{2}{3}v^2,$$

being expressed as follows:

$$v_{,}^2 = \frac{m}{4} S\,(5r^{-7}\Delta x^4 - r^{-3}),$$

$$v^2 = \frac{3m}{8} S\,(r^{-3} - 5r^{-7}\Delta x^4).^*$$

In conclusion, he offered reasons for believing that the law of action of the particles of the ether on each other resembles more the law which Poisson has in one of his memoirs proposed as likely to express the mutual action of the particles of ordinary and solid bodies, being perhaps of some such form as the following:†

$$rf(r) = -a\,.\,b^{-\left(\frac{r}{g\epsilon}\right)^h} + a_{,}\,.\,b_{,}^{-\left(\frac{r}{g_{,}\epsilon}\right)^{h_{,}}}; \qquad (18.$$

b and $b_{,}$ being each greater than unity, and g, $g_{,}$, h, $h_{,}$ being some large positive numbers, while a and $a_{,}$ are constant and positive multipliers, and ϵ is, as before, the mean or average interval between two adjacent particles. With such a law there would be a nearly constant repulsion, if a be greater than $a_{,}$, and if g be less than $g_{,}$, as long as $\frac{r}{g\epsilon}$ is sensibly less than unity; but the force would rapidly change, as the distance r approached to $g\epsilon$, and would then become a nearly constant attraction, until r became nearly $=g_{,}\epsilon$; it would then diminish rapidly, and soon become insensible. Sir William Hamilton did not, however, intend to exclude the hypothesis, that the function $rf(r)$ may contain several alternations of such repulsive and attractive terms,—much less did he deny that at great distances it may reduce itself to the law of the inverse square.

---

* [There is another horizontal, but not transverse, vibration for which $v_{,,}^2 = \frac{m}{24} S\,(15r^{-7}\Delta x^4 - 11r^{-3})$. In general, if n is positive and greater than 4, $v_{,}^2 + \frac{2}{3}v^2$ is negative.]

† [In Note Book 39 (1835) Hamilton made lengthy calculations founded on the function $f(r) = 10^{\left(-\frac{r}{10}\right)^n}$ for all values of r such that $r^2 = i^2 + k^2 + l^2$, where i, k, l are positive or negative integers or zero. He collects his results in the following table:

n	$S_{r=0}^{r=\infty} f(r)$	$S_{r=0}^{r=\infty} -rf'(r)$	$S_{r=0}^{r=\infty} rf(r)$	$S_{r=0}^{r=\infty} -r^2f'(r)$
80	3955·83	11647·45	29118·79	114831·59
100	3989·28	12135·58	29443·83	120019·00
200	4066·47	13110·81	30202·70	130165·27
400	4118·33	11075·69	30716·79	110244·49
1000	4140·91	7990·94	30941·14	79854·47

and concludes with the remark: "All these calculations have been made carefully by myself & verified with care, & I think that the results may be depended on."]

ON THE PROPAGATION OF LIGHT IN CRYSTALS

Read August, 1838.

[*British Association Report*, Newcastle 1838, p. 6.]

By continuing to modify the analysis of M. Cauchy* in the manner already explained, he had succeeded in deducing, more satisfactorily than had in his opinion been done before, from dynamical principles, a large and important class of the phenomena of light in crystals; though much still remained to be done before it could be said that a perfect theory of light was obtained. He had employed, for the purposes of calculation, the supposition that the arrangement of the particles of the ether in a crystal differs from an exactly cubical arrangement only by very small displacements, caused by the action of the particles of the crystalline body; and had attended only to those indirect or reflex effects of the latter particles which are owing to the disturbances which they produce in the arrangement of the former particles: but he did not mean to assert that he had established any strong physical probability for this being the true *modus operandi* in crystals, though he thought the hypothesis had explained so much already that it deserved to be still further developed.

---

* [Cauchy, *Œuvres*, II^e Série, Tome IX, pp. 390–450; I^e Série, Tome II, pp. 111–177.]

XIX.

RESEARCHES RESPECTING VIBRATION CONNECTED WITH THE THEORY OF LIGHT

[1839.]

[Note Book 52.]

[The dynamical system consists of a number $(n+2)$ of particles $(P_0, P_1, \dots P_{n-1})$ each of unit mass and in equilibrium, spaced at unit distances along a straight line. The end particles are fixed and each particle is attracted by a force (a^2) by the one immediately before and immediately after. The system executes small transverse vibrations and these are studied in five Problems. Each Problem is worked out in great detail with examples and Hamilton is led to various results, some of which must have been independent discoveries such as sequence equations and asymptotic values of Bessel Functions and others were many years ahead of their time such as the Reciprocal Theorem in Dynamics and the distinction between Phase-velocity and other types of velocity. The idea of a "fluctuating" function is first mentioned also here.

Problem I (pp. 451–463). P_0, P_{n+1} fixed, $P_1, P_2, \dots P_n$ having any assigned initial displacements and velocities.

Problem II (pp. 463–487). All initial displacements and velocities zero except for $P_{j_,}, P_{j_,+1}, \dots P_{j-1}, P_j$ and their displacements and velocities to correspond to the ith mode of vibration.

Problem III (pp. 487–503). The initial displacements and velocities of a number of particles to correspond to those of a progressive sinusoidal wave.

Problem IV (pp. 503–510). Discussion of previous case for large values of t.

Problem V (pp. 511–526). A single particle is constrained to move in an assigned manner.]

Problem I.

1. A finite number $(n+2)$ of equal particles $(P_0, P_1, \dots P_l, \dots P_n, P_{n+1})$ being supposed to be arranged in one plane, and nearly in one straight line, at finite and very nearly equal intervals (each $=1$); the two extreme particles (P_0 and P_{n+1}) being also supposed to be fixed and each of the (n) intermediate particles (as P_l) to be acted on only by the attractions (each $=a^2$) of the two (P_{l-1} & P_{l+1}) which immediately precede and follow it in the series; it is required to determine the laws of the transversal vibrations of the system: that is, to express the transversal displacement $(y_{l,t})$, at any time (t), of any intermediate and moveable particle (P_l) from the right line or axis (of x) connecting the two extreme and fixed particles (P_0 and P_{n+1}), for any given but arbitrary set of (n) small initial transversal displacements $(y_{l,0})$, and of (n) small initial transversal velocities $(y'_{l\,0})$.

2. (*Solution.*) This problem is equivalent to that of integrating generally a system of n simultaneous differential equations, of the second order, and of the form

$$y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t}), \tag{1}$$

or

$$\left(D_t^2 - \frac{a^2 \Delta_l^2}{1+\Delta_l}\right) y_{l,t} = 0; \tag{2}$$

the integer l taking in succession all values from 1 to n; and $y_{0,t}$, $y_{n+1,t}$ being supposed to be each equal to zero. It is easy to effect this integration by the known methods. We have only to assume

$$L_{l,k} = \sin \frac{kl\pi}{n+1},$$

$$r_k = 2a \sin \frac{\frac{1}{2}k\pi}{n+1},$$

$$Y_{k,t} = L_{1,k} y_{1,t} + \dots + L_{l,k} y_{l,t} + \dots + L_{n,k} y_{n,t},$$

k being an integer which takes in succession all values from 1 to n; and to observe that these assumptions give

$$-r_k^2 L_{l,k} = a^2 (L_{l+1,k} - 2L_{l,k} + L_{l-1,k}),$$

$$L_{0,k} = 0, \quad L_{n+1,k} = 0,$$

$$y_{l,t} = \frac{2}{n+1} (L_{l,1} Y_{1,t} + \dots + L_{l,k} Y_{k,t} + \dots + L_{l,n} Y_{n,t}).$$

For thus we easily transform the differential system (1) into another, which may be thus denoted,

$$Y''_{k,t} + r_k^2 Y_{k,t} = 0, \tag{3}$$

and which gives, by integration,

$$Y_{k,t} = Y_{k,0} \cos tr_k + Y'_{k,0} r_k^{-1} \sin tr_k; \tag{4}$$

so that the sought expression for $y_{l,t}$ may be thus written

$$y_{l,t} = \frac{2}{n+1} \left\{ \begin{aligned} &L_{l,1} Y_{1,0} \cos tr_1 + \dots + L_{l,k} Y_{k,0} \cos tr_k + \dots + L_{l,n} Y_{n,0} \cos tr_n \\ &\qquad + L_{l,1} Y'_{1,0} \frac{\sin tr_1}{r_1} + \dots + L_{l,k} Y'_{k,0}{}^{\sin tr_k} + \dots + L_{l,n} Y'_{n,0}{}^{\sin tr_n} \end{aligned} \right\}, \tag{5}$$

or, more concisely,

$$y_{l,t} = \frac{2}{n+1} \Sigma^n_{(k)1} L_{l,k} (Y_{k,0} \cos tr_k + Y'_{k,0} r_k^{-1} \sin tr_k); \tag{6}$$

in which we are to remember that

$$Y_{k,0} = \Sigma^n_{(l)1} L_{l,k} y_{l,0}, \quad Y'_{k,0} = \Sigma^n_{(l)1} L_{l,k} y'_{l,0}.$$

3. (*Corollary* 1.) If there be but one particle, P_j, displaced at the time 0, and if no particle have at that time any velocity, we may write $Y_{k,0} = y_{j,0} L_{j,k}$, $Y'_{k,0} = 0$, and the expression for the displacement $y_{l,t}$ of any particle P_l at the time t becomes

$$y_{l,t} = \frac{2y_{j,0}}{n+1} \Sigma^n_{(k)1} L_{j,k} L_{l,k} \cos tr_k$$

$$= \frac{2y_{j,0}}{n+1} \Sigma^n_{(k)1} \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1}\right).$$

4. (*Corollary* 2.) In like manner, if only one particle, P_j, have an initial velocity, $y'_{j,0}$, and if no particle have any initial displacement, we may write

$$Y_{k,0}=0, \quad Y'_{k,0}=y'_{j,0}L_{j,k},$$

and

$$\begin{aligned} y_{l,t} &= \frac{2y'_{j,0}}{n+1}\Sigma^n_{(k)1}L_{j,k}L_{l,k}\int_0^t dt\cos tr \\ &= \frac{2y'_{j,0}}{n+1}\Sigma^n_{(k)1}\sin\frac{jk\pi}{n+1}\sin\frac{lk\pi}{n+1}\int_0^t dt\cos\left(2at\sin\frac{\frac{1}{2}k\pi}{n+1}\right). \end{aligned}$$

5. (*Corollary* 3.) The general solution (6) may therefore be put under the form

$$y_{l,t}=\frac{2}{n+1}\Sigma^n_{(j)1}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\Sigma^n_{(k)1}\sin\frac{jk\pi}{n+1}\sin\frac{lk\pi}{n+1}\cos\left(2at\sin\frac{\frac{1}{2}k\pi}{n+1}\right) \tag{7}$$

6. (*Corollary* 4.) If the initial displacements and velocities be of the forms

$$y_{l,0}=\eta_i\sin\frac{il\pi}{n+1}, \quad y'_{l,0}=\eta'_i\sin\frac{il\pi}{n+1},$$

i being any integer from 1 to n and η_i, η'_i being constants, we shall have $Y_{i,0}=\frac{n+1}{2}\eta_i$, $Y'_{i,0}=\frac{n+1}{2}\eta'_i$, and all the other values of $Y_{k,0}$ and $Y'_{k,0}$ will vanish; therefore, in this case, the general expression (6) reduces itself to the following:

$$y_{l,t}=\sin\frac{il\pi}{n+1}\left(\eta_i+\eta'_i\int_0^t dt\right)\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}\right).$$

7. (*Corollary* 5.) By taking

$$\eta_i=\frac{2}{n+1}\Sigma^n_{(l)1}y_{l,0}\sin\frac{il\pi}{n+1}, \quad \eta'_i=\frac{2}{n+1}\Sigma^n_{(l)1}y'_{l,0}\sin\frac{il\pi}{n+1},$$

we may express any arbitrary initial displacements $y_{l,0}$ and velocities $y'_{l,0}$ by developements of the forms

$$y_{l,0}=\Sigma^n_{(i)1}\eta_i\sin\frac{il\pi}{n+1}, \quad y'_{l,0}=\Sigma^n_{(i)1}\eta'_i\sin\frac{il\pi}{n+1};$$

if then we had found otherwise the expression given in the last corollary for $y_{l,t}$, corresponding to the particular suppositions

$$y_{l,0}=\eta_i\sin\frac{il\pi}{n+1}, \quad y'_{l,0}=\eta'_i\sin\frac{il\pi}{n+1},$$

we might have thence deduced the general expression (6) under the form

$$y_{l,t}=\Sigma^n_{(i)1}\sin\frac{il\pi}{n+1}\left(\eta_i+\eta'_i\int_0^t dt\right)\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}\right). \tag{8}$$

8. (*Corollary* 6.) If we write, according to a notation already employed,

$$r_i=2a\sin\frac{\frac{1}{2}i\pi}{n+1}$$

and introduce two new constants, B_i and β_i, such that

$$B_i\cos\beta_i=\eta_i, \quad B_i\sin\beta_i=\eta'_i r_i^{-1},$$

we may employ this other expression

$$y_{l,t} = \Sigma_{(i)1}^{n} B_i \cos(tr_i - \beta_i) \sin \frac{il\pi}{n+1}; \tag{9}$$

in which r_i is a known function of the index (or integer) i, but B_i and β_i are in general arbitrary functions of that index.

9. (*Corollary* 7.) The general system of total displacements $y_{l,t}$ may be considered as the sum of n component systems of partial displacements,

$$y_{l,t} = B_i \cos(tr_i - \beta_i) \sin \frac{il\pi}{n+1},$$

of which each is separately possible, & of which all are mutually superposed. Each system of displacements, by itself, may be called a *simple movement* or *mode of simple vibration*. It corresponds to some one integer value of i (from 1 to n inclusive), and to one corresponding *periodic time*

$$\frac{2\pi}{r_i} = \frac{\pi}{a} \operatorname{cosec} \frac{\frac{1}{2}i\pi}{n+1};$$

involving also two arbitrary constants, or arbitrary functions of i, namely η_i and η_i', or B_i and β_i, which latter may be called *constants of amplitude and of epoch.*

10. (*Corollary* 8.) In any one such simple movement, corresponding to any one value of i, the displacements all attain extreme values when $t = r_i^{-1}\beta_i$; and these simultaneous and extreme values are all expressed by the formula

$$y_{l, r_i^{-1}\beta_i} = B_i \sin \frac{il\pi}{n+1}.$$

If $i = 1$, these extreme displacements (relatively to t) increase in magnitude with l from $l = 1$ till $l = \frac{n+1}{2}$ if n be odd, or till $l = \frac{n}{2}$ if n be even; and afterwards decrease from $l = \frac{n+1}{2}$ or from $l = \frac{n+2}{2}$ to $l = n$; being all of the same sign as B_1. But if $i = 2$, the displacements $B_2 \sin \frac{2l\pi}{n+1}$ increase in magnitude with l from $l = 1$ till $l = \frac{n+1}{4}$, or $= \frac{n}{4}$, or $= \frac{n-1}{4}$, or $= \frac{n+2}{4}$, according as n is of the form $4\nu - 1$, or 4ν, or $4\nu + 1$, or $4\nu + 2$, ν being an integer & $\not< 0$; they afterwards decrease and become negative when l is between $\frac{n+1}{2}$ and $n+1$, if $B_2 > 0$.

In general the formula $B_i \sin \frac{il\pi}{n+1}$ may be considered as corresponding to $i - 1$ *nodes* $N_1, N_2, \ldots N_{i-1}$, for which l (though integer for each actual particle) is supposed to receive the (perhaps fractional) values $\frac{(n+1)}{i}, \frac{2(n+1)}{i}, \ldots \frac{(i-1)(n+1)}{i}$. Between P_0 and N_1 the sine of $\frac{il\pi}{n+1}$ is positive; between N_1 and N_2 negative; and so on alternately. The i intermediate points $V_1, V_2, \ldots V_{i-1}, V_i$ for which $l = \frac{n+1}{2i}, \frac{3(n+1)}{2i}, \ldots \frac{(2i-3)(n+1)}{2i}, \frac{(2i-1)(n+1)}{2i}$ are *venters* or points of extreme excursion, alternately positive and negative (if $B_i > 0$).

11. (*Example* 1.) If $n=1$, so that there is but one moveable particle P_1, attracted equally to two fixed centres, P_0 and P_2, and slightly and transversally displaced from the middle of the line ($=2$) which joins them; then the variable displacement of this particle P_1 at the time t is represented by the formula

$$y_{1,t}=B_1\cos(tr_1-\beta_1),$$

because $\sin\frac{\pi}{2}=1$; in which formula $r_1=2a\sin\frac{\pi}{4}=a\sqrt{2}$. The extreme displacement is B_1 and the law is that of the cycloidal pendulum.

$$\eta_1=y_{1,0}=B_1\cos\beta_1,\quad \eta_1'=y_{1,0}'=r_1B_1\sin\beta_1:$$

and

$$y_{1,t}=y_{1,0}\cos(at\sqrt{2})+y_{1,0}'\cdot\frac{\sin(at\sqrt{2})}{a\sqrt{2}}.$$

12. (*Example* 2.) If $n=2$, so that there are two moveable particles P_1 and P_2 between two fixed particles P_0 and P_3; then the variable displacements $y_{1,t}$ and $y_{2,t}$ of P_1 and P_2 are

$$y_{1,t}=\frac{\sqrt{3}}{2}\{B_1\cos(tr_1-\beta_1)+B_2\cos(tr_2-\beta_2)\},\quad y_{2,t}=\frac{\sqrt{3}}{2}\{B_1\cos(tr_1-\beta_1)-B_2\cos(tr_2-\beta_2)\},$$

because

$$\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}=\sin\frac{2\pi}{3}=-\sin\frac{4\pi}{3},$$

also

$$r_1=2a\sin\frac{\pi}{6}=a,\quad r_2=2a\sin\frac{\pi}{3}=a\sqrt{3}.$$

The two simple modes of vibration, which are here superposed, are

1st $$y_{1,t}=y_{2,t}=\frac{\sqrt{3}}{2}B_1\cos(at-\beta_1),$$

and

2nd $$y_{1,t}=-y_{2,t}=\frac{\sqrt{3}}{2}B_2\cos(at\sqrt{3}-\beta_2).$$

The periodic time of the first mode is greater than that of the 2nd in the ratio of $\sqrt{3}$ to 1. The displacements of the two particles P_1 and P_2 are equal and on the same side in the 1st mode, but equal and opposite in the 2nd.

$$B_1\cos\beta_1=\eta_1=\frac{1}{\sqrt{3}}(y_{1,0}+y_{2,0}),\quad B_1\sin\beta_1=a^{-1}\eta_1'=\frac{1}{a\sqrt{3}}(y_{1,0}'+y_{2,0}'),$$

$$B_2\cos\beta_2=\eta_2=\frac{1}{\sqrt{3}}(y_{1,0}-y_{2,0}),\quad B_2\sin\beta_2=(a\sqrt{3})^{-1}\eta_2'=\frac{1}{3a}(y_{1,0}'-y_{2,0}').$$

13. (*Example* 3.) If $n=3$, then

$$y_{1,t}=\sqrt{\tfrac{1}{2}}B_1\cos(tr_1-\beta_1)+B_2\cos(tr_2-\beta_2)+\sqrt{\tfrac{1}{2}}B_3\cos(tr_3-\beta_3),$$

$$y_{2,t}=\quad B_1\cos(tr_1-\beta_1)-B_3\cos(tr_3-\beta_3),$$

$$y_{3,t}=\sqrt{\tfrac{1}{2}}B_1\cos(tr_1-\beta_1)-B_2\cos(tr_2-\beta_2)+\sqrt{\tfrac{1}{2}}B_3\cos(tr_3-\beta_3);$$

$$r_1=2a\sin\frac{\pi}{8}=a\sqrt{2-\sqrt{2}},\quad r_2=2a\sin\frac{2\pi}{8}=a\sqrt{2},\quad r_3=2a\sin\frac{3\pi}{8}=a\sqrt{2+\sqrt{2}};$$

$$B_1 \cos \beta_1 = \tfrac{1}{2} (\sqrt{\tfrac{1}{2}} y_{1,0} + y_{2,0} + \sqrt{\tfrac{1}{2}} y_{3,0}), \quad B_1 \sin \beta_1 = \frac{1}{2r_1} (\sqrt{\tfrac{1}{2}} y'_{1,0} + y'_{2,0} + \sqrt{\tfrac{1}{2}} y'_{3,0}),$$

$$B_2 \cos \beta_2 = \tfrac{1}{2} (y_{1,0} - y_{3,0}), \quad B_2 \sin \beta_2 = \frac{1}{2r_2} (y'_{1,0} - y'_{3,0}),$$

$$B_3 \cos \beta_3 = \tfrac{1}{2} (\sqrt{\tfrac{1}{2}} y_{1,0} - y_{2,0} + \sqrt{\tfrac{1}{2}} y_{3,0}), \quad B_3 \sin \beta_3 = \frac{1}{2r_3} (\sqrt{\tfrac{1}{2}} y'_{1,0} - y'_{2,0} + \sqrt{\tfrac{1}{2}} y'_{3,0});$$

there are three simple modes of vibration, with periods which are respectively $\frac{2\pi}{r_1}, \frac{2\pi}{r_2}, \frac{2\pi}{r_3}$, that is $\frac{\pi}{a} \operatorname{cosec} \frac{\pi}{8}, \frac{\pi}{a} \operatorname{cosec} \frac{2\pi}{8}, \frac{\pi}{a} \operatorname{cosec} \frac{3\pi}{8}$, or finally $\frac{\pi}{a} \frac{2}{\sqrt{2-\sqrt{2}}}, \frac{\pi}{a}\sqrt{2}, \frac{\pi}{a} \frac{2}{\sqrt{2+\sqrt{2}}}$; they may also be thus written $\frac{\pi\sqrt{2}}{a}\sqrt{2+\sqrt{2}}, \frac{\pi\sqrt{2}}{a}, \frac{\pi\sqrt{2}}{a}\sqrt{2-\sqrt{2}}$; in the 1st or slowest mode, the 3 displacements have all the same sign & are proportional to $\sqrt{\tfrac{1}{2}}$, 1, $\sqrt{\tfrac{1}{2}}$, that is, to 1, $\sqrt{2}$, 1, the second particle being a *venter*; in the 2nd mode, the 1st & 3rd displacements are equal and opposite, & the 2nd displacement vanishes, so that the middle particle P_2 remains at rest and forms what is called a *node*, the first and third particles being venters; in the 3rd or quickest mode, the 1st and 3rd displacements are equal and of a common sign, while the 2nd is of an opposite sign and greater in the ratio of $\sqrt{2}$ to 1; so that, in this mode there may be considered to be *two nodes*, one between P_1 and P_2 but nearer to P_1 and the other between P_2 and P_3 but nearer to P_3; in fact the abscissae of these two nodes are $\frac{4}{3}$ and $\frac{8}{3}$ respectively, the abscissae of the 3 vibrating particles P_1, P_2, P_3 being 1, 2, 3; and in the same third mode, there are *three venters* of which the first and third have for abscissae $\frac{2}{3}$ and $\frac{10}{3}$, so that they are near P_1 & P_3, but between P_0 and P_1 and P_3 and P_4 respectively, while the second venter coincides with the particle P_2.

14. (*Corollary* 9.) If there be but one particle P_j which at the time 0 has any displacement or velocity, we shall have

$$B_i \cos \beta_i = \frac{2y_{j,0}}{n+1} \sin \frac{ij\pi}{n+1}, \quad B_i \sin \beta_i = \frac{2r_i^{-1}}{n+1} y'_{j,0} \sin \frac{ij\pi}{n+1};$$

and therefore

$$B_i = \frac{2}{n+1} \sqrt{y_{j,0}^2 + r_i^{-2} y'^2_{j,0}} \sin \frac{ij\pi}{n+1}, \quad \tan \beta_i = \frac{r_i^{-1} y'_{j,0}}{y_{j,0}}.$$

15. (*Example* 4.) If $n=2$, then

$$B_i \cos \beta_i = \tfrac{2}{3} y_{j,0} \sin \frac{ij\pi}{3}, \quad B_i \sin \beta_i = \tfrac{2}{3} r_i^{-1} y'_{j,0} \sin \frac{ij\pi}{3};$$

therefore, more particularly,

$$B_1 \cos \beta_1 = \tfrac{2}{3} y_{j,0} \sin \frac{j\pi}{3}, \quad B_1 \sin \beta_1 = \frac{2}{3a} y'_{j,0} \sin \frac{j\pi}{3},$$

$$B_2 \cos \beta_2 = \tfrac{2}{3} y_{j,0} \sin \frac{2j\pi}{3}, \quad B_2 \sin \beta_2 = \frac{2}{3a\sqrt{3}} y'_{j,0} \sin \frac{2j\pi}{3}.$$

Still more particularly, if $j=1$,

$$B_1 \cos \beta_1 = \frac{1}{\sqrt{3}} y_{1,0}, \quad B_1 \sin \beta_1 = \frac{1}{a\sqrt{3}} y'_{1,0},$$

$$B_2 \cos \beta_2 = \frac{1}{\sqrt{3}} y_{1,0}, \quad B_2 \sin \beta_2 = \frac{1}{3a} y'_{1,0},$$

and, if $j=2$,

$$B_1\cos\beta_1=\frac{1}{\sqrt{3}}y_{2,0}, \qquad B_1\sin\beta_1=\frac{1}{a\sqrt{3}}y'_{2,0},$$

$$B_2\cos\beta_2=-\frac{1}{\sqrt{3}}y_{2,0}, \quad B_2\sin\beta_2=-\frac{1}{3a}y'_{2,0}.$$

In this manner we determine the coefficients of the two coexisting simple modes of vibration in the system $P_0\,P_1\,P_2\,P_3$, corresponding to any initial displacement and velocity of P_1 alone, or of P_2 alone.

16. (*Example* 5.) In the system $P_0\,P_1\,P_2\,P_3\,P_4$, $n=3$ and

$$B_i\cos\beta_i=\tfrac{1}{2}y_{j,0}\sin\frac{ij\pi}{4}, \quad B_i\sin\beta_i=\tfrac{1}{2}r_i^{-1}y'_{j,0}\sin\frac{ij\pi}{4};$$

that is,

$$B_1\cos\beta_1=\tfrac{1}{2}y_{j,0}\sin\frac{j\pi}{4}, \qquad B_1\sin\beta_1=\frac{\sqrt{2}+\sqrt{2}}{2a\sqrt{2}}y'_{j,0}\sin\frac{j\pi}{4},$$

$$B_2\cos\beta_2=\tfrac{1}{2}y_{j,0}\sin\frac{2j\pi}{4}, \quad B_2\sin\beta_2=\frac{1}{2a\sqrt{2}}y'_{j,0}\sin\frac{2j\pi}{4},$$

$$B_3\cos\beta_3=\tfrac{1}{2}y_{j,0}\sin\frac{3j\pi}{4}, \quad B_3\sin\beta_3=\frac{\sqrt{2}-\sqrt{2}}{2a\sqrt{2}}y'_{j,0}\sin\frac{3j\pi}{4}.$$

17. (*Corollary* 10.) By last corollary or by corollary 3, article 5, the whole effect at the time t on the particle P_l, of the initial state of P_j, is

$$\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)f(j,l,t),$$

in which

$$f(j,\ l,\ t)=\frac{2}{n+1}\Sigma_{(i)\,1}^{n}\sin\frac{ij\pi}{n+1}\sin\frac{il\pi}{n+1}\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}\right.$$

If $\frac{n+1}{2}$ be much larger than j or l, this finite sum is nearly = the definite integral

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta).$$

If then we consider the case of a very numerous system of particles, we shall have, *nearly*, for those which are much *nearer to one end than to the middle*,

$$y_{l,t}=\frac{4}{\pi}\Sigma_{(j)\,1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta); \qquad (10)$$

$y_{j,0}$ and $y'_{j,0}$ being supposed $=0$ unless $\frac{2j}{n+1}$ be small; and this expression corresponds *rigorously* to the limit $n=\infty$, j and l remaining finite.

18. (*Corollary* 11.) If nothing be neglected, we have

$$4\sin\alpha\sin\beta\cos\gamma=2(\cos\overline{\alpha-\beta}-\cos\overline{\alpha+\beta})\cos\gamma$$
$$=\cos(\alpha-\beta+\gamma)+\cos(\alpha-\beta-\gamma)-\cos(\alpha+\beta-\gamma)-\cos(\alpha+\beta+\gamma);$$

therefore*

$$f(j,\, l,\, t)=\frac{1}{2(n+1)}\Sigma^{n}_{(i)\,1}\left\{\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}+\frac{i(l-j)\pi}{n+1}\right)\right.$$
$$+\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}-\frac{i(l-j)\pi}{n+1}\right)$$
$$-\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}+\frac{i(l+j)\pi}{n+1}\right)$$
$$\left.-\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}-\frac{i(l+j)\pi}{n+1}\right)\right\}$$

and therefore

$$y_{l,t}=\frac{1}{2(n+1)}\Sigma^{n}_{(j)\,1}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\Sigma^{n}_{(i)\,1}\left\{\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}+\frac{i(l-j)\pi}{n+1}\right)\right.$$
$$+\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}-\frac{i(l-j)\pi}{n+1}\right)-\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}+\frac{i(l+j)\pi}{n+1}\right)$$
$$\left.-\cos\left(2at\sin\frac{\frac{1}{2}i\pi}{n+1}-\frac{i(l+j)\pi}{n+1}\right)\right\} \tag{11}$$

If, now, j, l and n all tend to ∞ but so that $\frac{l+j}{n+1}$ is nearly $=1$, and that $\frac{l-j}{n+1}$ is nearly $=0$, or in other words so that $\frac{2l}{n+1}$ and $\frac{2j}{n+1}$ are each nearly $=1$, though $2l-n-1$ and $2j-n-1$ may both be large numbers positive or negative; in short, if we consider only particles P_j and P_l which are much *nearer to the middle than to the ends* of the very long line P_0P_{n+1}, although they are not necessarily near to one another; we may then neglect those sums of rapidly fluctuating cosines which involve $\frac{i(l+j)\pi}{n+1}$† and may transform the other sums into definite integrals by making $\frac{\frac{1}{2}i\pi}{n+1}=\theta$; and thus we obtain, as a very approximate formula,

$$y_{l,t}=\frac{2}{\pi}\Sigma^{\infty}_{(h)-\infty}\left(y_{l+h,0}+y'_{l+h,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}}d\theta\cos 2h\theta\cos(2at\sin\theta). \tag{12}$$

Accordingly this expression gives

$$y_{l\pm1,t}=\frac{2}{\pi}\Sigma^{\infty}_{(h)-\infty}\left(y_{l+h,0}+y'_{l+h,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}}d\theta\cos(2h\theta\mp 2\theta)\cos(2at\sin\theta),$$

---

* [$f(j,\, l,\, t)=\{J_{2(l-j)}(2at)-J_{2(l+j)}(2at)\}$.

The following note appears on the opposite page of the manuscript. "It is remarkable that this function $f(j, l, t)$ is symmetric relatively to j and l, even if n be not large. Indeed each part, corresponding to any one value of i, or to any one mode of simple vibration, is symmetric also. Thus, the effect (and even that part of the effect which corresponds to any given number i of *venters*) of the initial state of P_j or the state of P_l at the time t, is the same as the effect of a like initial state of P_l on the state of P_j at the time t; even though P_j may be near one extremity and P_l near the middle of the system. It will be important to try whether a similar result holds good for other attracting or repelling systems."]

† [This can be inferred from the value given in the previous note for $f(j, l, t)$. Hamilton's paper on Fluctuating Functions did not appear until 1843, *Trans. R.I.A.* XIX, pp. 264–321, although there is a short note in *Proceedings R.I.A.* I (1841), pp. 475–477.]

therefore

$$y_{l+1,t}+y_{l-1,t}=\frac{4}{\pi}\Sigma_{(h)=-\infty}^{\infty}\left(y_{l-h,0}+y'_{l-h,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\cos 2h\theta\cos 2\theta\cos(2at\sin\theta)$$

and

$$y_{l+1,t}-2y_{l,t}+y_{l-1,t}=-\frac{8}{\pi}\Sigma_{(h)=-\infty}^{\infty}\left(y_{l-h,0}+y'_{l-h,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\cos 2h\theta\sin\theta^2\cos(2at\sin\theta);$$

the function (12) therefore satisfies rigorously and indefinitely the equation in mixed differences (1), and is the complete integral of that indefinite equation because it reproduces the arbitrary initial data $y_{l,0}$ and $y'_{l,0}$, as the values of $y_{l,t}$ and $y'_{l,t}$, for $t=0$.

19. (*Remark.*) Thus the expression (10) corresponds rigorously to the transversal vibrations of an indefinite line of equal particles extending in one direction from the fixed point P_0; or if in both directions, then so that $y_{-l,t}=-y_{l,t}$; and the expression (12) corresponds rigorously to the transversal vibrations of an indefinite line of particles extending in two opposite directions, & having no point fixed.

As applied to the theory of light, the expression (12) seems adapted to illustrate the *internal propagation* of luminiferous vibration, and the expression (10) to illustrate the *reflexion* of such vibration. And this expression (10) may be thus written

$$y_{l,t}=\frac{2}{\pi}\Sigma_{(j)=-\infty}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta), \tag{13}$$

if we consider $y_{-j,0}$ and $y'_{-j,0}$ as equal to $-y_{j,0}$ and $-y'_{j,0}$.

20. When n is finite, if we put for abridgement

$$\phi=\frac{\pi}{2(n+1)},\quad\text{and therefore}\quad r_i=2a\sin i\phi,$$

we have, for any simple vibration, the formula

$$y_{l,t}=B_i\cos(2at\sin i\phi-\beta_i)\sin 2il\phi,$$

which may be put under the form

$$y_{l,t}=\tfrac{1}{2}B_i\sin(2il\phi-2at\sin i\phi+\beta_i)$$
$$+\tfrac{1}{2}B_i\sin(2il\phi+2at\sin i\phi-\beta_i).$$

It may therefore be considered as the sum (or resultant) of two *conjugate* simple movements, of which the *phases* are respectively $2il\phi-2at\sin i\phi+\beta_i$ and $2il\phi+2at\sin i\phi-\beta_i$; the *amplitudes* are each $=\frac{1}{2}B_i$; and the *velocities of transmission of phase* (from particle to particle) are respectively $\frac{a\sin i\phi}{i\phi}$ and $\frac{-a\sin i\phi}{i\phi}$; that is, they are equal in amount but opposite in direction. The positive velocity is $<a$ and $>\frac{2a}{\pi}$, because $i\phi>0$ but $<\frac{\pi}{2}$. The *epochs* β_i and $-\beta_i$ are, in like manner, equal and opposite.

21. Each of the two conjugate simple movements, described in the last article, satisfies the indefinite equation in mixed differences (1) whatever i and ϕ may be; but the advantage of combining them, & of supposing $\phi=\frac{\pi}{2(n+1)}$, is that we thereby satisfy also the conditions

$y_{0,t}=0$, $y_{n+1,t}=0$, for all integer values of i. If we omit the last condition ($y_{n+1,t}=0$) we may take any values for i and ϕ; but we must still combine the two conjugate formulae. If we omit both of the extreme conditions, we may use either formula alone, and may assign any value to i and ϕ $\left(\text{between } \phi=0 \text{ and } \phi=\frac{\pi}{2}\right)$.

22. In this manner then we might perceive that at the limit considered in article **17**, which corresponds to the integration of the original equation (1), subject only to the one condition $y_{0,t}=0$, we may write

$$y_{l,t}=\int_{\theta_1}^{\theta_2} d\theta\, B_\theta \cos(2at\sin\theta-\beta_\theta)\sin 2l\theta, \qquad (14)$$

the limits θ_1 and θ_2 being arbitrary quantities and B_θ and β_θ being arbitrary functions of θ. But in order to reproduce in this case the initial values of $y_{l,t}$ and $y'_{l,t}$ we must (if possible) determine these arbitraries so as to have

$$y_{l,0}=\int_{\theta_1}^{\theta_2} d\theta\, B_\theta \cos\beta_\theta \sin 2l\theta, \quad y'_{l,0}=\int_{\theta_1}^{\theta_2} 2a\, d\theta\, B_\theta \sin\beta_\theta \sin\theta \sin 2l\theta;$$

and these conditions accordingly are satisfied, as in the formula (13), by supposing

$$\theta_1=0, \quad \theta_2=\frac{\pi}{2}, \quad B_\theta\cos\beta_\theta=\frac{2}{\pi}\,\Sigma_{(j)-\infty}^{\infty}\, y_{j,0}\sin 2j\theta,$$

$$y_{-j,0}=-y_{j,0}, \quad y'_{-j,0}=-y'_{j,0}, \quad B_\theta\sin\beta_\theta=\frac{1}{a\pi}\,\Sigma_{(j)-\infty}^{\infty}\, y'_{j,0}\sin 2j\theta \operatorname{cosec}\theta.$$

23. We might also, in like manner, have perceived, that at the other limit considered in article **18**, corresponding merely to the indefinite integration of the equation (1), we may write

$$y_{l,t}=\int_{\theta_1}^{\theta_2} d\theta\, B_\theta \sin(2l\theta-2at\sin\theta+\beta_\theta)+\int_{\iota_1}^{\iota_2} d\iota\, C_\iota \sin(2l\iota+2at\sin\iota+\gamma_\iota), \qquad (15)$$

the limits θ_1, θ_2 and ι_1, ι_2 being arbitrary quantities, while B_θ, β_θ are arbitrary functions of θ, and C_ι, γ_ι are arbitrary functions of ι. To reproduce the initial values we must endeavour to determine these arbitrary quantities and functions, so as to have

$$y_{l,0}=\int_{\theta_1}^{\theta_2} d\theta\, B_\theta \sin(2l\theta+\beta_\theta)+\int_{\iota_1}^{\iota_2} d\iota\, C_\iota \sin(2l\iota+\gamma_\iota),$$

$$y'_{l,0}=-2a\int_{\theta_1}^{\theta_2} d\theta\, B_\theta \cos(2l\theta+\beta_\theta)\sin\theta+2a\int_{\iota_1}^{\iota_2} d\iota\, C_\iota \cos(2l\iota+\gamma_\iota)\sin\iota;$$

conditions which may be satisfied, as in the formula (12), by supposing

$$B_\theta\cos\beta_\theta=\frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\sin 2h\theta-\frac{y'_{h,0}\cos 2h\theta}{2a\sin\theta}\right),$$

$$B_\theta\sin\beta_\theta=\frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\cos 2h\theta+\frac{y'_{h,0}\sin 2h\theta}{2a\sin\theta}\right),$$

$$C_\iota\cos\gamma_\iota=\frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\sin 2h\iota+\frac{y'_{h,0}\cos 2h\iota}{2a\sin\iota}\right),$$

$$C_\iota\sin\gamma_\iota=\frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\cos 2h\iota-\frac{y'_{h,0}\sin 2h\iota}{2a\sin\iota}\right),$$

and $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, $\iota_1 = 0$, $\iota_2 = \frac{\pi}{2}$. In fact these last suppositions give

$$B_\theta \sin(2l\theta + \beta_\theta) = \frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\cos(2h\theta - 2l\theta) + \frac{y'_{h,0}}{2a\sin\theta}\sin(2h\theta - 2l\theta)\right),$$

$$C_\iota \sin(2l\iota + \gamma_\iota) = \frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\cos(2h\iota - 2l\iota) - \frac{y'_{h,0}}{2a\sin\iota}\sin(2h\iota - 2l\iota)\right)$$

and therefore

$$B_\theta \sin(2l\theta + \beta_\theta) + C_\theta \sin(2l\theta + \gamma_\theta) = \frac{2}{\pi}\Sigma_{(h)-\infty}^{\infty}\, y_{h,0}\cos(2h\theta - 2l\theta),$$

so that, performing on this the operation $\int_0^{\frac{\pi}{2}} d\theta$ we get $y_{l,0}$; also

$$B_\theta \cos(2l\theta + \beta_\theta) = \frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\sin(2h\theta - 2l\theta) - \frac{y'_{h,0}}{2a\sin\theta}\cos(2h\theta - 2l\theta)\right),$$

$$C_\iota \cos(2l\iota + \gamma_\iota) = \frac{1}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{h,0}\sin(2h\iota - 2l\iota) + \frac{y_{h,0}}{2a\sin\iota}\cos(2h\iota - 2l\iota)\right),$$

and therefore

$$2a\sin\theta\{-B_\theta\cos(2l\theta + \beta_\theta) + C_\theta\cos(2l\theta + \gamma_\theta)\} = \frac{2}{\pi}\Sigma_{(h)-\infty}^{\infty}\, y'_{h,0}\cos(2h\theta - 2l\theta),$$

so that the operation $\int_0^{\frac{\pi}{2}} d\theta$, performed on this, reduces it to $y'_{l,0}$: the initial values are therefore reproduced. At the same time, the expression (15) becomes

$$\begin{aligned} y_{l,t} &= \int_0^{\frac{\pi}{2}} d\theta\,[\cos(2at\sin\theta)\{B_\theta\sin(2l\theta + \beta_\theta) + C_\theta\sin(2l\theta + \gamma_\theta)\} \\ &\qquad\qquad + \sin(2at\sin\theta)\{-B_\theta\cos(2l\theta + \beta_\theta) + C_\theta\cos(2l\theta + \gamma_\theta)\}] \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\Sigma_{(h)-\infty}^{\infty}\cos(2h\theta - 2l\theta)\left(y_{h,0}\cos(2at\sin\theta) + y'_{h,0}\frac{\sin(2at\sin\theta)}{2a\sin\theta}\right) \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\Sigma_{(h)-\infty}^{\infty}\cos(2h\theta - 2l\theta)\left(y_{h,0} + y'_{h,0}\int_0^t dt\right)\cos(2at\sin\theta) \\ &= \frac{2}{\pi}\Sigma_{(h)-\infty}^{\infty}\left(y_{l+h,0} + y'_{l+h,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\cos 2h\theta\cos(2at\sin\theta), \end{aligned}$$

so that the formula (12) is re-deduced.

24. One element in the solution of the problem of article **1** has been the theorem that

$$\Sigma_{(k)1}^{n}\sin\frac{jk\pi}{n+1}\sin\frac{lk\pi}{n+1} = 0, \text{ or } = \frac{n+1}{2},$$

according as j and l, being both integer numbers > 0 & $< n+1$, are unequal or equal to each other. As we shall have several analogous summations to perform in these researches, it may be well to give here the process of proof in full.

The equation $2\sin\alpha\cos(2k\alpha + \beta) = \sin(2k\alpha + \alpha + \beta) - \sin(2k\alpha - \alpha + \beta)$ gives, when it is summed with reference to k,

$$2\sin\alpha\,\Sigma_{(k)k_1}^{k_2}\cos(2k\alpha + \beta) = \sin(2k_2\alpha + \alpha + \beta) - \sin(2k_1\alpha - \alpha + \beta).$$

Let
$$\alpha = \alpha_1 \pm \alpha_2, \quad \beta = 0,$$

then
$$\Sigma^{k_2}_{(k)\,k_1} \cos (2k\alpha_1 \pm 2k\alpha_2) = \frac{\sin (2k_2+1)(\alpha_1 \pm \alpha_2) - \sin (2k_1-1)(\alpha_1 \pm \alpha_2)}{2 \sin (\alpha_1 \pm \alpha_2)};$$

$\because$
$$\Sigma^{k_2}_{(k)\,k_1} \cos 2k\alpha_1 \cos 2k\alpha_2 = \frac{\sin (2k_2+1)(\alpha_1 - \alpha_2) - \sin (2k_1-1)(\alpha_1 - \alpha_2)}{4 \sin (\alpha_1 - \alpha_2)}$$
$$+ \frac{\sin (2k_2+1)(\alpha_1 + \alpha_2) - \sin (2k_1-1)(\alpha_1 + \alpha_2)}{4 \sin (\alpha_1 + \alpha_2)}$$

and
$$\Sigma^{k_2}_{(k)\,k_1} \sin 2k\alpha_1 \sin 2k\alpha_2 = \frac{\sin (2k_2+1)(\alpha_1 - \alpha_2) - \sin (2k_1-1)(\alpha_1 - \alpha_2)}{4 \sin (\alpha_1 - \alpha_2)}$$
$$- \frac{\sin (2k_2+1)(\alpha_1 + \alpha_2) - \sin (2k_1-1)(\alpha_1 + \alpha_2)}{4 \sin (\alpha_1 + \alpha_2)}$$

Hence

$$\Sigma^{n}_{(k)\,1} \sin 2k\alpha_1 \sin 2k\alpha_2$$
$$= \frac{\sin (2n+1)(\alpha_1 - \alpha_2) - \sin (\alpha_1 - \alpha_2)}{4 \sin (\alpha_1 - \alpha_2)} - \frac{\sin (2n+1)(\alpha_1 + \alpha_2) - \sin (\alpha_1 + \alpha_2)}{4 \sin (\alpha_1 + \alpha_2)}$$
$$= \frac{\sin (\alpha_1 + \alpha_2) \sin (2n+1)(\alpha_1 - \alpha_2) - \sin (\alpha_1 - \alpha_2) \sin (2n+1)(\alpha_1 + \alpha_2)}{4 \sin (\alpha_1 + \alpha_2) \sin (\alpha_1 - \alpha_2)}$$
$$= \frac{\cos (2n\alpha_1 - \overline{2n+2}\alpha_2) - \cos (\overline{2n+2}\alpha_1 - 2n\alpha_2) + \cos (\overline{2n+2}\alpha_1 + 2n\alpha_2) - \cos (2n\alpha_1 + \overline{2n+2}\alpha_2)}{8 \sin (\alpha_1 + \alpha_2) \sin (\alpha_1 - \alpha_2)}$$
$$= \frac{\sin 2n\alpha_1 \sin \overline{2n+2}\alpha_2 - \sin 2n\alpha_2 \sin \overline{2n+2}\alpha_1}{2 (\cos 2\alpha_2 - \cos 2\alpha_1)};$$

this sum therefore vanishes, if $\alpha_1 = \frac{j\pi}{2n+2}$, $\alpha_2 = \frac{l\pi}{2n+2}$, unless $\cos 2\alpha_2 = \cos 2\alpha_1$, that is, in the present question, unless $j = l$. But, for that particular case, the sum may be found by differentiating numerator and denominator relatively to α_1, & then making $\alpha_1 = \alpha_2$; it is $\because$

$$= -\frac{n+1}{2} \cos l\pi \sin \frac{nl\pi}{n+1} \operatorname{cosec} \frac{l\pi}{n+1} = \frac{n+1}{2}.$$

25. The same theorem of summation shows that, in the notation of articles 7 and 8,

$$\Sigma^{n}_{(i)\,1} B_i^2 \cos \beta_i^2 = \Sigma^{n}_{(i)\,1} \eta_i^2 = \frac{2}{n+1} \Sigma^{n}_{(l)\,1} y_{l,0}^2;$$

$$\Sigma^{n}_{(i)\,1} r_i^2 B_i^2 \sin \beta_i^2 = \Sigma^{n}_{(i)\,1} \eta_i'^2 = \frac{2}{n+1} \Sigma^{n}_{(l)\,1} y_{l,0}'^2.$$

It is interesting to calculate also $\Sigma^{n}_{(i)\,1} r_i^2 B_i^2 \cos \beta_i^2 = \Sigma^{n}_{(i)\,1} r_i^2 \eta_i^2$. Since $r_i = 2a \sin \frac{\frac{1}{2} i\pi}{n+1}$, we have $r_i^2 = 2a^2 - 2a^2 \cos \frac{i\pi}{n+1}$; we have therefore only to calculate

$$\left(\frac{n+1}{2}\right)^2 \Sigma^{n}_{(i)\,1} \eta_i^2 \cos \frac{i\pi}{n+1} = \Sigma^{n}_{(i)\,1} \cos \frac{i\pi}{n+1} \left(\Sigma^{n}_{(l)\,1} y_{l,0} \sin \frac{il\pi}{n+1}\right)^2.$$

For this purpose we have to calculate

$$2\,\Sigma^n_{(i)1}\cos\frac{i\pi}{n+1}\sin\frac{il\pi}{n+1}\sin\frac{ji\pi}{n+1}=\Sigma^n_{(i)1}\left(\sin\frac{i\,(j+1)\,\pi}{n+1}+\sin\frac{i\,(j-1)\,\pi}{n+1}\right)\sin\frac{il\pi}{n+1}=0,$$

unless $j=l\pm 1$. But if $j=l\pm 1$, j & l being each >0 and $<n+1$, then the last sum $= n+1$. Hence

$$\Sigma^n_{(i)1}\cos\frac{i\pi}{n+1}\left(\Sigma^n_{(l)1}\,y_{l,0}\sin\frac{il\pi}{n+1}\right)^2=\frac{n+1}{2}\,\Sigma^{n-1}_{(l)1}\,y_{l,0}\,y_{l-1,0};$$

therefore

$$\Sigma^n_{(i)1}\,\eta_i^2\cos\frac{i\pi}{n+1}=\frac{2}{n+1}\,\Sigma^{n-1}_{(l)1}\,y_{l,0}\,y_{l-1,0};$$

and finally, because $y_{n+1,0}=0$,

$$\Sigma^n_{(i)1}\,r_i^2\,B_i^2\cos\beta_i^2=\Sigma^n_{(i)1}\,r_i^2\,\eta_i^2=\frac{4a^2}{n+1}\,\Sigma^n_{(l)1}\,y_{l,0}\,(y_{l,0}-y_{l+1,0}).$$

Hence

$$\Sigma^n_{(i)1}\,r_i^2\,B_i^2=\frac{2}{n+1}\,\Sigma^n_{(l)1}\,\{y'^2_{l,0}+2a^2\,y_{l,0}\,(y_{l,0}-y_{l-1,0})\}.$$

26. (*Examples.*) When $n=1$, then

$$r_1=a\sqrt{2},\quad B_1\cos\beta_1=\eta_1=y_{1,0};\quad r_1\,B_1\sin\beta_1=\eta_1'=y_{1,0}'.$$

When $n=2$, then

$$r_1=a,\quad r_2=a\sqrt{3},\quad B_1^2\cos\beta_1^2+B_2^2\cos\beta_2^2=\tfrac{2}{3}\,(y_{1,0}^2+y_{2,0}^2),$$

$$r_1^2\,B_1^2\sin\beta_1^2+r_2^2\,B_2^2\sin\beta_2^2=\tfrac{2}{3}\,(y'^2_{1,0}+y'^2_{2,0}),$$

$$r_1^2\,B_1^2\cos\beta_1^2+r_2^2\,B_2^2\cos\beta_2^2=\frac{4a^2}{3}\,(y_{1,0}^2+y_{2,0}^2-y_{1,0}y_{2,0}).$$

When $n=3$, then

$$r_1=a\sqrt{2-\sqrt{2}},\quad r_2=a\sqrt{2},\quad r_3=a\sqrt{2+\sqrt{2}},$$

$$B_1^2\cos\beta_1^2+B_2^2\cos\beta_2^2+B_3^2\cos\beta_3^2=\tfrac{1}{2}\,(y_{1,0}^2+y_{2,0}^2+y_{3,0}^2),$$

$$r_1^2\,B_1^2\sin\beta_1^2+r_2^2\,B_2^2\sin\beta_2^2+r_3^2\,B_3^2\sin\beta_3^2=\tfrac{1}{2}\,(y'^2_{1,0}+y'^2_{2,0}+y'^2_{3,0}),$$

$$r_1^2\,B_1^2\cos\beta_1^2+r_2^2B_2^2\cos\beta_2^2+r_3^2\,B_3^2\cos\beta_3^2=a^2\,(y_{1,0}^2+y_{2,0}^2+y_{3,0}^2-y_{1,0}y_{2,0}-y_{2,0}y_{3,0}).$$

27. The non-periodical part of $\Sigma^n_{(l)1}\,y'^2_{l,t}$ is $\dfrac{n+1}{4}\,\Sigma^n_{(i)1}\,r_i^2\,B_i^2$; this non-periodical part is therefore equal to the sum

$$Q=\Sigma^n_{(l)1}\,\{\tfrac{1}{2}y'^2_{l,0}+a^2y_{l,0}\,(y_{l,0}-y_{l+1,0})\}. \tag{16}$$

This part Q appears to be in some sense a measure* of the *quantity of vibration* of the system (the mass of each particle being unity).

Problem II.

28. It is required to apply the general solution of the 1st Problem to the case where, at the time 0, all the displacements & velocities vanish except those of the $j-j_{\prime}+1$ consecutive particles $P_{j_\prime}$, $P_{j_\prime+1}$, ... P_{j-1}, P_j; supposing also that the initial displacements and velocities of these are such as to agree with a simple mode of vibration, in such a manner that, if l be $> j, -1$ and $< j+1$, we have

---

* [Sum of initial kinetic and potential energies.]

$$y_{l,0} = B_k^{\backprime} \cos \beta_k^{\backprime} \sin 2kl\phi, \quad y'_{l,0} = B_k^{\backprime} r_k \sin \beta_k^{\backprime} \sin 2kl\phi,$$

ϕ being $= \dfrac{\frac{1}{2}\pi}{n+1}$, and $r_k = 2a \sin k\phi$; but that

$$y_{l,0} = 0, \quad y'_{l,0} = 0,$$

if l be $< j$, or $> j$.

29. (*Solution.*) The general expressions

$$B_i \cos \beta_i = \eta_i = \frac{2}{n+1} \Sigma_{(l)1}^{n} y_{l,0} \sin 2il\phi,$$

$$r_i B_i \sin \beta_i = \eta'_i = \frac{2}{n+1} \Sigma_{(l)1}^{n} y'_{l,0} \sin 2il\phi,$$

become now

$$\eta_i = \frac{2}{n+1} B_k^{\backprime} \cos \beta_k^{\backprime} \Sigma_{(l)j,}^{j} \sin 2il\phi \sin 2kl\phi,$$

$$\eta'_i = \frac{2r_k}{n+1} B_k^{\backprime} \sin \beta_k^{\backprime} \Sigma_{(l)j,}^{j} \sin 2il\phi \sin 2kl\phi.$$

But, by article **24**,

$$\Sigma_{(l)j}^{j} \sin 2il\phi \sin 2kl\phi = \frac{\sin (2j+1)(i-k)\phi - \sin (2j, -1)(i-k)\phi}{4 \sin (i-k)\phi}$$

$$- \frac{\sin (2j+1)(i+k)\phi - \sin (2j, -1)(i+k)\phi}{4 \sin (i+k)\phi} = \frac{F(j) - F(j, -1)}{2};$$

$$F(j) = \frac{\sin (2j+1)(i-k)\phi}{2 \sin (i-k)\phi} - \frac{\sin (2j+1)(i+k)\phi}{2 \sin (i+k)\phi} = \frac{\sin 2ji\phi \sin 2\overline{j+1}\, k\phi - \sin 2jk\phi \sin 2\overline{j+1} i\phi}{\cos 2k\phi - \cos 2i\phi};$$

therefore

$$2\, \Sigma_{(l)j,}^{j} \sin 2il\phi \sin 2kl\phi = \frac{1}{\cos 2k\phi - \cos 2i\phi} \{\sin 2ji\phi \sin 2\overline{j+1} k\phi - \sin 2jk\phi \sin 2\overline{j+1} i\phi$$

$$- \sin 2j, k\phi \sin 2\overline{j, -1} i\phi + \sin 2j, i\phi \sin 2\overline{j, -1} k\phi\};$$

$$2\, \Sigma_{(l)j,}^{j} (\sin 2kl\phi)^2 = j - j, +1 + \frac{\cos 2j, k\phi \sin 2\overline{j, -1} k\phi - \sin 2jk\phi \cos 2\overline{j+1} k\phi}{\sin 2k\phi}.$$

Hence, by the general formula (9), we have, in the present question,

$$y_{l,t} = \frac{B_k^{\backprime}}{n+1} \Sigma_{(i)1}^{n} \left(\cos \beta_k^{\backprime} \cos (2at \sin i\phi) + \sin \beta_k^{\backprime} \frac{\sin k\phi}{\sin i\phi} \sin (2at \sin i\phi) \right) \frac{\sin 2il\phi}{\cos 2k\phi - \cos 2i\phi}$$

$$\times \{\sin 2ji\phi \sin 2\overline{j+1} k\phi - \sin 2jk\phi \sin 2\overline{j+1} i\phi$$

$$- \sin 2j, k\phi \sin 2\overline{j, -1} i\phi + \sin 2j, i\phi \sin 2\overline{j, -1} k\phi\}; \quad (17)$$

and the part corresponding to $i=k$ is

$$\frac{B'_k}{n+1}\cos(2at\sin k\phi-\beta'_k)\sin 2kl\phi\left\{j-j_,+1+\frac{\cos 2j_,k\phi\sin 2\overline{j_,-1}k\phi-\sin 2jk\phi\cos 2\overline{j+1}k\phi}{\sin 2k\phi}\right\}.$$

30. If $j_,=1$, so that the j first particles $P_1, \ldots P_j$ are all disturbed originally in the way above supposed, then

$$y_{l,t}=\frac{B'_k}{n+1}\Sigma^n_{(i)1}\left(\cos\beta'_k\cos(2at\sin i\phi)+\sin\beta'_k\frac{\sin k\phi}{\sin i\phi}\sin(2at\sin i\phi)\right)$$
$$\times\frac{\sin 2il\phi\{\sin 2ji\phi\sin 2j+1k\phi-\sin 2jk\phi\sin 2j+1i\phi\}}{\cos 2k\phi-\cos 2i\phi}; \qquad (18)$$

and the part corresponding to $i=k$ is

$$\frac{B'_k}{n+1}\cos(2at\sin k\phi-\beta'_k)\sin 2kl\phi\left\{j-\frac{\sin 2jk\phi\cos 2\overline{j+1}k\phi}{\sin 2k\phi}\right\}.$$

31. If j and l be each much smaller than n, so that n is treated as infinite, while j and l, though perhaps large, are finite, then the expression in article **30** becomes a definite integral, namely

$$y_{l,t}=\frac{2}{\pi}B'_k\int_0^{\frac{\pi}{2}}d\theta\left(\cos\beta'_k\cos(2at\sin\theta)+\sin\beta'_k\frac{\sin\alpha}{\sin\theta}\sin(2at\sin\theta)\right)$$
$$\times\frac{\sin 2l\theta\{\sin 2j\theta\sin 2(j+1)\alpha-\sin 2j\alpha\sin 2(j+1)\theta\}}{\cos 2\alpha-\cos 2\theta}; \qquad (19)$$

in which $\alpha=k\phi$. This expression satisfies the equation in mixed differences (1): and gives $y_{l,0}=0$, $y'_{l,0}=0$, if $l>j$; but $y_{l,0}=B'_k\cos\beta'_k\sin 2l\alpha$, $y'_{l,0}=2a\sin\alpha\, B'_k\sin\beta'_k\sin 2l\alpha$, if $l<j+1$, l being a positive integer: it gives also $y_{0,t}=0$.*

32. In the formula (18), making $i\phi=\theta$ and $k\phi=\alpha$, we are led to consider the product

$$\sin 2l\theta\{\sin 2j\theta\sin 2\overline{j+1}\alpha-\sin 2j\alpha\sin 2\overline{j+1}\theta\}$$
$$=\tfrac{1}{2}\sin 2l\theta\{\cos(2j\overline{\alpha-\theta}+2\alpha)-\cos(2j\overline{\alpha+\theta}+2\alpha)-\cos(2j\overline{\theta-\alpha}+2\theta)+\cos(2j\overline{\theta+\alpha}+2\theta)\}$$
$$=\sin 2l\theta\{\sin(\theta+\alpha)\sin(2j+1)(\theta-\alpha)-\sin(\theta-\alpha)\sin(2j+1)(\theta+\alpha)\}$$
$$=\tfrac{1}{2}\sin(\theta+\alpha)\{\cos(\overline{2j-2l+1}\theta-\overline{2j+1}\alpha)-\cos(\overline{2j+2l+1}\theta-\overline{2j+1}\alpha)\}$$
$$-\tfrac{1}{2}\sin(\theta-\alpha)\{\cos(\overline{2j-2l+1}\theta+\overline{2j+1}\alpha)-\cos(\overline{2j+2l+1}\theta+\overline{2j+1}\alpha)\}.$$

If now we suppose, as in article **18**, that $\frac{j+l}{n+1}$ is nearly $=1$ but that $\frac{j-l}{n+1}$ is nearly $=0$, we may neglect those sums which involve cosines of $2(j+l)\theta+$ const., unless they be divided by something which vanishes or becomes very small in the course of the summation; and may reduce (under the sign of summation) the recent product to

$$\cos(2j-2l+1)\theta\,.\cos(2j+1)\alpha\,.\cos\theta\,.\sin\alpha+\sin(2j-2l+1)\theta\,.\sin(2j+1)\alpha\,.\sin\theta\,.\cos\alpha$$
$$=\tfrac{1}{2}\sin(\theta+\alpha)\cos\{(2j-2l+1)\theta-(2j+1)\alpha\}-\tfrac{1}{2}\sin(\theta-\alpha)\cos\{(2j-2l+1)\theta+(2j+1)\alpha\};$$

reserving, however, the part

$$-\tfrac{1}{2}\sin(\theta+\alpha)\cos\{(2j+2l+1)\theta-(2j+1)\alpha\}$$

* $\left[\frac{\sin 2j\theta\sin 2(j+1)\alpha-\sin 2j\alpha\sin 2(j+1)\theta}{\cos 2\alpha-\cos 2\theta}=\Sigma_{s=0}^{j-1}2\sin 2(j-s)\theta\sin 2(j-s)\alpha.\right]$

for special consideration. It may however be instructive, before thus passing to these limits, to resume the formula (18), & to study first the consequences of it in the case when the number n of moveable particles is finite but even, & when $j=\frac{n}{2}$.

33. The expression, $j-\frac{\sin 2j\alpha \cos 2(j+1)\alpha}{\sin 2\alpha}$, which occurs in article **30**, may be put under the form $j+\frac{1}{2}-\frac{\sin 2(2j+1)\alpha}{2\sin 2\alpha}$; it reduces itself therefore to $\frac{n+1}{2}$, if $j=\frac{n}{2}$, $\left(\text{because } \alpha=\frac{\frac{1}{2}k\pi}{n+1}\right)$, that is, if exactly half of the whole number n of moveable particles have such original displacements and velocities as correspond to a simple movement of any one kind; & consequently, in this case, that part of the whole resultant movement which is of the same period is

$$\tfrac{1}{2}B_k^{\backprime}\cos(2at\sin\alpha-\beta_k^{\backprime})\sin 2l\alpha;$$

it is therefore exactly half of that other movement

$$B_k^{\backprime}\cos(2at\sin\alpha-\beta_k^{\backprime})\sin 2l\alpha$$

which would reproduce the initial displacements and velocities, not only for half but for the whole of the system of moveable particles. In other words, we have the theorem:

If the initial state of half the system $P_1, P_2, \ldots P_{\frac{n}{2}}$ correspond to one simple movement

$$\tfrac{1}{2}B_k^{\backprime}\cos(2at\sin\alpha-\beta_k^{\backprime})\sin 2l\alpha,$$

and if the initial state of the other half $P_{\frac{n}{2}+1}, \ldots P_{n-1}, P_n$ correspond to another simple movement of the same period and amplitude, but with an epoch differing by an odd multiple of π,

$$-\tfrac{1}{2}B_k^{\backprime}\cos(2at\sin\alpha-\beta_k^{\backprime})\sin 2l\alpha,$$

in which $\alpha=\frac{\frac{1}{2}k\pi}{n+1}$, then the resultant vibration of the system will be composed entirely of simple movements of other orders, that is, with other periodic times. (The next article will show that the indices i which mark these orders differ by odd numbers from the index k.)

34. To express this resultant vibration, we may employ the formula (18), under the form

$$y_{l,t}=\frac{B_k^{\backprime}}{2j+1}\Sigma_{(i)}\left(\cos\beta_k^{\backprime}\cos(2at\sin\theta)+\sin\beta_k^{\backprime}\frac{\sin\alpha}{\sin\theta}\sin(2at\sin\theta)\right)$$
$$\times\frac{\sin 2l\theta\{\sin(\theta+\alpha)\sin(2j+1)(\theta-\alpha)-\sin(\theta-\alpha)\sin(2j+1)(\theta+\alpha)\}}{\cos 2\alpha-\cos 2\theta}; \qquad (20)$$

in which the part corresponding to $i=k$ is now to be omitted; we have also, now,

$$\theta=\frac{\frac{1}{2}i\pi}{2j+1}, \quad \alpha=\frac{\frac{1}{2}k\pi}{2j+1},$$

$$\because \qquad (2j+1)(\theta\mp\alpha)=\tfrac{1}{2}(i\mp k)\pi;$$

we need therefore attend only to those values of i which differ from k by odd numbers, positive or negative. Let $l=j+h$; then

$$\sin 2l\theta=\sin(2j\theta+2h\theta)=\sin\left(\frac{i\pi}{2}+(2h-1)\theta\right)$$
$$=\sin\left(\frac{(i-k)\pi}{2}+\frac{k\pi}{2}+(2h-1)\theta\right)=\sin\frac{(i-k)\pi}{2}\cos\left(\frac{k\pi}{2}+(2h-1)\theta\right),$$

because $\cos\frac{(i-k)\pi}{2}=0$. For the same reason

$$\left(\sin\frac{(i-k)\pi}{2}\right)^2=1,\quad \sin\frac{(i-k)\pi}{2}\sin\frac{(i+k)\pi}{2}=\cos k\pi;$$

$$\frac{\sin 2l\theta\,\{\sin(\theta+\alpha)\sin(2j+1)(\theta-\alpha)-\sin(\theta-\alpha)\sin(2j+1)(\theta+\alpha)\}}{\cos 2\alpha-\cos 2\theta}$$

$$=\frac{\cos\left(\frac{k\pi}{2}+(2h-1)\theta\right)}{\cos 2\alpha-\cos 2\theta}\{\sin(\theta+\alpha)-\cos k\pi\sin(\theta-\alpha)\}$$

$$=\tfrac{1}{2}\cos\left(\frac{k\pi}{2}+(2h-1)\theta\right)\left(\frac{1}{\sin(\theta-\alpha)}+\frac{(-1)^{k+1}}{\sin(\theta+\alpha)}\right).$$

(If k be odd, this becomes

$$\tfrac{1}{2}(-1)^{\frac{k+1}{2}}\sin(2h-1)\theta\left(\frac{1}{\sin(\theta-\alpha)}+\frac{1}{\sin(\theta+\alpha)}\right);$$

if k be even,

$$\tfrac{1}{2}(-1)^{\frac{k}{2}}\cos(2h-1)\theta\left(\frac{1}{\sin(\theta-\alpha)}-\frac{1}{\sin(\theta+\alpha)}\right).\Big)$$

Hence the expression for $y_{l,t}$ becomes

$$y_{j+h,t}=\frac{\frac{1}{2}B^{\backprime}_k}{2j+1}\Sigma_{(i)}\left(\cos\beta^{\backprime}_k\cos(2at\sin\theta)+\sin\beta^{\backprime}_k\frac{\sin\alpha}{\sin\theta}\sin(2at\sin\theta)\right)$$
$$\times\cos\left(\frac{k\pi}{2}+(2h-1)\theta\right)\left(\frac{1}{\sin(\theta-\alpha)}+\frac{(-1)^{k+1}}{\sin(\theta+\alpha)}\right): \quad (21)$$

in which the summation is to be performed relatively to i for all values of that index which differ from k by odd (integer) differences, being also >0 and $<n+1$; and

$$\theta=\frac{\frac{1}{2}i\pi}{2j+1},\quad \alpha=\frac{\frac{1}{2}k\pi}{2j+1}.$$

35. For example, if $j=1$, $k=1$, we must take $i=2$, $\theta=\frac{\pi}{3}$, $\alpha=\frac{\pi}{6}$, $\theta-\alpha=\frac{\pi}{6}$, $\theta+\alpha=\frac{\pi}{2}$, $\sin(\theta-\alpha)=\frac{1}{2}$, $\sin(\theta+\alpha)=1$, $\sin\alpha=\frac{1}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}$, and

$$y_{1+h,t}=\tfrac{1}{2}B^{\backprime}_1\left(\cos\beta^{\backprime}_1\cos\overline{at\sqrt{3}}+\sin\beta^{\backprime}_1\frac{\sin at\sqrt{3}}{\sqrt{3}}\right)\sin(1-2h)\theta;$$

that is,

$$y_{1,t}=B^{\backprime}_1\frac{\sqrt{3}}{4}\left(\cos\beta^{\backprime}_1\cos\overline{at\sqrt{3}}+\sin\beta^{\backprime}_1\frac{\sin at\sqrt{3}}{\sqrt{3}}\right)=-y_{2,t}.$$

These values accordingly result from the more general formulae of article **12** by supposing

$$y_{1,0}=\frac{\sqrt{3}}{4}B^{\backprime}_1\cos\beta^{\backprime}_1,\quad y'_{1,0}=\overset{a\sqrt{3}}{}\,B^{\backprime}_1\sin\beta^{\backprime}_1,$$

$$y_{2,0}=-\frac{\sqrt{3}}{4}B^{\backprime}_1\cos\beta^{\backprime}_1,\quad y'_{2,0}=-\frac{a\sqrt{3}}{4}B^{\backprime}_1\sin\beta^{\backprime}_1.$$

And if we had supposed $j=1$, $k=2$, we should have been obliged to take $i=1$, $\alpha=\frac{\pi}{3}$, $\theta=\frac{\pi}{6}$, $)-\alpha=-\frac{\pi}{6}$, $\theta+\alpha=\frac{\pi}{2}$,

$$y_{1+h,t}=\tfrac{1}{2}B_{2}^{\backprime}(\cos\beta_{2}^{\backprime}\cos at+\sqrt{3}\sin\beta_{2}^{\backprime}\sin at)\cos(2h-1)\frac{\pi}{6};$$

that is,

$$y_{1,t}=y_{2,t}=\frac{\sqrt{3}}{4}B_{2}^{\backprime}(\cos\beta_{2}^{\backprime}\cos at+\sqrt{3}\sin\beta_{2}^{\backprime}\sin at).$$

Accordingly these expressions result from the formulae of article **12** by supposing

$$y_{1,0}=y_{2,0}=\frac{\sqrt{3}}{4}B_{2}^{\backprime}\cos\beta_{2}^{\backprime},\quad y'_{1,0}=y'_{2,0}=\frac{3a}{4}B_{2}^{\backprime}\sin\beta_{2}^{\backprime}.$$

36. Whatever j may be, if we take k odd and $=2\kappa-1$, we must take i even, and of the form 2ι; κ and ι being each some one of the integers 1, 2, … j. Hence, in this case,

$$\begin{aligned}y_{j+h,t}=\frac{2(-1)^{\kappa}}{2j+1}B_{2\kappa-1}^{\backprime}\Sigma_{(\iota)1}^{j}\Bigg\{&\cos\beta_{2\kappa-1}^{\backprime}\cos\left(2at\sin\frac{\iota\pi}{2j+1}\right)\sin\frac{\iota\pi}{2j+1}\\&+\sin\beta_{2\kappa-1}^{\backprime}\sin\left(2at\sin\frac{\iota\pi}{2j+1}\right)\sin\frac{(\kappa-\frac{1}{2})\pi}{2j+1}\Bigg\}\frac{\sin\frac{(2h-1)\iota\pi}{2j+1}\cos\frac{(\kappa-\frac{1}{2})\pi}{2j+1}}{\cos\frac{(2\kappa-1)\pi}{2j+1}-\cos\frac{2\iota\pi}{2j+1}}.\end{aligned}\tag{22}$$

And if, on the other hand, we take $k=2\kappa$ and $i=2\iota-1$, we have

$$\begin{aligned}y_{j+h,t}=\frac{2(-1)^{\kappa}}{2j+1}B_{2\kappa}^{\backprime}\Sigma_{(\iota)1}^{j}\Bigg\{&\cos\beta_{2\kappa}^{\backprime}\cos\left(2at\sin\frac{(\iota-\frac{1}{2})\pi}{2j+1}\right)\sin\frac{(\iota-\frac{1}{2})\pi}{2j+1}\\&+\sin\beta_{2\kappa}^{\backprime}\sin\left(2at\sin\frac{(\iota-\frac{1}{2})\pi}{2j+1}\right)\sin\frac{\kappa\pi}{2j+1}\Bigg\}\frac{\cos\frac{(2h-1)(\iota-\frac{1}{2})\pi}{2j+1}\sin\frac{\kappa\pi}{2j+1}\operatorname{cotan}\frac{(\iota-\frac{1}{2})\pi}{2j+1}}{\cos\frac{2\kappa\pi}{2j+1}-\cos\frac{(2\iota-1)\pi}{2j+1}}.\end{aligned}\tag{23}$$

And these formulae may be considered as rigorous with reference to the present question.

37. Supposing now that j increases without limit, but that k so increases with it as to leave $\alpha=$ some finite arc, between 0 and $\frac{\pi}{2}$; we shall have, as the limits of the two last formulae, the following:

$$\begin{aligned}y_{j+h,t}=\frac{2}{\pi}(-1)^{\kappa}B_{2\kappa-1}^{\backprime}\int_{0}^{\frac{\pi}{2}}d\theta\,\{&\cos\beta_{2\kappa-1}^{\backprime}\cos(2at\sin\theta)\sin\theta+\sin\beta_{2\kappa-1}^{\backprime}\sin(2at\sin\theta)\sin\alpha\}\\&\times\frac{\sin(2h\theta-\theta)\cos\alpha}{\cos2\alpha-\cos2\theta};\end{aligned}\tag{24}$$

and

$$\begin{aligned}y_{j+h,t}=\frac{2}{\pi}(-1)^{\kappa}B_{2\kappa}^{\backprime}\int_{0}^{\frac{\pi}{2}}d\theta\,\{&\cos\beta_{2\kappa}^{\backprime}\cos(2at\sin\theta)\sin\theta+\sin\beta_{2\kappa}^{\backprime}\sin(2at\sin\theta)\sin\alpha\}\\&\times\frac{\cos(2h\theta-\theta)\sin\alpha\operatorname{cotan}\theta}{\cos2\alpha-\cos2\theta};\end{aligned}\tag{25}$$

the first corresponding to the case $k=2\kappa-1$, & the second to $k=2\kappa$. It is evident that both these expressions satisfy the equation of mixed differences; & to show that they also reproduce the initial displacements and velocities, we must show that they give, according as the integer h is $>$ or not >0,

$$y_{j+h,0}=\mp\tfrac{1}{2}B'_k\cos\beta'_k\sin 2(j+h)\alpha,$$
$$y'_{j+h,0}=\mp aB'_k\sin\beta'_k\sin 2(j+h)\alpha\sin\alpha;$$

in which $\alpha=\dfrac{\frac{1}{2}k\pi}{2j+1}$, so that

$$\sin 2(j+h)\alpha=\sin\{\tfrac{1}{2}k\pi+(2h-1)\alpha\}=\sin\tfrac{1}{2}k\pi\cos(2h-1)\alpha+\cos\tfrac{1}{2}k\pi\sin(2h-1)\alpha$$
$$=(-1)^{\kappa+1}\cos(2h-1)\alpha,\quad\text{or}\quad=(-1)^{\kappa}\sin(2h-1)\alpha,$$

according as k is of the form $2\kappa-1$, or of the form 2κ.

38. There are, therefore, for a verification, or for an à posteriori proof of the formulae of the last article, the 2 following equations to be proved:*

$$\pm\frac{\pi}{4}\frac{\cos(2h\alpha-\alpha)}{\cos\alpha}=\int_0^{\frac{\pi}{2}}d\theta\,\frac{\sin\theta\sin(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta};$$

$$\mp\frac{\pi\sin(2h\alpha-\alpha)}{\sin\alpha}=\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos\theta\cos(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}.$$

the upper signs corresponding to positive values, and the lower signs corresponding to negative values, of the odd integer $2h-1$.

Now, if we put

$$c_h=\frac{2}{\pi}\sin 2\alpha\int_0^{\frac{\pi}{2}}\frac{\cos 2h\theta}{\cos 2\theta-\cos 2\alpha}d\theta,$$

we shall have, for *all* values of h,

$$c_{h+1}+c_{h-1}-2\cos 2\alpha\,c_h=\frac{4}{\pi}\sin 2\alpha\int_0^{\frac{\pi}{2}}\cos 2h\theta\,d\theta=\frac{2}{h\pi}\sin 2\alpha\sin h\pi;$$

therefore this function vanishes, if h be any integer $>$ or <0; but $c_1+c_{-1}-2\cos 2\alpha\,c_0=2\sin 2\alpha$. Again $c_h=c_{-h}$; and $c_0=0$. To prove this last relation, we may set out with the evident relation† $0=\int_{-\infty}^{\infty}\frac{dx}{x}$, which gives $0=\int_{-\infty}^{\infty}\frac{dx}{x-a}$, a being real; (though the complete discussion of the value of this definite integral belongs to the theory of singular integrals, considered first by Cauchy;) therefore

$$0=\int_0^{\infty}\left(\frac{dx}{x-a}-\frac{dx}{x+a}\right)=2a\int_0^{\infty}\frac{dx}{x^2-a^2},$$

---

* [The integrals which follow are to be interpreted as being Cauchy's Principal Values, i.e.

$$\int_0^{\frac{\pi}{2}}=\lim_{\epsilon=0}\left(\int_0^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{\frac{\pi}{2}}\right).]$$

† $\left[\text{I.e. }\int_{\epsilon}^{\infty}\frac{dx}{x}+\int_{-\infty}^{-\epsilon}\frac{dx}{x}=0.\right]$

and therefore $0=\int_0^\infty \frac{dx}{x^2-a^2}$, if a be real and different from 0. Make $x=\tan 2\theta$, $a=\tan 2\alpha$, and suppose $(\cos 2\alpha)^2>0$; then

$$0=2\int_0^{\frac{\pi}{4}}\frac{d\theta\,(\sec 2\theta)^2\sec 2\alpha}{(\sec 2\theta)^2-(\sec 2\alpha)^2}=\int_0^{\frac{\pi}{4}}\left(\frac{d\theta}{\cos 2\alpha+\cos 2\theta}+\frac{d\theta}{\cos 2\alpha-\cos 2\theta}\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\cos 2\alpha-\cos 2\theta},$$

which is what was to be proved. (However, it is to be observed that we have here supposed $\cos 2\alpha$ to be different from 0. Yet even if it were $=0$, so that we had to consider the integral $\int_0^{\frac{\pi}{2}}\frac{d\theta}{\cos 2\theta}$, we might consider this as being $=\int_0^{\frac{\pi}{4}}(\sec 2\theta-\sec 2\theta)\,d\theta$, and therefore as being $=0$.) Admitting then that $c_0=0$, we have $c_{-1}=c_1=\sin 2\alpha$. Hence $c_h=\sin 2h\alpha$, if h be any integer not less than 0; and $c_h=-\sin 2h\alpha$, if h be any integer not greater than 0. That is,

$$\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos 2h\theta}{\cos 2\theta-\cos 2\alpha}=\pm\frac{\pi}{2}\frac{\sin 2h\alpha}{\sin 2\alpha},\quad\text{or}\quad =0,$$

according as the integer h is $\gtrless 0$, or $=0$. Hence, if $h>0$,

$${}_{0}^{\frac{\pi}{2}}d\theta\,\frac{\cos 2h\theta-\cos(2h\theta-2\theta)}{\cos 2\theta-\cos 2\alpha}=\frac{\pi}{2}\frac{\sin 2h\alpha-\sin(2h\alpha-2\alpha)}{\sin 2\alpha};$$

that is, dividing by $+2$,

$${}_{0}^{\frac{\pi}{2}}d\theta\,\frac{\sin\theta\sin(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}=\frac{\pi}{4}\frac{\cos(2h\alpha-\alpha)}{\cos\alpha},$$

if the integer $2h-1$ be >0; from which, without any new calculation, we see that

$$\int_0^{\frac{\pi}{2}}d\theta\,\frac{\sin\theta\sin(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}=\frac{\pi}{4}\frac{\cos(2h\alpha-\alpha)}{\cos\alpha},$$

if the integer $2h-1$ be <0.

In like manner, if h be >0 (being integer), we have

$$\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos 2h\theta+\cos(2h\theta-2\theta)}{\cos 2\theta-\cos 2\alpha}=\frac{\pi}{2}\frac{\sin 2h\alpha+\sin(2h\alpha-2\alpha)}{\sin 2\alpha},$$

that is,

$$\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos\theta\cos(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}=-\frac{\pi}{4}\frac{\sin(2h\alpha-\alpha)}{\sin\alpha},$$

if the integer $2h-1$ be >0. And hence, without any new calculation, we see that

$$\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos\theta\cos(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}=\frac{\pi}{4}\frac{\sin(2h\alpha-\alpha)}{\sin\alpha},$$

if the integer $2h-1$ be <0. The initial conditions are therefore satisfied.

39. The same analysis shows that, if the integer h be > 0,

$$\frac{\pi}{2}\cos 2h\alpha = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos 2\alpha - \cos 2\theta}\{2\cos\alpha^2 \sin\theta \sin(2h\theta - \theta) + 2\sin\alpha^2 \cos\theta\cos(2h\theta - \theta)\}$$

$$= \int^{\frac{\pi}{2}} d\theta \frac{\cos(2h\theta - 2\theta) - \cos 2\alpha \cos 2h\theta}{\cos 2\alpha - \cos 2\theta} \qquad \int^{\frac{\pi}{2}} \frac{\sin 2\theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} d\theta;$$

and accordingly, if we denote this last integral by f_h, we have

$$f_{h+1} + f_{h-1} - 2f_h \cos 2\alpha = \int_0^{\frac{\pi}{2}} d\theta \{\cos(2h\theta + 2\theta) - \cos(2h\theta - 2\theta)\} = 0,$$

if $h > 1$, and $= -\frac{\pi}{2}$, if $h = 1$; also $f_0 = 0$, and

$$f_1 = \int_0^{\frac{\pi}{2}} (\cos 2\alpha + \cos 2\theta)\, d\theta = \frac{\pi}{2}\cos 2\alpha;$$

therefore

$$f_2 = \pi\{(\cos 2\alpha)^2 - \tfrac{1}{2}\} = \frac{\pi}{2}\cos 4\alpha, \quad \text{and} \quad f_h = \frac{\pi}{2}\cos 2h\alpha,$$

if $h > 0$. The same integral vanishes (as we have just remarked) when $h = 0$, and since it changes sign with h, it must become $= -\frac{\pi}{2}\cos 2h\alpha$, if $h < 0$.

We have therefore the discontinuous equation

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta \sin 2h\theta\, d\theta}{\cos 2\alpha - \cos 2\theta} = \pm \cos 2h\alpha, \quad \text{or} \quad = 0,$$

according as the integer h is $\gtrless 0$, or $= 0$; & we found, in the last article, that

$$\frac{2}{\pi}\sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{\cos 2h\theta\, d\theta}{\cos 2\alpha - \cos 2\theta} = \mp \sin 2h\alpha, \quad \text{or} \quad = 0,$$

according as the integer h is $\gtrless 0$, or $= 0$. Indeed, we may consider both the two last equations as included in either of the two which occur at the beginning of article **38**; & as conducting reciprocally to those two, by easy combinations.

40. We see then that if we assume

$$y_{h,t} = -\frac{2}{\pi}\cos\alpha \int_0^{\frac{\pi}{2}} \frac{d\theta \sin(2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta}\{b \sin\theta \cos(2at \sin\theta) + c \sin\alpha \sin(2at\sin\theta)\},$$

b and c being any constants, & h being any integer number, we shall satisfy the indefinite equation in mixed differences

$$a^2(y_{h+1,t} + y_{h-1,t} - 2y_{h,t}) = y''_{h,t},$$

and also the initial conditions

$$\frac{y_{h,0}}{b} = \frac{y'_{h,0}}{2ac\sin\alpha} = \mp \tfrac{1}{2}\cos(2h\alpha - \alpha),$$

according as $2h-1$ is $>$ or <0. In like manner, if we assume

$$y_{h,t}=-\frac{2}{\pi}\sin\alpha\int_0^{\frac{\pi}{2}}\frac{d\theta\cos\theta\cos(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}\left\{b\cos(2at\sin\theta)+\frac{c\sin\alpha}{\sin\theta}\sin(2at\sin\theta)\right\},$$

we shall satisfy the same indefinite equation in mixed differences, and the conditions

$$\frac{y_{h,0}}{b}=\frac{y'_{h,0}}{2ac\sin\alpha}=\pm\tfrac{1}{2}\sin(2h\alpha-\alpha),$$

according as $2h-1$ is $\gtrless 0$. If we assume, in the third place,

$$y_{h,t}=-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2h\theta}{\cos 2\alpha-\cos 2\theta}\{b\sin\theta\cos(2at\sin\theta)+c\sin\alpha\sin(2at\sin\theta)\},$$

we shall satisfy the equation in differences, & the conditions

$$\frac{y_{h,0}}{b}=\frac{y'_{h,0}}{2ac\sin\alpha}=\mp\tfrac{1}{2}\cos 2h\alpha,\quad\text{or }=0,$$

according as h is $\gtrless 0$, or $=0$. And if we assume, in the fourth place,

$$y_{h,t}=-\frac{1}{\pi}\sin 2\alpha\int_0^{\frac{\pi}{2}}\frac{d\theta\cos 2h\theta}{\cos 2\alpha-\cos 2\theta}\left\{b\cos(2at\sin\theta)+\frac{c\sin\alpha}{\sin\theta}\sin(2at\sin\theta)\right\},$$

we shall satisfy the same equation in differences, and the conditions

$$\frac{y_{h,0}}{b}=\frac{y'_{h,0}}{2ac\sin\alpha}=\pm\tfrac{1}{2}\sin 2h\alpha,\quad\text{or }=0,$$

according as h is $\gtrless 0$, or $=0$.

41. It follows that the first expression of article **40** corresponds to the effect, at the time t, of an initial state represented by

$$y_{h,dt}=\mp\tfrac{1}{2}\{b\cos(2a\,dt\sin\alpha)+c\sin(2a\,dt\sin\alpha)\}\cos(2h\alpha-\alpha),$$

and the second expression of the same article to the effect of an initial state represented by

$$y_{h,dt}=\pm\tfrac{1}{2}\{b\cos(2a\,dt\sin\alpha)+c\sin(2a\,dt\sin\alpha)\}\sin(2h\alpha-\alpha),$$

the upper or the lower signs being taken according as $2h>$ or <1.

It follows also that the third expression of the same article corresponds to the effect of an initial state represented by

$$y_{h,dt}=\mp\tfrac{1}{2}\{b\cos(2a\,dt\sin\alpha)+c\sin(2a\,dt\sin\alpha)\}\cos 2h\alpha,\quad\text{or }=0,$$

and the fourth expression to the effect of the initial state

$$y_{h,dt}=\pm\tfrac{1}{2}\{b\cos(2a\,dt\sin\alpha)+c\sin(2a\,dt\sin\alpha)\}\sin 2h\alpha,\quad\text{or }=0,$$

according as h is $\gtrless 0$, or $=0$. The system of particles is here supposed to extend indefinitely in two opposite directions from the particle P_0, so that no account is taken of any fixity of the extreme particles.

42. Resuming then the consideration of the case where half only of the system is agitated at the time 0, we see that if this system be indefinite in both directions, and if its initial state be represented by the formula

$$y_{h,dt}=\cos(2h\alpha-\alpha)\,.\,(b\cos+c\sin)(2a\sin\alpha\,dt),\quad\text{or }=0,$$

according as h is not greater than 0 or is greater than 0, its state at the time t is represented by the formula

$$y_{h,t} = \tfrac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\cos(2h\alpha - \alpha)$$

$$-\frac{2}{\pi}\cos\alpha\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin(2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta).$$

And if the initial state be

$$y_{h,dt} = -(b\cos + c\sin)(2a\sin\alpha\,dt)\sin(2h\alpha - \alpha), \quad \text{or } = 0,$$

according as h is not greater than 0, or greater than 0, then the state at the time t will be

$$y_{h,t} = -\tfrac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\sin(2h\alpha - \alpha)$$

$$-\frac{2}{\pi}\sin\alpha\int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos\theta}{\sin\theta}\,\frac{\cos(2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta).$$

for all (integer) values of h. And hence, by an easy combination, we find that if the initial state be

$$y_{h,dt} = (b\cos + c\sin)(2a\sin\alpha\,dt)\sin 2h\alpha, \quad \text{or } = 0,$$

according as h is $\not> 0$ or > 0, the state at the time t is

$$y_{h,t} = \tfrac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\sin 2h\alpha$$

$$+\frac{\sin 2\alpha}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta}\,\frac{\cos 2h\theta}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta).$$

In like manner, if the initial state be

$$y_{h,dt} = (b\cos + c\sin)(2a\sin\alpha\,dt)\cos 2h\alpha, \quad \text{or } = 0, \text{ as } h \not> 0 \text{ or } > 0,$$

then the state at the time t is

$$y_{h,t} = \tfrac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\cos 2h\alpha$$

$$+\frac{\cos 2\alpha}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta}\,\frac{\cos 2h\theta}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta)$$

$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta}\,\frac{\cos(2h\theta - 2\theta)}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta)$$

$$= \tfrac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\cos 2h\alpha$$

$$-\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos\theta\sin 2h\theta}{\cos 2\alpha - \cos 2\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta)$$

$$+\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos 2h\theta}{\sin\theta}(b\sin\theta\cos + c\sin\alpha\sin)(2at\sin\theta).$$

43. The third conclusion of article **42** might also easily have been deduced from the fourth conclusion of **41**. And the fourth conclusion of article **42** might have been deduced from the third conclusion of **41**, namely from the theorem that if the initial state be

$$y_{h,dt} = \mp\tfrac{1}{2}(b\cos + c\sin)(2a\sin\alpha\,dt)\cos 2h\alpha, \quad \text{or } = 0,$$

according as h is $\gtrless 0$ or $=0$, then the state at the time t is

$$y_{h,t} = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta \cos\theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} (b \sin\theta \cos + c \sin\alpha \sin)(2at \sin\theta).$$

For we have only to add to this the term $\frac{1}{2}(b\cos + c\sin)(2at\sin\alpha)\cos 2h\alpha$, to allow for the additional parts of the initial states of all the particles except P_0; and then to allow for the remaining part of the initial state of that one particle, namely the term

$$y_{0,dt} = \tfrac{1}{2}(b\cos + c\sin)(2a\sin\alpha\, dt) = \tfrac{1}{2}b + ac\sin\alpha\, dt,$$

by means of the formula (12) of article **18**; which shows that, in the indefinite system here considered, the effect of this initial state $y_{0,dt}$ of the particle P_0 on any other particle P_h at any time t is

$$\frac{1}{\pi}\left(b + 2ac\sin\alpha \int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \cos(2at\sin\theta)$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \left(b\cos + \frac{\sin\alpha}{\sin\theta} c \sin\right)(2at\sin\theta).$$

44. To treat now the question proposed in article **32**, we are to suppose, in passing to the limit there required, that for *all* integer values of $h > 0$, we have

$$y_{j+h,0} = 0, \quad y'_{j+h,0} = 0,$$

j being some very large integer number which however is to be treated as given; (but as infinite; that is, in one part of the calculation we are not to consider it as varying, but in another part of the same calculation we are to treat it as increasing without limit;) and for *all* integer values of $h \ngtr 0$,

$$y_{j+h,0} = B^{\backprime}_k \cos\beta^{\backprime}_k \sin 2(j+h)\alpha,$$
$$y'_{j+h,0} = 2a\sin\alpha\, B^{\backprime}_k \sin\beta^{\backprime}_k \sin 2(j+h)\alpha,$$

$B^{\backprime}_k$ and $\beta^{\backprime}_k$ being arbitrary constants, and $\alpha = \frac{\frac{1}{2}k\pi}{n+1} =$ some given and finite arc. And the problem may be considered as being to find a function $y_{j+h,t}$ of h and t, which shall satisfy the initial conditions just now mentioned, & also the indefinite equation in mixed differences

$$y''_{j+h,t} = a^2(y_{j+h+1,t} + y_{j+h-1,t} - 2y_{j+h,t}).$$

Now, the initial state $y_{j+h,dt}$ here proposed may be considered as the sum of two others, of which one is expressed by the formula

$$B^{\backprime}_k \cos 2j\alpha (\cos\beta^{\backprime}_k \cos + \sin\beta^{\backprime}_k \sin)(2a\,dt\sin\alpha)\sin 2h\alpha, \quad \text{or } 0,$$

and the other by the formula

$$B^{\backprime}_k \sin 2j\alpha (\cos\beta^{\backprime}_k \cos + \sin\beta^{\backprime}_k \sin)(2a\,dt\sin\alpha)\cos 2h\alpha, \quad \text{or } 0,$$

according as the integer h is $\ngtr$ or > 0. The first part of the initial state gives, for its own effect at the time t, by article **42**,

$$\tfrac{1}{2}B^{\backprime}_k \cos 2j\alpha (\cos\beta^{\backprime}_k \cos + \sin\beta^{\backprime}_k \sin)(2at\sin\alpha)\sin 2h\alpha$$
$$+ \frac{1}{\pi}B^{\backprime}_k \cos 2j\alpha \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}\frac{\cos 2h\theta}{\cos 2\alpha - \cos 2\theta}(\cos\beta^{\backprime}_k \sin\theta\cos + \sin\beta^{\backprime}_k \sin\alpha\sin)(2at\sin\theta);$$

and the second part of the initial state gives, by the same article,

$$\tfrac{1}{2}B^{\backprime}_k \sin 2j\alpha\,(\cos\beta^{\backprime}_k \cos + \sin\beta^{\backprime}_k \sin)\,(2at\sin\alpha)\cos 2h\alpha$$

$$-\frac{2}{\pi}B^{\backprime}_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos\theta\sin 2h\theta}{\cos 2\alpha-\cos 2\theta}\,(\cos\beta^{\backprime}_k \sin\theta\cos + \sin\beta^{\backprime}_k \sin\alpha\sin)\,(2at\sin\theta)$$

$$+\frac{1}{\pi}B^{\backprime}_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta\cos 2h\theta\left(\cos\beta^{\backprime}_k\cos + \frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta);$$

the last line of which last expression would have disappeared if the initial values $y_{j,0}$ and $y'_{j,0}$ had been only half as great as they are here supposed to be. The whole effect at the time t, or the expression for $y_{j+h,t}$, is, therefore, in the present question,

$$y_{j+h,t} = \tfrac{1}{2}B^{\backprime}_k \sin 2(j+h)\alpha\,.\,(\cos\beta^{\backprime}_k\cos + \sin\beta^{\backprime}_k\sin)\,(2at\sin\alpha)$$

$$+\frac{1}{\pi}B^{\backprime}_k \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos 2h\theta\sin 2(j+1)\alpha - \sin 2j\alpha\cos 2(h-1)\theta}{\cos 2\alpha-\cos 2\theta}$$

$$\times\left(\cos\beta^{\backprime}_k\cos + \frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta). \quad (26)$$

Accordingly it is evident that this expression satisfies the indefinite equation in mixed differences; & it satisfies also the initial conditions, because the theorem of article 38,

$$\int_0^{\frac{\pi}{2}} \frac{\cos 2h\theta\,d\theta}{\cos 2\alpha-\cos 2\theta} = \mp\frac{\pi}{2}\,\frac{\sin 2h\alpha}{\sin 2\alpha}, \text{ or } 0,$$

according as the integer h is $\gtrless 0$, or $=0$, gives

$$\frac{1}{\pi}B^{\backprime}_k \int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos 2h\theta\sin 2(j+1)\alpha - \sin 2j\alpha\cos 2(h-1)\theta}{\cos 2\alpha-\cos 2\theta}$$

$$= \mp\tfrac{1}{2}B^{\backprime}_k\,\frac{\sin 2h\alpha\sin 2(j+1)\alpha - \sin 2j\alpha\sin 2(h-1)\alpha}{\sin 2\alpha}$$

$$= \mp\tfrac{1}{2}B^{\backprime}_k\,(\sin 2h\alpha\cos 2j\alpha + \sin 2j\alpha\cos 2h\alpha)$$

$$= \mp\tfrac{1}{2}B^{\backprime}_k \sin 2(j+h)\alpha,$$

according as h is $>$ or $\ngtr 0$.

45. We ought also to be able to verify the expression obtained in the last article, by deducing from it those of article **37**. Suppose then $k=2\kappa-1$ and $n=2j$; we shall have $\alpha = \frac{(\kappa-\frac{1}{2})\pi}{2j+1}$; therefore

$$\cos(2j\alpha+\alpha)=0, \quad \sin(2j\alpha+\alpha)=(-1)^{\kappa+1};$$

therefore

$$\sin 2j\alpha = \sin(2j\alpha+2\alpha) = (-1)^{\kappa+1}\cos\alpha; \quad \cos 2j\alpha = (-1)^{\kappa+1}\sin\alpha;$$

$$\sin 2(j+h)\alpha = (-1)^{\kappa+1}\cos(2h\alpha-\alpha);$$

and

$$y_{j+h,t} = \tfrac{1}{2}(-1)^{\kappa+1}B^{\backprime}_{2\kappa-1}\cos(2h\alpha-\alpha)\,.\,(\cos\beta^{\backprime}_{2\kappa-1}\cos + \sin\beta^{\backprime}_{2\kappa-1}\sin)\,(2at\sin\alpha)$$

$$-\frac{2}{\pi}B^{\backprime}_{2\kappa-1}(-1)^{\kappa+1}\cos\alpha\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}$$

$$\times(\cos\beta^{\backprime}_{2\kappa-1}\sin\theta\cos + \sin\beta^{\backprime}_{2\kappa-1}\sin\alpha\sin)\,(2at\sin\theta);$$

an expression which accordingly differs from the corresponding one (24) of article **37** only by the addition of the first line, which was there purposely suppressed. Again, if $k=2\kappa$, $\alpha=\frac{\kappa\pi}{2j+1}$, then $\sin(2j\alpha+\alpha)=0$, $\cos(2j\alpha+\alpha)=(-1)^{\kappa}$, $\sin(2j\alpha+2\alpha)=(-1)^{\kappa}\sin\alpha=-\sin 2j\alpha$,

$$\cos 2j\alpha=(-1)^{\kappa}\cos\alpha,\quad \sin 2(j+h)\alpha=(-1)^{\kappa}\sin(2h\alpha-\alpha),$$

and $y_{j+h,t}=\frac{1}{2}(-1)^{\kappa}B^{\backprime}_{2\kappa}\sin(2h\alpha-\alpha)\,.\,(\cos\beta^{\backprime}_{2\kappa}\cos+\sin\beta^{\backprime}_{2\kappa}\sin)(2at\sin\alpha)$

$$+\frac{2}{\pi}B^{\backprime}_{2\kappa}(-1)^{\kappa}\sin\alpha\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos\theta\cos(2h\theta-\theta)}{\cos 2\alpha-\cos 2\theta}\left(\cos\beta^{\backprime}_{2\kappa}\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_{2\kappa}\sin\right)(2at\sin\theta);$$

an expression agreeing, as closely as it ought, with (25) of article **37**.

46. The results obtained in recent articles may be used so as to throw light upon the analysis begun in article **32**. In fact we may now easily perceive that, by admitting the transformation in article **31** of sums into integrals, an expression for $y_{l,t}$ or for $y_{j+h,t}$ is deduced, involving functions of $l\pm j$, namely an expression consisting of the two following parts*:

$$\frac{B^{\backprime}_k}{\pi}\int_0^{\frac{\pi}{2}}d\theta\,\frac{\cos 2(j-l)\theta\,.\sin 2(j+1)\alpha-\cos 2(j-l+1)\theta\,.\sin 2j\alpha}{\cos 2\alpha-\cos 2\theta}$$
$$\times\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta);$$

and

$$-\frac{B^{\backprime}_k}{\pi}\int_0^{2}d\theta\,\frac{\cos 2(j+l)\theta\,.\sin 2(j+1)\alpha-\cos 2(j+l+1)\theta\,.\sin 2j\alpha}{\cos 2\alpha-\cos 2\theta}$$
$$\times\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta).$$

The first of these two parts coincides with the second part of the expression for $y_{j+h,t}$ in article **44**, when we change l to $j+h$. With respect to the second of the two parts assigned in the present article, it may be remarked that (see article **32**)

$$-\cos 2(j+l)\theta\,.\sin 2(j+1)\alpha+\cos 2(j+l+1)\theta\,.\sin 2j\alpha$$
$$=-\sin(\theta+\alpha)\,.\cos\{(2j+2l+1)(\theta-\alpha)+2l\alpha\}+\sin(\theta-\alpha)\,.\cos\{(2j+2l+1)(\theta+\alpha)-2l\alpha\};$$

and that $\cos 2\alpha-\cos 2\theta=2\sin(\theta-\alpha)\sin(\theta+\alpha)$; therefore, dividing by this latter function, and neglecting the terms which have no small divisors and those which change sign with $\theta-\alpha$, we find, for the part still to be considered, the expression

$$\frac{1}{2\pi}B^{\backprime}_k\int_0^{2}d\theta\,\frac{\sin(2j+2l+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin 2l\alpha\,(\cos\beta^{\backprime}_k\cos+\sin\beta^{\backprime}_k\sin)(2at\sin\alpha)$$
$$=\tfrac{1}{2}B^{\backprime}_k\sin 2(j+h)\alpha\,.\cos(2at\sin\alpha-\beta^{\backprime}_k),$$

coinciding with the first part of the expression in article **44**, because $j+l$ increases without limit.

---

* [If these two parts are joined together under the integral sign, the resulting integrand does not become infinite when $\theta=\alpha$ and can be evaluated by ordinary methods. The process here consists of taking each part separately and interpreting each integral by Cauchy's method and the method of fluctuation.]

47. Resuming now the analysis of articles **30** and **32**, we see, in the first place, that the expression (18) may be decomposed into two parts, namely the two values of the expression

$$\frac{2(j\mp l)\,i\phi\,.\sin 2(j+1)\,k\phi-\cos 2(j\mp l+1)\,i\phi\,.\sin 2jk\phi}{\cos 2k\phi-\cos 2i\phi}$$

$$\times\left(\cos\beta_k^i\cos+\frac{\sin k\phi}{\sin i\phi}\sin\beta_k^i\sin\right)(2at\sin i\phi);$$

in which $\phi=\frac{\frac{1}{2}\pi}{n+1}$, and j, k, l, n are finite. We may also change

$$y_{l,t}=\Sigma^n_{(i)1}C_i \quad \text{to} \quad C_k+\Sigma^{k-1}_{(i)1}C_i+\Sigma^n_{(i)k+1}C_i,$$

in which

$$C_i=\frac{\frac{1}{2}B_k^i}{n+1} \qquad \frac{\sin 2(2j+1)k\phi}{\sin 2k\phi};$$

this last expression being rigoro rously the periodic time $\frac{\pi}{a}\operatorname{cosec} k\phi$. Let j, k, l, n be very large, but such that the ratios $\frac{j}{n}, \frac{k}{n}, \frac{l}{n}$ are sensibly >0 and <1, and that $\frac{j-l}{n}$ is sensibly $=0$. Then ϕ is extremely small, and so is even $2(j-l)\phi=\frac{j-l}{n+1}\pi$, although the number $j-l$ may be considerable. Thus, the part C_i' of C_i which involves $j-i$ alters very little when i is changed to $i+1$: unless the denominator becomes small by i being nearly $=k$, or at least by $i\phi-k\phi$ being small, which may be while $i-k$ is considerable; and therefore, with respect to this part C_i', the conversion of summation into integration is permitted unless it shall be found that this conversion is invalid near the critical value $i\phi=k\phi$. To examine what happens near this value, let $i=k+g$, g being an integer >0 or <0 which may be considerable itself but is to be so chosen that the product $g\phi$ may be moderately small: & let us calculate $C'_{k+g}+C'_{k-g}$. This sum is found to involve $\frac{\sin 2(l-j)\,g\phi}{\sin g\phi}$ and $\frac{\sin(2at\cos k\phi\sin g\phi)}{\sin g\phi}$; that is, we have to sum expressions of the form

$$\phi\times f(g\phi), \quad \phi\times\frac{\sin 2(l-j)\,g\phi}{\sin g\phi}f(g\phi), \quad \phi\times\frac{\sin(2at\cos k\phi\sin g\phi)}{\sin g\phi}f(g\phi),$$

from $g=1$ to $g=$ a large integer, and the functions $f(g\phi)$ not varying rapidly near the lower limit of this summation, while ϕ is still extremely small and tends to 0. But such summations $\Sigma^g_{(g)1}\phi\times(\&c.)$ may be replaced by the definite integrations $\int_0^{g\phi}d(g\phi)\,(\&c.)$; & therefore, (on account chiefly of $\sin\frac{\pi(l-j)}{n+1}$ and $\sin\left(2at\cos\alpha\sin\frac{\frac{1}{2}\pi}{n+1}\right)$ bearing determined limits to $\sin\frac{\frac{1}{2}\pi}{n+1}$ when n tends to ∞,) it is permitted to change the summation $(\Sigma^{k-1}_{(i)1}+\Sigma^n_{(i)k+1})\,\phi$ into a definite integration $\int d\theta$, for that part C_i' of C_i which depends on $j-l$. With respect to the other part C_i'' of C_i which involves $j+l$, we see that this part involves cosines of arcs which receive a finite increment

$$=2(j+l)\,\phi=\frac{(j+l)\,\pi}{n+1},$$

when i is changed to $i+1$; while these cosines are multiplied by functions which receive, by the same change of i, only infinitely small alterations, except near the critical value $i\phi = k\phi$. It is therefore permitted to reject all values of i which do not render $g\phi = i\phi - k\phi$ small; and the limit of this smallness is 0. We may therefore, after employing the transformation indicated in article **32**, change i into k, or θ into α, except in

$$-\frac{\cos\{2l\alpha + (2j+2l+1)\,g\phi\} + \cos\{2l\alpha - (2j+2l+1)\,g\phi\}}{\sin g\phi}$$

which may be reduced to

$$\frac{2\sin 2l\alpha \sin(2j+2l+1)\,g\phi}{g\phi}.$$

It remains therefore to calculate the sum

$$\Sigma_{(g)1}^{\infty} \frac{1}{g} \sin\frac{(j+l+\frac{1}{2})\,g\pi}{n+1};$$

for we shall have

$$(\Sigma_{(i)1}^{k-1} + \Sigma_{(i)k+1}^{n})\,C_i^{\backprime\backprime} = \frac{B_k^{\backprime}}{\pi} \sin 2l\alpha \cos(2at\sin\alpha - \beta_k^{\backprime})\,\Sigma_{(g)1}^{\infty} \frac{1}{g} \sin\frac{g\,(j+l+\frac{1}{2})\,\pi}{n+1}.$$

But $\dfrac{(j+l+\frac{1}{2})\,\pi}{n+1}$ is >0, $<2\pi$; therefore, by a known theorem,

$$\Sigma_{(g)1}^{\infty} \frac{1}{g} \sin\frac{g\,(j+l+\frac{1}{2})\,\pi}{n+1} = \frac{\pi}{2}\left(1 - \frac{j+l+\frac{1}{2}}{n+1}\right)$$

accurately; we may therefore write, for this sum, $\dfrac{\pi}{2}\left(1 - \dfrac{j}{n}\right)$, & we have for that part of $\Sigma_{(i)}\left({}_1^{k-1} + {}_{k+1}^{n}\right) C_i$ (observe this notation) which depends on $j+l$, the expression

$$\left(\tfrac{1}{2} - \frac{j}{n}\right) B_k^{\backprime} \sin 2l\alpha \cos(2at\sin\alpha - \beta_k^{\backprime}).$$

And since $C_k = \dfrac{j}{n} B_k^{\backprime} \sin 2l\alpha \cos(2at\sin\alpha - \beta_k^{\backprime})$, we have, upon the whole,

$$\begin{aligned} y_{l,t} = {} & \tfrac{1}{2} B_k^{\backprime} \sin 2l\alpha \cos(2at\sin\alpha - \beta_k^{\backprime}) \\ & + \frac{1}{\pi} B_k^{\backprime} \int_0^{\frac{\pi}{2}} d\theta\, \frac{\cos 2(l-j)\,\theta \,.\, \sin 2(j+1)\,\alpha - \cos 2(l-j-1)\,\theta \,.\, \sin 2j\alpha}{\cos 2\alpha - \cos 2\theta} \\ & \qquad \times \left(\cos\beta_k^{\backprime} \cos + \frac{\sin\alpha}{\sin\theta} \sin\beta_k^{\backprime} \sin\right)(2at\sin\theta); \qquad (27) \end{aligned}$$

an expression which coincides with that marked (26) in article **44**.

48. The analysis of the foregoing article shows, at the same time, by what steps we may pass back from this expression (27) or (26) to that marked (18) in article **30**; that is, from the supposition of n infinite to that of n finite. In this return, we are 1st to restore for α its value $\dfrac{\frac{1}{2}k\pi}{n+1}$; 2nd to change $\dfrac{1}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} d\theta\, F(\theta)$ to $\dfrac{1}{2(n+1)} \Sigma_{(i)1}^{n} F\left(\dfrac{\frac{1}{2}i\pi}{n+1}\right)$; 3rd in the part free from the sign of integration, namely $\frac{1}{2} B_k^{\backprime} \sin 2l\alpha \cos(2at\sin\alpha - \beta_k^{\backprime})$, ($\alpha$ retaining for abridgement its meaning just recited,) to resolve the factor $\frac{1}{2}$ into two parts which are nearly $\dfrac{j}{n}$ and $\frac{1}{2} - \dfrac{j}{n}$; or, more precisely, to substitute for this part free of the sign $\int$ the sum of the two following functions:

$$C_k = \frac{B_k^{\backprime}}{n+1} \cos(2at\sin\alpha - \beta_k^{\backprime}) \sin 2l\alpha \left\{\frac{2j+1}{2} - \frac{\sin 2(2j+1)\,\alpha}{2\sin 2\alpha}\right\};$$

and

$$\frac{\grave{k}}{4(n+1)}\Sigma_{(i)}\left({}_1^{k-1}+{}_{k+1}^{n}\right)\left(\frac{\cos\{(2j+2l+1)(\theta+\alpha)-2l\alpha\}}{\sin(\theta+\alpha)}-\frac{\cos\{(2j+2l+1)(\theta-\alpha)+2l\alpha\}}{\sin(\theta-\alpha)}\right)$$

$$\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta);$$

in which $\theta=\dfrac{\frac{1}{2}i\pi}{n+1}$.

49. With respect to the physical meaning of this last resolution of the factor $\frac{1}{2}$ into the two parts $\frac{j}{n}$ and $\frac{1}{2}-\frac{j}{n}$, the foregoing analysis shows that the part $\frac{j}{n}$ corresponds to the immediate effect of the initial state, namely

$$y_{l,dt}=B^{\backprime}_k\sin 2l\alpha\cos(2a\sin\alpha\, dt-\beta^{\backprime}_k),\quad\text{or }=0,$$

according as $l \ngtr$ or $>j$, in producing the part

$$\frac{j}{n}B^{\backprime}_k\sin 2l\alpha\cos(2at\sin\alpha-\beta^{\backprime}_k),$$

for all values of l (from 1 to n) or, more precisely (when n is finite) the part

$$\frac{2j+1-\dfrac{\sin 2(2j+1)\alpha}{\sin 2\alpha}}{2(n+1)}\cdot B^{\backprime}_k\sin 2l\alpha\cos(2at\sin\alpha-\beta^{\backprime}_k),$$

with the same periodic time $\frac{\pi}{a}\operatorname{cosec}\alpha$, & the same number of venters $k=\frac{2\alpha}{\pi}(n+1)$, as there would be in the initial state, if that were extended to all values of l and t. In such a manner that if one third part only of the system (supposed numerous) be originally agitated so as to correspond with a given simple mode of vibration, or with a given value of k, then the whole system becomes agitated with all possible simple modes superposed upon each other, corresponding to all possible values of i (from 1 to n), but the amplitude of the k^{th} mode is $\frac{1}{3}$ of the initial amplitude. And the modes for which i is nearly equal to k, or more precisely for which θ is nearly equal to α, so that their periodic times $\frac{\pi}{a}\operatorname{cosec}\theta$ are only a little less or a little greater than $\frac{\pi}{a}\operatorname{cosec}\alpha$, besides producing the effect expressed by the definite integral in the formula (27) or (26), produce also a resultant mode which (if n be large) coincides nearly with the simple initial mode k & has an amplitude which bears to the initial amplitude the ratio of $\frac{1}{2}-\frac{j}{n}$ to 1. Thus, when (as in the case just now mentioned) the initial agitation occupied only the third part of the (numerous) system, so that $\frac{j}{n}=\frac{1}{3}$, we have $\frac{1}{2}-\frac{j}{n}=\frac{1}{6}$, and the indirect effect (extending to the whole system) increases by $\frac{1}{6}$ of the initial amplitude the immediate or direct effect which had been found to amount to $\frac{1}{3}$. This indirect effect is produced in an indefinitely short time t, and then is permanent; so that if n be very large, there is at once produced for the whole system a permanent mode of vibration which coincides with the initial simple mode in all respects except that of having an amplitude only half as great; which ratio does not require for its establishment that the part originally agitated should be exactly or nearly half of the whole system. The remaining effect, expressed by the definite integral, corresponds to a complex mode of vibration formed by the superposition of infinitely many simple modes; but when the time elapsed is very small, it reduces itself sensibly to a single mode, namely the initial mode k; or rather to two coexisting movements with the initial period & epoch, but with amplitudes which bear to the initial amplitude the ratios of $\frac{1}{2}$ and $-\frac{1}{2}$, according as l is $<$ or $>j$, and so reproduce the given initial discontinuity. We shall soon consider whether any and what reduction of the same sort takes place when the time elapsed is large.

50. The definite integral in (26) or (27) may be transformed by observing that of the two parts

$$\sin(\theta+\alpha)\cos\{(2j-2l+1)\theta-(2j+1)\alpha\} \quad \text{and} \quad -\sin(\theta-\alpha)\cos\{(2j-2l+1)\theta+(2j+1)\alpha\},$$

into which (as was remarked in article **32**) the numerator

$$\cos 2(l-j)\theta \,.\, \sin 2(j+1)\alpha - \cos 2(l-j-1)\theta \,.\, \sin 2j\alpha$$

may be decomposed, the second results from the first by changing θ to $\pi-\theta$; while $\cos 2\theta$ in the denominator & $\sin\theta$ do not alter by such change. In this manner we find that the formula (27) may be thus written:

$$y_{l,t} = \tfrac{1}{2}B_k^{\backprime}\sin 2l\alpha\cos(2at\sin\alpha-\beta_k^{\backprime})$$
$$+\frac{1}{2\pi}B_k^{\backprime}\int_0^{\pi} d\theta\,\frac{\cos\{(2l-2j-1)(\theta-\alpha)+2l\alpha\}}{\sin(\theta-\alpha)}\left(\cos\beta_k^{\backprime}\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta_k^{\backprime}\sin\right)(2at\sin\theta). \qquad (28)$$

(As this is a decided simplification of the integral (27), it will be interesting to inquire whether we cannot find a similar simplification of the sum (18).)

51. Under this last form, as under those found before, we see clearly that the function $y_{l,t}$ satisfies the indefinite equation in mixed differences; and to show that it satisfies also the initial conditions, we ought to be able to show that

$$\frac{1}{\pi}\int_0^{\pi} d\theta\,\frac{\cos\{(2h-1)(\theta-\alpha)+2l\alpha\}}{\sin(\theta-\alpha)} = \mp\sin 2l\alpha,$$

according as $2h-1$ is $\gtrless 0$. This discontinuous equation appears to resolve itself into the two following:

$$\int_0^{\pi} d\theta\,\frac{\cos(2h-1)(\theta-\alpha)}{\sin(\theta-\alpha)} = 0,$$

for all integer values of h; and

$$\int_0^{\pi} \frac{\sin(2h-1)(\theta-\alpha)}{\sin(\theta-\alpha)} = \pm\pi,$$

according as the integer h is $>$ or $\ngtr 0$. Accordingly

$$\Delta_h\int_0^{\pi} d\theta\,\frac{\cos(2h-1)(\theta-\alpha)}{\sin(\theta-\alpha)} = -2\int_0^{\pi} d\theta\sin 2h(\theta-\alpha) = 0,$$
$$\Delta_h\int_0^{\pi} d\theta\,\frac{\sin(2h-1)(\theta-\alpha)}{\sin(\theta-\alpha)} = 2\int_0^{\pi} d\theta\cos 2h(\theta-\alpha) = 0, \quad \text{or } 2\pi,$$

according as h is $\gtrless$ or $=0$; so that it only remains to prove that $\int_0^{\pi}\frac{d\theta}{\tan(\theta-\alpha)} = 0$, and this integral

$$= \int_0^{\frac{\pi}{2}} d\theta\{\operatorname{cotan}(\theta-\alpha)-\operatorname{cotan}(\theta+\alpha)\} = \int_0^{\frac{\pi}{2}}\frac{d\theta\sin 2\alpha}{\sin(\theta+\alpha)\sin(\theta-\alpha)} = 2\sin 2\alpha\int_0^{\frac{\pi}{2}}\frac{d\theta}{\cos 2\alpha-\cos 2\theta} = 0$$

by article **38**.*

---

* [See note, p. 469.]

52. It is evident also from inspection of the integral in (28) that this integral reduces itself to $-\frac{1}{2}B^{\backprime}_k \sin 2l\alpha \cos(2at\sin\alpha - \beta^{\backprime}_k)$, that is to the part free from the sign $\int$, taken negatively, (so that the one part of the formula destroys the other,) if we take $j = -\infty$, or $h = l - j = \infty$. A result which might have been expected, because, by throwing indefinitely far back in the system the origin of the disturbance we must render the effect of that disturbance insensible for any finite values of l and t. And we have thus a new explanation of the term independent of integration; namely that it is the negative of the value of the integral term for $j = -\infty$. We may therefore write the formula (28) as follows:

$$y_{l,t} = \frac{B^{\backprime}_k}{2\pi}\int_0^\pi \frac{d\theta}{\sin(\theta-\alpha)}\left[\cos\{(2l-2j-1)(\theta-\alpha)+2l\alpha\} - \cos\{(2l-2j_{,}+1)(\theta-\alpha)+2l\alpha\}\right]$$
$$\times\left(\cos\beta^{\backprime}_k\cos + \frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta); \qquad (29)$$

in which $j_{,} = -\infty$. And if we treat $j_{,}$ as finite, we may then consider this last formula as expressing the solution of the question:

To find a function $y_{l,t}$ which shall satisfy the indefinite equation in mixed differences (1), and also the initial conditions

$$y_{l,dt} = B^{\backprime}_k \sin 2l\alpha\cos(2a\sin\alpha\, dt - \beta^{\backprime}_k), \quad \text{or } = 0,$$

according as l is, or is not, one of the $j - j_{,} + 1$ successive integers $j_{,}, j_{,}+1, \dots j-1, j$.

53. This question might have been resolved by the help of the formula (12), which, when applied to it, becomes

$$y_{l,t} = \frac{2}{\pi}B^{\backprime}_k\int_0^{\frac{\pi}{2}} d\theta\,\{\Sigma^j_{(j)j_{,}}\sin 2j\alpha\cos 2(j-l)\theta\}\left(\cos\beta^{\backprime}_k\cos + \frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta); \qquad (30)$$

in which

$$2\Sigma^j_{(j)j_{,}}\sin 2j\alpha\cos 2(j-l)\theta = \Sigma^j_{(j)j_{,}}\{\sin 2(j\theta+j\alpha-l\theta) - \sin 2(j\theta-j\alpha-l\theta)\}$$
$$= \frac{\cos\{(2j_{,}-1)(\theta+\alpha)-2l\theta\} - \cos\{(2j+1)(\theta+\alpha)-2l\theta\}}{2\sin(\theta+\alpha)}$$
$$-\frac{\cos\{(2j_{,}-1)(\theta-\alpha)-2l\theta\} - \cos\{(2j+1)(\theta-\alpha)-2l\theta\}}{2\sin(\theta-\alpha)};$$

so that the formula (30) reduces itself to (29). And because the formula (12) admits of a very easy proof, and may almost be said to be obviously true, it might have been a better or at least a more elementary mode of proceeding to have begun by deducing (30) from it & to have then transformed (30) into (29) in the manner just now indicated; after which it would have been easy to pass to (28) as the limit corresponding to the supposition $j_{,} = -\infty$.

54. The formula (29) may be thus written:

$$y_{l,t} = \frac{1}{\pi}B^{\backprime}_k\int_0^\pi d\theta\,\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin\{(2l-j-j_{,})(\theta-\alpha)+2l\alpha\}$$
$$\times\left(\cos\beta^{\backprime}_k\cos + \frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta); \qquad (31)$$

in which we may remember that $j - j_{,} + 1$ is the number of particles $P_{j_{,}}, \dots P_j$ originally agitated;

and $l-\frac{j+j_{,}}{2}$ is the distance of the particle P_l from the middle of the initial agitation. If $j_{,}=j$, that is, if there be but one particle originally agitated, this last expression becomes

$$y_{l,t}=\frac{1}{\pi}B^{\backprime}_k\int_0^{\pi}d\theta\sin\{2(l-j)\theta+2j\alpha\}\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta)$$

$$=\frac{2}{\pi}B^{\backprime}_k\sin 2j\alpha\int_0^{\frac{\pi}{2}}d\theta\cos\{2(l-j)\theta\}\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta),$$

as it ought to do.

55. Article **32** and all the subsequent articles have had reference chiefly to the case of a system or series of particles extending indefinitely in *both* directions from the particle P_l, of which the motion is to be examined; but it is easy to deduce analogous results for the case considered in article **31**, in which the system is indefinite in *one* direction only, the particle P_0 being fixed. The formula (19) for this last mentioned case may be thus written:

$$y_{l,t}=\frac{1}{\pi}B^{\backprime}_k\int_0^{\pi}d\theta\frac{\sin(2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin 2l\theta\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta); \qquad (32)$$

and might have been obtained from (10) under the form

$$y_{l,t}=\frac{4}{\pi}B^{\backprime}_k\int_0^{\frac{\pi}{2}}d\theta\,(\Sigma^{j}_{(j)1}\sin 2j\alpha\sin 2j\theta)\sin 2l\theta\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta). \qquad (33)$$

And if, instead of $\Sigma^{j}_{(j)1}$, we take $\Sigma^{j}_{(j)j_{,}}$, that is, if we suppose only the $j-j_{,}+1$ consecutive particles $P_{j_{,}}, \dots P_j$ to be originally agitated, we have then

$$y_{l,t}=\frac{2}{\pi}B^{\backprime}_k\int_0^{\pi}d\theta\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\cos(j+j_{,})(\theta-\alpha)\sin 2l\theta$$

$$:\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta). \qquad (34)$$

If only one particle P_j be originally agitated, then $j_{,}=j$, and the last formula becomes

$$y_{l,t}=\frac{2}{\pi}B^{\backprime}_k\int_0^{\pi}d\theta\cos 2j(\theta-\alpha)\sin 2l\theta\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta)$$

$$=\frac{4}{\pi}B^{\backprime}_k\sin 2j\alpha\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\left(\cos\beta^{\backprime}_k\cos+\frac{\sin\alpha}{\sin\theta}\sin\beta^{\backprime}_k\sin\right)(2at\sin\theta),$$

agreeing evidently with (10).

56. It is worth observing that the formula (32) may be obtained from (31) by changing $j_{,}$ to $-j$. And it is easy to explain this circumstance. In fact, instead of supposing P_0 fixed by any external cause, we may suppose it to be originally at rest, and to remain so because $y_{-l,0}=-y_{l,0}$ and $y'_{-l,0}=-y'_{l,0}$, as was remarked in article **19**. But in this view we must suppose, in the question of articles **31** and **55**, that the $2j+1$ particles $P_{-j}, \dots P_j$ are all originally agitated according to the law $B^{\backprime}_k\sin 2l\alpha\cos(2a\sin\alpha\,dt-\beta^{\backprime}_k)$; except the particle P_0, which fulfils this law by being undisturbed, & any others which in like manner have $\sin 2l\alpha=0$.

57. An analogous reasoning may be employed to deduce the solution of the 1st Problem from the formula (12), or the laws of vibration of a finite from those of an infinite system. To illustrate this transition, let us begin by considering the case where, for all values of j, positive & negative,

$$y_{j,0}=B_1\sin\frac{j\pi}{2},\quad y'_{j,0}=0;$$

so that

$$y_{2j,0}=0\quad\text{and}\quad y_{4j+1,0}=-y_{4j-1,0}=B_1.$$

The formula (12) may in general be thus written:

$$y_{l,t}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\,\Sigma_{(j)-\infty}^{\infty}\cos(2l\theta-2j\theta)\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\cos(2at\sin\theta);\tag{35}$$

& in the present case it becomes

$$y_{l,t}=\frac{4B_1}{\pi}\int_0^{\frac{\pi}{2}}d\theta\left(\Sigma_{(j)1}^{\infty}\sin 2j\theta\sin\frac{j\pi}{2}\right)\sin 2l\theta\cos(2at\sin\theta);$$

in which

$$2\,\Sigma_{(j)1}^{\infty}\sin 2j\theta\sin\frac{j\pi}{2}=2(\sin 2\theta-\sin 6\theta+\sin 10\theta-\&c.)$$

$$=\lim_{j=\infty}2\,\Sigma_{(j)1}^{j}\sin 2j\theta\sin\frac{j\pi}{2}=\lim_{\lambda=\infty}\frac{(-1)^{\lambda+1}\sin 4\lambda\theta}{\cos 2\theta}=\lim_{\lambda=\infty}\frac{\sin 4\lambda\left(\theta-\frac{\pi}{4}\right)}{\sin 2\left(\theta-\frac{\pi}{4}\right)};$$

therefore

$$y_{l,t}=\frac{2B_1}{\pi}\lim_{\lambda=\infty}\int_0 d\theta\,\frac{\sin 4\lambda\left(\theta-\frac{\pi}{4}\right)}{\sin 2\left(\theta-\frac{\pi}{4}\right)}\sin 2l\theta\cos(2at\sin\theta)$$

$$=B_1\sin\frac{l\pi}{2}\cos(at\sqrt{2})=y_{1,0}\sin\frac{l\pi}{2}\cos(at\sqrt{2}),$$

as found in the 1st example, article **11**, for the case of a single moveable particle. Indeed, we there considered, on the one hand, only the value $l=1$; &, on the other hand, supposed $y'_{1,0}$ not to vanish. But with respect to this last part of the conditions of article **11**, if we now suppose $y'_{j,0}=y'_{1,0}\sin\frac{j\pi}{2}$, we get, by the analysis of the present article, the additional term

$$\frac{2y'_{1,0}}{\pi}\lim_{\lambda=\infty}\int_0^{\frac{\pi}{2}}d\theta\,\frac{\sin 4\lambda\left(\theta-\frac{\pi}{4}\right)}{\sin 2\left(\theta-\frac{\pi}{4}\right)}\sin 2l\theta\int_0^t dt\cos(2at\sin\theta)$$

$$=y'_{1,0}\sin\frac{l\pi}{2}\int_0^t dt\cos(at\sqrt{2})=y'_{1,0}\sin\frac{l\pi}{2}\frac{\sin(at\sqrt{2})}{a\sqrt{2}},$$

which completes the agreement with the results of the 1st example. In fact

$$\lim_{\lambda=\infty}\int_0^{\frac{\pi}{2}}d\theta\,\frac{\sin 4\lambda\left(\theta-\frac{\pi}{4}\right)}{\sin 2\left(\theta-\frac{\pi}{4}\right)}F(\theta)=\frac{\pi}{2}F\left(\frac{\pi}{4}\right),$$

if the function $F(\theta)$ remain finite for the whole extent of the integral.

58. We might even have considered it as evident à priori that the indefinite integration of the equation in mixed differences

$$= \frac{a^2\Delta_l^2}{1+\Delta_l} y_{l,t}, \tag{2}$$

combined with the initial conditions

$$y_{l,0} = y_{1,0} \sin\frac{l\pi}{2}, \quad y'_{l,0} = y'_{1,0} \sin\frac{l\pi}{2},$$

if these be supposed to hold good for all values of the integer l, from a large negative to a large positive value, must conduct nearly, and more and more nearly as these initial conditions hold good for a greater extent of l, to an expression of the form

$$y_{l,t} = y_{1,t} \sin\frac{l\pi}{2},$$

in which $y_{1,t}$ is a function determined by the differential equation

$$y''_{1,t} = -2a^2 y_{1,t};$$

and therefore that the integral of this equation, namely

$$y_{1,t} = y_{1,0} \cos(at\sqrt{2}) + y'_{1,0} \frac{\sin(at\sqrt{2})}{a\sqrt{2}},$$

when multiplied by $\sin\frac{l\pi}{2}$, must express the limit to which the expression

$$y_{l,t} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \left(\Sigma_{(j)-h}^{h} \cos(2l\theta - 2j\theta) \sin\frac{j\pi}{2}\right)\left(y_{1,0}\cos(2at\sin\theta) + y'_{1,0}\frac{\sin(2at\sin\theta)}{2a\sin\theta}\right)$$

tends as h increases without limit. And since the sum

$$\Sigma_{(j)-h}^{h} \cos(2l\theta - 2j\theta) \sin\frac{j\pi}{2} = \sin 2l\theta \cdot \frac{\sin 4\lambda\left(\theta - \frac{\pi}{4}\right)}{\sin 2\left(\theta - \frac{\pi}{4}\right)},$$

in which $\lambda = \frac{h}{2}$ or $\frac{h+1}{2}$ according as h is even or odd, we might thus be led *by the consideration of the differential equations* to discover the following *limiting values of definite integrals*:

$$\lim_{\lambda=\infty}\int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda\left(\theta - \frac{\pi}{4}\right)}{\sin 2\left(\theta - \frac{\pi}{4}\right)} \sin 2l\theta \cos(2at\sin\theta) = \frac{\pi}{2}\sin\frac{l\pi}{2}\cos(at\sqrt{2});$$

$$\lim_{\lambda=\infty}\int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda\left(\theta - \frac{\pi}{4}\right)}{\sin 2\left(\theta - \frac{\pi}{4}\right)} \frac{\sin 2l\theta}{\sin\theta} \sin(2at\sin\theta) = \frac{\pi}{\sqrt{2}}\sin\frac{l\pi}{2}\sin(at\sqrt{2}).$$

59. In like manner, if the initial conditions be

$$y_{j,0} = \eta_i \sin\frac{ij\pi}{n+1}, \quad y'_{j,0} = \eta'_i \sin\frac{ij\pi}{n+1},$$

and if we use the formula (35) of article **57**, we have the expression

$$y_{l,t} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \left(\Sigma_{(j)-\infty}^{\infty} \sin 2j\theta \sin \frac{ij\pi}{n+1}\right) \sin 2l\theta \left(\eta_i + \eta_i' \int_0^t dt\right) \cos(2at \sin\theta);$$

in which $\Sigma_{(j)-\infty}^{\infty} = 2 \lim\limits_{j=\infty} \Sigma_{(j)1}^{j}$; and, by article **24**,

$$4\Sigma_{(j)1}^{j} \sin 2j\theta \sin 2j\alpha = \frac{\sin(2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} - \frac{\sin(2j+1)(\theta+\alpha)}{\sin(\theta+\alpha)};$$

therefore

$$y_{l,t} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \lim_{j=\infty}\left(\frac{\sin(2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} - \frac{\sin(2j+1)(\theta+\alpha)}{\sin(\theta+\alpha)}\right) \sin 2l\theta \left(\eta_i + \eta_i' \int_0^t dt\right) \cos(2at\sin\theta),$$

in which $\alpha = \frac{i\pi}{2(n+1)}$, so that α is > 0 & $< \frac{\pi}{2}$, if $i > 0$, $< n+1$. We may therefore neglect the part depending on the rapidly fluctuating term $\sin(2j+1)(\theta+\alpha)$ as being rigorously null at the limit $j=\infty$; and in the part depending on $\sin(2j+1)(\theta-\alpha)$ may confine ourselves to the consideration of infinitely small values, positive or negative, of $\theta-\alpha$. We find therefore, as the limit sought,

$$y_{l,t} = \sin 2l\alpha \left(\eta_i + \eta_i' \int_0^t dt\right) \cos(2at\sin\alpha),$$

the initial conditions being, for all integer values of j,

$$y_{j,0} = \eta_i \sin 2j\alpha, \quad y_{j,0}' = \eta_i' \sin 2j\alpha;$$

and thus the formula of the 4th corollary, article **6**, for the case of a finite number of particles is deduced from that of an infinite number. And hence by reasoning similar to that of article **7**, we may infer, for an infinite system, that if the initial conditions be, for all integer values of j,

$$y_{j,0} = \Sigma_{(i)1}^{n} \eta_i \sin \frac{ij\pi}{n+1}, \quad y_{j,0}' = \Sigma_{(i)1}^{n} \eta_i' \sin \frac{ij\pi}{n+1},$$

which require only that (if i be integer) we should have the kind & degree of initial periodicity expressed by the formula

$$y_{j,0} = -y_{2n+2-j,0} = y_{2n+2+j,0}, \quad y_{j,0}' = -y_{2n+2-j,0}' = y_{2n+2+j,0}',$$

we shall then have, for all integer values of l & for all values of t, the same kind & degree of periodicity, which may be expressed as follows:

$$y_{l,t} = \Sigma_{(i)1}^{n} \sin \frac{il\pi}{n+1} \left(\eta_i + \eta_i' \int_0^t dt\right) \cos\left(2at \sin \frac{i\pi}{2n+2}\right); \qquad (8)$$

$$y_{l,t} = -y_{2n+2-l,t} = y_{2n+2+l,t}, \quad y_{l,t}' = -y_{2n+2-l,t}' = y_{2n+2+l,t}'.$$

Thus the theory of a finite system is included in that of an infinite system, since the formula (8) has been deduced from the formula (12).

60. The reasoning of the foregoing article shows also that if the initial conditions be, for the whole extent of an infinite system, or for all integer values of j,

$$y_{j,0} = \eta_\alpha \sin 2j\alpha, \quad y_{j,0}' = \eta_\alpha' \sin 2j\alpha,$$

α being any real arc between 0 and $\frac{\pi}{2}$, (and therefore also if α be any real arc), we shall have, for the whole extent of the same system at any time t,

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos (2at \sin \alpha).$$

In fact it has been shown, & is evident, that this expression satisfies the indefinite equation in mixed differences, whatever α may be. And here we might commence, from a new point of view, reasonings analogous to those of article **23**; but it seems desirable to pass on to other things.

61. By a transformation analogous to that of article **50**, we may simplify the formula (18) of article **30**. For if, in that formula, we extend the summation relatively to i as far as the value $i = 2n+1$, we merely double the whole expression, because $(2n+2-i)\phi = \pi - i\phi$, and the value $i = n+1$ gives $\sin 2il\phi = \sin l\pi = 0$; & the two lines of the last expression in the 1st sentence of article **32** are changed each into the other by changing θ to $\pi - \theta$; so that by confining ourselves to one alone we again halve the expression. In this manner we find that the formula (18) may be thus written:

$$y_{l,t} = \frac{B_k^{\backprime}}{2(n+1)} \Sigma_{(i)1}^{2n+1} \sin 2l\theta \frac{\sin (2j+1)(\theta - \alpha)}{\sin (\theta - \alpha)} \left(\cos \beta_k^{\backprime} \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k^{\backprime} \sin \right) (2at \sin \theta); \quad (36)$$

in which $\theta = \frac{i\pi}{2(n+1)}$, $\alpha = \frac{k\pi}{2(n+1)}$. It is evident that this expression for $y_{l,t}$ satisfies the equation in differences, & gives $y_{0,t} = 0$, $y_{n+1,t} = 0$; it ought also to give

$$y_{l,0} = B_k^{\backprime} \cos \beta_k^{\backprime} \sin 2l\alpha, \quad y'_{l,0} = 2a \sin \alpha \, B_k^{\backprime} \sin \beta_k^{\backprime} \sin 2l\alpha,$$

if l be 1, 2, ... j; but $y_{l,0} = 0$, $y'_{l,0} = 0$, if $l = j+1, j+2, \dots n$. We ought therefore to find that

$$\Sigma_{(i)1}^{2n+1} \frac{\sin 2l\theta}{2n+2} \frac{\sin (2j+1)(\theta - \alpha)}{\sin (\theta - \alpha)} = \sin 2l\alpha, \quad \text{or } = 0,$$

according as l is not greater, or greater, than j; l and j being integers which are each > 0 and $< n+1$; and θ, α having their recent values. In fact

$$\frac{\sin (2j+1)(\theta - \alpha)}{\sin (\theta - \alpha)} = \Sigma_{(j)-j}^{j} \cos 2j(\theta - \alpha) = 1 + 2\, \Sigma_{(j)1}^{j} \cos 2j(\theta - \alpha);$$

& making $\theta = i\phi$, we have

$$2 \sin 2l\theta \cos 2j(\theta - \alpha) = \sin \{2(j+l)i\phi - 2j\alpha\} - \sin \{2(j-l)i\phi - 2j\alpha\};$$

$$\Sigma_{(i)1}^{2n+1} \sin \{2(j \pm l)i\phi - 2j\alpha\} = \frac{\cos \{(j \pm l)\phi - 2j\alpha\} - \cos \{(4n+3)(j \pm l)\phi - 2j\alpha\}}{2 \sin (j \pm l)\phi}$$

$$= \frac{\sin \{(2n+1)(j \pm l)\phi\}}{\sin (j \pm l)\phi} \sin \{(2n+2)(j \pm l)\phi - 2j\alpha\} = \sin 2j\alpha,$$

unless denominator $= 0$; therefore

$$\Sigma_{(i)1}^{2n+1} \sin 2li\phi \cos 2j(i\phi - \alpha) = 0$$

unless $j = l$, if j & l be each > 0, $< n+1$; but when $j = l$, this sum becomes $(2n+2)\sin 2l\alpha$.

62. Another mode of verifying, and indeed proving the formula (36), is to show that the expression of which it is (relatively to j) the sum, namely

$$y_{l,t} = \frac{B^{\backprime}_k}{n+1} \Sigma^{2n+1}_{(i)1} \sin 2l\theta \cos 2j\,(\theta - \alpha) \left(\cos \beta^{\backprime}_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta^{\backprime}_k \sin \right) (2at \sin \theta),$$

corresponds to the case of only one particle P_j initially disturbed,

$$y_{j,0} = B^{\backprime}_k \cos \beta^{\backprime}_k \sin 2j\alpha, \quad y'_{j,0} = 2a \sin \alpha\, B^{\backprime}_k \sin \beta^{\backprime}_k \sin 2j\alpha.$$

Accordingly the last expression for $y_{l,t}$ may be thus written:

$$y_{l,t} = \frac{2}{n+1} \Sigma^{n}_{(i)1} \sin \frac{il\pi}{n+1} \sin \frac{ij\pi}{n+1} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2} i\pi}{n+1} \right);$$

and under this form it agrees with (7). Reciprocally, in (7), if we suppress the sign of summation $\Sigma^{n}_{(j)1}$ so as to attend only to the effect of the initial state of a single particle P_j; & if we represent this state by the formulae

$$y_{j,0} = \eta_i \sin \frac{ij\pi}{n+1}; \quad y'_{j,0} = \eta'_i \sin \frac{ij\pi}{n+1};$$

we may write

$$y_{l,t} = \frac{1}{n+1} \Sigma^{2n+1}_{(k)1} \sin \frac{lk\pi}{n+1} \cos \frac{j\,(k-i)\,\pi}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2} k\pi}{n+1} \right)$$

$$= \frac{1}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \Sigma^{2n+1}_{(k)1} \sin \frac{lk\pi}{n+1} \cos \frac{j\,(k-i)\,\pi}{n+1} \cos \left(2at \sin \frac{\frac{1}{2} k\pi}{n+1} \right).$$

Interchanging i & k to conform more closely to the notation of article **28**, and summing relatively to j from 1 to j, we get this other formula, equivalent to (18) or (36):

$$y_{l,t} = \frac{1}{2n+2} \left(\eta_k + \eta'_k \int_0^t dt \right) \Sigma^{2n+1}_{(i)1} \frac{\sin \dfrac{il\pi}{n+1} \sin \dfrac{(2j+1)\,(i-k)\,\pi}{2n+2}}{\sin \dfrac{(i-k)\,\pi}{2n+2}} \cos \left(2at \sin \frac{i\pi}{2n+2} \right). \quad (37)$$

And if we sum, instead, from $j_{,}$ to j, we get this transformation of (17) & therefore this other form of the solution of Problem II:

$$y_{l,t} = \frac{1}{n+1} \left(\eta_k + \eta'_k \int_0^t dt \right) \Sigma^{2n+1}_{(i)1} \frac{\sin\,(j - j_{,} + 1)\,(\theta - \alpha)}{\sin\,(\theta - \alpha)}$$

$$\times \sin 2l\theta \cos\,(j + j_{,})\,(\theta - \alpha) \cos\,(2at \sin \theta); \quad (38)$$

in which, as in many former equations, $\theta = \frac{i\pi}{2n+2}$, $\alpha = \frac{k\pi}{2n+2}$. It is evident that this includes the formula (34).

Problem III.

63. It is proposed to determine the consequences of the supposition that the initial states of some number of successive particles correspond to one of the two conjugate components of a simple movement, (considered in article **20**,) that is to the uniform transmission of phase in one direction.

64. Before passing to this determination, it will be convenient to review & recapitulate the chief results already obtained.

(I). The number n of moveable particles being *finite*, so that the differential equations to be satisfied are n in number, namely

$$y''_{1,t}=a^2(0-2y_{1,t}+y_{2,t}),\quad y''_{2,t}=a^2(y_{1,t}-2y_{2,t}+y_{3,t}),$$

$$y''_{3,t}=a^2(y_{2,t}-2y_{3,t}+y_{4,t}),\quad \ldots,\quad y''_{n-1,t}=a^2(y_{n-2,t}-2y_{n-1,t}+y_{n,t}),\quad y''_{n,t}=a^2(y_{n-1,t}-2y_{n,t}),$$

we found that we might satisfy these equations, & therefore also the dynamical conditions of the question, by supposing all the displacements to correspond to that simple mode of vibration, which is expressed by the formula

$$y_{l,t}=\sin 2l\alpha_i\left(\eta_i+\eta'_i\int_0^t dt\right)\cos(2at\sin\alpha_i);$$

in which $\alpha_i=\dfrac{\imath\pi}{2n+2}$, and η_i, η'_i are constants. In this mode, the n moveable or intermediate and the two fixed or extreme particles are, at any moment t, arranged all upon the i alternate branches of a *sinusoid*, which has 2 extreme & $i-1$ intermediate nodes and i venters. This sinusoid varies with the time, & oscillates between two extreme positions determined by those of the first venter. The sinusoidal form is expressed by the factor $\sin 2l\alpha_i$, and the oscillation of the first venter by the factor

$$\left(\eta_i+\eta'_i\int_0^t dt\right)\cos(2at\sin\alpha_i)$$

$$=\left(\eta_i\cos+\frac{\eta'_i\sin}{2a\sin\alpha_i}\right)(2at\sin\alpha_i)=(\eta_i\cos+r_i^{-1}\eta'_i\sin)(tr_i),\quad \text{if } r_i=2a\sin\alpha_i.$$

The greatest positive excursion of the venter is attained at those moments, (succeeding each other after equal intervals or periods of time, each period T_i being

$$=\frac{2\pi}{r_i}=\frac{\pi}{a}\operatorname{cosec}\alpha_i=\frac{\pi}{a}\operatorname{cosec}\frac{\imath\pi}{2n+2}\Big),$$

when

$$\cos tr_i=\frac{\eta_i}{\sqrt{\eta_i^2+r_i^{-2}\eta_i'^2}},\quad \sin tr_i=\frac{r_i^{-1}\eta'_i}{\sqrt{\eta_i^2+r_i^{-2}\eta_i'^2}},$$

and this greatest positive excursion $=B_i=\sqrt{\eta_i^2+r_i^{-2}\eta_i'^2}$. The greatest negative excursion $=-B_i$, & is attained at moments which follow or precede, by exactly half the periodic time T_i, the moments of greatest positive excursion; so that if these last be of the form $\epsilon_i+\nu T_i$, in which ν is any integer, positive, negative or null, while ϵ_i is such that

$$\cos r_i\epsilon_i=\frac{\eta_i}{\sqrt{(\eta_i^2+r_i^{-2}\eta_i'^2)}},\quad \sin r_i\epsilon_i=\frac{r_i^{-1}\eta'_i}{\sqrt{(\eta_i^2+r_i^{-2}\eta_i'^2)}},$$

the moments of greatest negative excursion are expressed by the formula $\epsilon_i+(\nu+\frac{1}{2})T_i$. The intermediate moments $\epsilon_i+(\nu\pm\frac{1}{4})T_i$ are such that in them the sinusoid reduces itself to a straight line, the displacements of the particles all vanishing; in such a manner that $y_{l,t}=0$, if $t=\epsilon_i+(\nu\pm\frac{1}{4})T_i$. In fact we have then $tr_i=r_i\epsilon_i+2\nu\pi\pm\dfrac{\pi}{2}$; therefore

$$\cos tr_i=\mp\sin r_i\epsilon_i,\quad \sin tr_i=\pm\cos r_i\epsilon_i,\quad (\eta_i\cos+r_i^{-1}\eta'_i\sin)(tr_i)=0.$$

The variable velocity $y'_{l,t}$ is expressed as follows, in this simple mode of vibration, for any particle P_l:

$$y'_{l,t}=\sin 2l\alpha_i\left(\eta'_i+\eta_i\frac{d}{dt}\right)\cos tr_i;$$

in which the first factor $\sin 2l\alpha_i$ corresponds to the sinusoidal relation between the several particles, & the other factor

$$\left.\eta'_i + \eta_i \frac{d}{dt}\right) \cos tr_i = (\eta'_i \cos - r_i \eta_i \sin)\, tr_i$$

expresses the velocity of the first venter. This velocity vanishes when $t = \epsilon_i + \nu T_i$ and when $t = \epsilon_i + (\nu + \frac{1}{2})\, T_i$, that is, at the moments of greatest positive or negative excursion; but at the moments when $t = \epsilon_i + (\nu \pm \frac{1}{4})\, T_i$, & when therefore $tr_i = r_i \epsilon_i + 2\nu\pi \pm \frac{\pi}{2}$, the velocity of the venter becomes $= \mp r_i \sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}$; it attains therefore at these moments a negative or positive maximum of amount & this greatest velocity is equal to $\sqrt{\eta_i'^2 + r_i^2 \eta_i^2} = \sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}$ multiplied by the coefficient r_i which multiplies the time t under the signs of periodicity. (In former articles $r_i \epsilon_i$ has been called β_i.)

(II). The foregoing being a *possible permanent mode* of vibration of the system, it follows that if at any one moment, such as the moment $t = 0$, the displacements $y_{l,0}$ & the velocities $y'_{l,0}$ are all such as to agree with it, then, at all subsequent moments t, the displacements & velocities $y_{l,t}$ & $y'_{l,t}$ will still agree with the same simple mode. In other words, if the particles are all arranged on a sinusoidal curve of the form $y_{x,0} = Y_0 \sin 2x\alpha_i$ at the moment 0 & also on another such curve $y_{x,dt} = Y_{dt} \sin 2x\alpha_i$ at the infinitely near moment dt, the coefficient α_i being still $= \frac{i\pi}{2n+2}$ and the coefficients Y_0 and Y_{dt} representing for these two near moments 0 and dt the displacements of the first venter, $\left(\text{for which } x = \frac{\pi}{4\alpha_i} = \frac{n+1}{2i}\right)$, then at any subsequent moment t the particles will all be arranged on a curve of the same kind, namely $y_{x,t} = Y_t \sin 2x\alpha_i$; in which the coefficient Y_t represents the displacement of the venter and satisfies the differential equation of the second order $Y''_t + r_i^2 Y_t = 0$, so that it may be deduced from Y_0 & from $Y_{dt} = Y_0 + Y'_0 dt$ by the formula

$$Y_t = \left(Y_0 + Y'_0 \int_0^t dt\right) \cos tr_i.$$

(III). By the linear form of the differential equations of the question, it is permitted to add together any number of particular integrals or to superpose any number of small motions of which each is separately possible. On the other hand, any single initial displacement $y_{j,0}$, of any one particle P_j, may be considered as the sum or resultant of n different initial sinusoidal displacements of the form $y_{l,0} = \eta_i \sin 2l\alpha_i$, of which each separately extends to all the particles P_l, but which destroy each other by interference or superposition for all the particles except P_j. For we may write $y_{j,0} = \Sigma_{(i)\,1}^{n} \eta_i \sin 2j\alpha_i$, if we so choose the n coefficients η_i as to have $\eta_i = \frac{2}{n+1} y_{j,0} \sin 2j\alpha_i$; because $\Sigma_{(i)\,1}^{n} (\sin 2j\alpha_i)^2 = \Sigma_{(i)\,1}^{n} \left(\sin \frac{ij\pi}{n+1}\right)^2 = \frac{n+1}{2}$. And with the same choice of the coefficients η_i we shall have as the resultant initial displacement of any other particle P_l the null expression

$$y_{l,0} = \Sigma_{(i)\,1}^{n} \eta_i \sin 2l\alpha_i = \frac{2}{n+1} y_{j,0} \Sigma_{(i)\,1}^{n} \sin 2j\alpha_i \sin 2l\alpha_i = 0,$$

l being different from j. The effect of a single initial displacement $y_{j,0}$ of any single particle P_j

is to produce, at the time t, a system of displacements, or a complex mode of vibration, represented by the formula $y_{l,t}=\frac{2}{n+1}y_{j,0}\Sigma_{(i)1}^{n}\sin 2j\alpha_i\sin 2l\alpha_i\cos tr_i$. In fact this complex mode is a possible permanent mode, because it is the sum of n simple possible & permanent modes; & it reproduces the initial conditions, giving $y_{l,0}=0$ or $=y_{j,0}$ according as l (being integer) is different from or equal to j; & giving $y'_{l,0}=0$ for all values of l. In like manner the effect of any single initial velocity $y'_{j,0}$ is to produce the complex mode of vibration represented as follows:

$$y_{l,t}=\frac{2}{n+1}y'_{j,0}\int_0^t dt\,\Sigma_{(i)1}^{n}\sin 2j\alpha_i\sin 2l\alpha_i\cos tr_i.$$

And therefore the effect of any arbitrary initial state, or the complete solution of Problem I, may be expressed thus:

$$y_{l,t}=\frac{2}{n+1}\Sigma_{(j)1}^{n}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\Sigma_{(i)1}^{n}\sin 2j\alpha_i\sin 2l\alpha_i\cos tr_i;$$

in which it is important to observe that the part of the state of any particle P_l at the time t, which corresponds to a given value of i and to any given initial displacement or velocity of any other particle P_j, is equal to that part of the state of the latter particle P_j at the same time t, which corresponds to the same value of i (or to the same mode of component & simple vibration), and to an equal initial displacement or velocity of the former particle P_l; because the product $\sin 2j\alpha_i\sin 2l\alpha_i$ is symmetric relatively to j and l.*

(IV). To pass to Problem II, we are to suppose that the initial states of some one or more successive particles correspond to the k^{th} mode of simple vibration, so that for one or more successive values of j we have

$$y_{j,0}=\eta_k\sin 2j\alpha_k,\quad y'_{j,0}=\eta'_k\sin 2j\alpha_k,\quad \left(\alpha_k=\frac{k\pi}{2n+2}\right);$$

but that, for all the other values of j, $y_{j,0}$ and $y'_{j,0}$ vanish. And it now is necessary to sum the product $\sin 2j\alpha_i\sin 2j\alpha_k$ between some given limits of j; or at least this is the operation which first presents itself. But because we have afterwards to multiply by $\sin 2l\alpha_i$ and to sum relatively to i, and because $\alpha_{2n+2-i}=\pi-\alpha_i$, $\alpha_{n+1}=\frac{\pi}{2}$, we may substitute $\cos 2j(\alpha_i-\alpha_k)$ for $\sin 2j\alpha_i\sin 2j\alpha_k$, if we afterwards change $2\Sigma_{(i)1}^{n}$ to $\Sigma_{(i)1}^{2n+1}$. In this manner we find, if j be confined to one value, the expression

$$y_{l,t}=\frac{1}{n+1}\left(\eta_k+\eta'_k\int_0^t dt\right)\Sigma_{(i)1}^{2n+1}\sin 2l\alpha_i\cos 2j(\alpha_i-\alpha_k)\cos tr_i;$$

and if we are to sum relatively to j from $j_{,}$ to j, then we find

$$y_{l,t}=\frac{1}{2n+2}\left(\eta_k+\eta'_k\int_0^t dt\right)\Sigma_{(i)1}^{2n+1}\sin 2l\alpha_i\left\{\frac{\sin(2j+1)(\alpha_i-\alpha_k)-\sin(2j_{,}-1)(\alpha_i-\alpha_k)}{\sin(\alpha_i-\alpha_k)}\right\}\cos tr_i$$

$$=\frac{1}{n+1}\left(\eta_k+\eta'_k\int_0^t dt\right)\Sigma_{(i)1}^{2n+1}\sin 2l\alpha_i\frac{\sin(j-j_{,}+1)(\alpha_i-\alpha_k)}{\sin(\alpha_i-\alpha_k)}\cos(j+j_{,})(\alpha_i-\alpha_k)\cos tr_i;$$

an expression which is the complete solution of the II$^{\text{nd}}$ Problem.

---

* [Rayleigh, *Theory of Sound*, I, pp. 150–157.]

When all the j first particles are originally in the k^{th} mode of vibration and all the others are originally without displacement or velocity, so that $j_{,}=1$, we may write more simply

$$y_{l,t}=\frac{1}{2n+2}\left(\eta_k+\eta'_k\int_0^t dt\right)\Sigma_{(i)1}^{2n+1}\frac{\sin(2j+1)(\alpha_i-\alpha_k)}{\sin(\alpha_i-\alpha_k)}\sin 2l\alpha_i\cos tr_i\,.$$

(V). The number n of moveable particles being still supposed finite, the most general mode of motion of the system may be considered (as we have seen) as the resultant of n simple modes, of the kind lately described: so that we may write generally

$$y_{l,t}=\Sigma_{(i)1}^{n}\sin 2l\alpha_i\left(\eta_i+\eta'_i\int_0^t dt\right)\cos tr_i$$
$$=\Sigma_{(i)1}^{n}B_i\sin 2l\alpha_i\cos r_i(t-\epsilon_i).$$

And because $\alpha_{2n+2-i}=\pi-\alpha_i$, $r_{2n+2-i}=r_i$, we may write

$$y_{l,t}=\tfrac{1}{2}\Sigma_{(i)1}^{2n+1}B_i\sin 2l\alpha_i\cos r_i(t-\epsilon_i),$$

if we assume $B_{2n+2-i}=-B_i$, but $\epsilon_{2n+2-i}=\epsilon_i$.

With these last assumptions, we may therefore write also

$$y_{l,t}=\tfrac{1}{2}\Sigma_{(i)1}^{2n+1}B_i\sin(2l\alpha_i-r_it+r_i\epsilon_i);$$

and consequently, (B_{n+1} being $=0$), may consider the general mode of complex vibration $y_{l,t}$ as the sum of n pairs of component vibrations, of which each pair might separately continue to exist, but not (in general) each component semi-mode of vibration itself, if taken without its conjugate semi-mode, which has the same periodic time for the vibration of any single particle.

Those n component semi-modes for which $i<n+1$ would correspond, if the system were indefinite, to a continual *transmission of phase in the forward or positive direction* with a *velocity* (for the i^{th} mode) $=\dfrac{r_i}{2\alpha_i}=a\dfrac{\sin\alpha_i}{\alpha_i}$; and those n other component semi-modes for which $i>n+1$ would correspond, if the system were indefinite, to a continual *transmission of phase in the backward or negative direction* with a *velocity* which, for the semi-mode conjugate to the i^{th}, is $=-a\dfrac{\sin\alpha_i}{\alpha_i}$, and is therefore equal in amount (though different in sign) to that just now determined for the i^{th} semi-mode itself. In fact, the parts of $y_{l,t}$ corresponding to these two conjugate semi-modes are

$$\tfrac{1}{2}B_i\sin(2l\alpha_i-r_it+r_i\epsilon_i)\quad\text{and}\quad\tfrac{1}{2}B_i\sin(2l\alpha_i+r_it-r_i\epsilon_i).$$

Their resultant vanishes for the extreme particles P_0 and P_{n+1}, whatever t may be; & for any intermediate particle P_l, it is, as before,

$$B_i\sin 2l\alpha_i\cos(r_it-r_i\epsilon_i).$$

In general, whatever may be the arbitrary initial state of the system, we may represent its state at the time t by the formula

$$y_{l,t}=\tfrac{1}{2}\Sigma_{(i)1}^{2n+1}(\eta_i\sin+r_i^{-1}\eta'_i\cos)(2l\alpha_i-2at\sin\alpha_i),$$

if we assume that $\eta_{2n+2-i}=-\eta_i$, $\eta'_{2n+2-i}=-\eta'_i$, and therefore that $\eta_{n+1}=0$, $\eta'_{n+1}=0$; α_i being still $=\dfrac{i\pi}{2n+2}$. And we shall still have, in this last formula for $y_{l,t}$, as in others,

$$\eta_i=\frac{2}{n+1}\Sigma_{(j)1}^{n}y_{j,0}\sin 2j\alpha_i,\quad \eta'_i=\frac{2}{n+1}\Sigma_{(j)1}^{n}y'_{j,0}\sin 2j\alpha_i.$$

Thus, generally, we have the expression

$$y_{l,t} = \frac{1}{n+1} \Sigma^{n}_{(j)1} \Sigma^{2n+1}_{(i)1} \sin 2j\alpha_i \,(y_{j,0} \sin + r_i^{-1} y'_{j,0} \cos)\,(2l\alpha_i - tr_i),$$

in which r_i is still $= 2a \sin \alpha_i$, and α_i is still $= \frac{i\pi}{2n+2}$.

Accordingly it is easy to prove à posteriori the truth of this last expression for $y_{l,t}$. And if in it we make $y_{j,0} = C_{i,j} \cos \gamma_{i,j}$, $r_i^{-1} y'_{j,0} = C_{i,j} \sin \gamma_{i,j}$, so that $C_{2n+2-i,j} = C_{i,j}$ and $\gamma_{2n+2-i,j} = \gamma_{i,j}$, we shall have

$$y_{l,t} = \frac{1}{n+1} \Sigma^{n}_{(j)1} \Sigma^{2n+1}_{(i)1} C_{i,j} \sin 2j\alpha_i \sin (2l\alpha_i + \gamma_{i,j} - tr_i).$$

But this last transformation does not seem to be attended with any advantage.

65. The foregoing article contains a recapitulation of the chief results obtained already in this manuscript for the case of a finite system. If the system be unlimited in one direction, so that only the condition $y_{0,t} = 0$ but not the condition $y_{n+1,t} = 0$ is to be attended to, we have then the following results:

(I)′. The differential equations to be satisfied are now infinite in number; they need involve only positive values of l, but l may be taken as great as we please; they may be written thus:

$$y''_{1,t} = a^2 (0 - 2y_{1,t} + y_{2,t}), \quad y''_{2,t} = a^2 (y_{1,t} - 2y_{2,t} + y_{3,t}), \ldots$$

$$y''_{l,t} = a^2 (y_{l-1,t} - 2y_{l,t} + y_{l+1,t}), \text{ \&c. ad infinitum.}$$

A particular integral, or possible permanent mode of motion of the system, which may also be considered as a simple mode, is expressed by the formula

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt\right) \cos (2at \sin \alpha)$$

in which α is any real arc & η_α, η'_α are any arbitrary real functions thereof. This formula indicates an arrangement of all the particles on a sinusoidal curve, containing indefinitely many alternate branches and varying with the time, but so that a first node is always at the fixed particle P_0 & a first venter at a distance, as measured on the axis of the system, $= \frac{\pi}{4\alpha}$. The whole space-period, or interval between two similar modes, is $S_\alpha = \frac{\pi}{\alpha}$; the whole time-period, or periodic time of vibration of any one particle, is connected therewith, being $T_\alpha = \frac{\pi}{a} \operatorname{cosec} \alpha$. The positive maximum of excursion of the first venter is $B_\alpha = \sqrt{\eta_\alpha^2 + r_\alpha^{-2} \eta_\alpha'^2}$, in which $r_\alpha = 2a \sin \alpha$; and is attained when $t = \epsilon_\alpha + \nu T_\alpha$, ν being any integer & ϵ_α being such that $\cos r_\alpha \epsilon_\alpha = \frac{\eta_\alpha}{B_\alpha}$, $\sin r_\alpha \epsilon_\alpha = \frac{r_\alpha^{-1} \eta'_\alpha}{B_\alpha}$. The negative maximum of excursion of the same venter is $= -B_\alpha$, & is attained when $t = \epsilon_\alpha + (\nu + \frac{1}{2}) T_\alpha$. At both these two sets of moments, the velocity of the venter vanishes (& so do therefore the velocities of all the particles); while, on the contrary, the velocity of the venter attains the negative or positive maximum $\mp r_\alpha B_\alpha$ at the intermediate moments when $t = \epsilon_\alpha + (\nu \pm \frac{1}{4}) T_\alpha$; & at these last mentioned moments the displacements all vanish, or the particles are all in the axis.

(II)′. A mode of vibration, such as has been just now described, if once established, will persist; if then, at each of any two near moments 0 & dt, we have

$$y_{l,0} = \eta_\alpha \sin 2l\alpha \quad \text{and} \quad y_{l,dt} = (\eta_\alpha + \eta'_\alpha dt) \sin 2l\alpha,$$

we shall have, for all subsequent moments,

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos tr_\alpha,$$

that is,

$$y_{l,t} = B_\alpha \sin 2l\alpha \cos \{ r_\alpha (t - \epsilon_\alpha) \}.$$

(III)′. The sum or integral of any number of such simple vibrations, that is the resultant of the superposition of any finite or infinite number of them, if once established, will persist; but any single initial displacement $y_{j,0}$ may be expressed by such an integral as follows:

$$y_{l,0} = \frac{4}{\pi} y_{j,0} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \left(= \frac{2}{\pi} y_{j,0} \int_0^\pi d\alpha \sin 2j\alpha \sin 2l\alpha \right),$$

because this integral becomes $= y_{j,0}$ or $= 0$, according as the positive integer l is equal to or different from j; the effect of any single initial displacement $y_{j,0}$ is therefore to produce, at the time t, the system of displacements represented by the formula

$$y_{l,t} = \frac{4}{\pi} y_{j,0} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta.$$

In like manner the effect of any single initial velocity $y'_{j,0}$ is to produce, at the time t, the system of displacements

$$y_{l,t} = \frac{4}{\pi} y'_{j,0} \int_0^t dt \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta;$$

and therefore the effect of any arbitrary initial state of the indefinite system of particles P_1, P_2, ... is to produce, at the time t, a state which may be thus expressed:

$$y_{l,t} = \frac{4}{\pi} \Sigma_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta.$$

This result may be connected with the corresponding one in the subdivision (III) of article **64** for the case of a finite system; & the same remark respecting the symmetry of $\sin 2j\alpha \sin 2l\alpha$ applies.

(IV)′. If, for some set of successive values of j, from $j = j_{,}$ to $j = j$, we have the initial conditions $y_{j,0} = \eta_\alpha \sin 2j\alpha$, $y'_{j,0} = \eta'_\alpha \sin 2j\alpha$, while $y_{j,0}$ and $y'_{j,0}$ vanish for all other (positive) values of j, we have then, by changing $2\int_0^{\frac{\pi}{2}} d\theta$ to $\int_0^\pi d\theta$ & summing $\cos 2j(\theta - \alpha)$ relatively to j,

$$y_{l,t} = \frac{1}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \left\{ \frac{\sin (2j+1)(\theta - \alpha) - \sin (2j_{,} - 1)(\theta - \alpha)}{\sin (\theta - \alpha)} \right\} \sin 2l\theta \cos tr_\theta$$

$$= \frac{2}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \frac{\sin (j - j_{,} + 1)(\theta - \alpha)}{\sin (\theta - \alpha)} \cos (j + j_{,})(\theta - \alpha) \sin 2l\theta \cos tr_\theta;$$

and if in particular $j, = 1$, so that the first j particles all satisfy the foregoing initial conditions, then

$$y_{l,t} = \frac{1}{\pi}\left(\eta_\alpha + \eta'_\alpha \int_0^t dt\right) \int_0^\pi d\theta \frac{\sin(2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} \sin 2l\theta \cos tr_\theta .$$

(V)′. The most general mode of vibration of the present system may be expressed as follows:

$$y_{l,t} = \int_0^{\frac{\pi}{2}} d\theta \sin 2l\theta \left(\eta_\theta + \eta'_\theta \int_0^t dt\right) \cos tr_\theta ,$$

in which η_θ and η'_θ are connected with the initial state of the system by the relations

$$y_{l,0} = \int_0^{\frac{\pi}{2}} \eta_\theta \sin 2l\theta \, d\theta, \quad y'_{l,0} = \int_0^{\frac{\pi}{2}} \eta'_\theta \sin 2l\theta \, d\theta,$$

so that, by what was lately shown (section (III)′), we have

$$\eta_\theta = \frac{4}{\pi} \Sigma_{(j)1}^{\infty} y_{j,0} \sin 2j\theta, \quad \eta'_\theta = \frac{4}{\pi} \Sigma_{(j)1}^{\infty} y'_{j,0} \sin 2j\theta .$$

If we extend these last expressions to all values of θ from 0 to π, we shall have

$$y_{l,0} = \tfrac{1}{2} \int_0^\pi \eta_\theta \sin 2l\theta \, d\theta, \quad y'_{l,0} = \tfrac{1}{2} \int_0^\pi \eta'_\theta \sin 2l\theta \, d\theta,$$

and

$$y_{l,t} = \tfrac{1}{2} \int_0^\pi d\theta \sin 2l\theta \left(\eta_\theta + \eta'_\theta \int_0^t dt\right) \cos tr_\theta .$$

And this expression again may be put under the form

$$y_{l,t} = \tfrac{1}{2} \int_0^\pi d\theta \, (\eta_\theta \sin + r_\theta^{-1} \eta'_\theta \cos)(2l\theta - tr_\theta),$$

or, substituting for η_θ and η'_θ their values,

$$y_{l,t} = \frac{2}{\pi} \Sigma_{(j)1}^{\infty} \int_0^\pi d\theta \sin 2j\theta \, (y_{j,0} \sin + r_\theta^{-1} y'_{j,0} \cos)(2l\theta - tr_\theta).$$

The general expression for the mode of vibration or complex motion of a system indefinite in one direction may therefore be considered as the sum of an infinite number of pairs of conjugate component motions, in each of which there is a continual & uniform transmission of phase in one of two opposite directions. In any one such component motion, corresponding to $\theta = \alpha$, if $\alpha > 0$, $< \frac{\pi}{2}$, there is as above a space-period $S_\alpha = \frac{\pi}{\alpha}$, and a time-period $T_\alpha = \frac{2\pi}{r_\alpha} = \frac{\pi}{a} \operatorname{cosec} \alpha$; and the velocity of transmission is $\frac{S_\alpha}{T_\alpha} = a \frac{\sin \alpha}{\alpha}$. In the conjugate component motion, corresponding to $\theta = \pi - \alpha$, we have the same length of space-period & of time-period, but the velocity of transmission is negative and may be represented by $-a \frac{\sin \alpha}{\alpha}$. The combination of the two is necessary in order to preserve the fixity of P_0.

66. To recapitulate in like manner the results already obtained relative to a system which extends indefinitely in two opposite directions, without any condition of fixity, we may observe that:

(I)″. The differential equations are now all those included in the formula

$$y''_{l,t} = a^2 (y_{l-1,t} - 2y_{l,t} + y_{l+1,t}),$$

l receiving all integer values. A particular integral is

$$y_{l,t} = \cos(2l\alpha - \gamma_\alpha)\left(\eta_\alpha + \eta'_\alpha \int_0^t dt\right)\cos tr_\alpha,$$

in which α is an arbitrary real quantity, $r_\alpha = 2a\sin\alpha$, and η_α, η'_α, γ_α are arbitrary real functions of α. The function γ_α is introduced, instead of the constant $\frac{\pi}{2}$ which occupied its place in (I) and (I)′, because we do not now suppose $y_{0,t}$ to vanish, & therefore retain the cosines as well as the sines of $2l\alpha$. This simple mode of vibration is still sinusoidal, but the particle P_0 is not now necessarily a node. With this exception the remarks of (I)′ apply to it.

(II)″. In this indefinite system also a sinusoidal mode of vibration, if once established, will be permanent. A node and venter may be assumed at pleasure, but when the space-period is thus determined, the time-period is so too. By making $\gamma_\alpha = \frac{\pi}{2} + 2\alpha\lambda_\alpha$, in which λ_α is a new arbitrary but real function of α, namely the abscissa of a node, we may write for any one mode of this sort

$$y_{l,t} = B_\alpha \sin\{2(l-\lambda_\alpha)\alpha\}\cos\{r_\alpha(t-\epsilon_\alpha)\}.$$

(III)″. The sum or integral of any number of such vibrations will be permanent; therefore the effect of any single initial displacement $y_{j,0}$ is

$$y_{l,t} = \frac{2}{\pi} y_{j,0}\int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta,$$

and the effect of any single initial velocity $y'_{j,0}$ is

$$y_{l,t} = \frac{2}{\pi} y'_{j,0}\int_0^t dt \int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta.$$

The effect therefore of an initial arbitrary state of the system is, at the time t, expressed by the formula

$$y_{l,t} = \frac{2}{\pi}\Sigma_{(j)-\infty}^{\infty}\left(y_{j,0} + y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta.$$

By supposing $y_{-j,0} = -y_{j,0}$ and $y'_{-j,0} = -y'_{j,0}$, we can reduce this general expression for a system indefinite in *both* directions to the corresponding expression in (III)′ for a system which is indefinite in one direction only, the particle P_0 being fixed. It is also possible, by consideration of limits, to connect the expression for a doubly infinite with that for a doubly finite system, so as to deduce each from the other. In deducing the infinite from the finite, we suppose j, l, n to increase indefinitely together, preserving finite ratios; in deducing the finite from the infinite, we suppose a certain periodicity of initial state, for greater and greater distances from the origin P_0, in each of two opposite directions.

(IV)″. In the recent expression for $y_{l,t}$ we may change $2\int_0^{\frac{\pi}{2}}$ to $\int_0^{\pi}$; & then, if

$$y_{j,0} = \eta_\alpha \cos(2j\alpha - \gamma_\alpha), \quad y'_{j,0} = \eta'_\alpha \cos(2j\alpha - \gamma_\alpha),$$

we may change

$$\cos\{2(l-j)\theta\}\cos(2j\alpha-\gamma_\alpha) \text{ to } \cos\{2l\theta\mp\gamma_\alpha-2j(\theta\mp\alpha)\};$$

in which it is remarkable that we may take at pleasure the upper or the lower signs. Summing relatively to j, we find

$$y_{l,t}=\frac{1}{2\pi}\left(\eta_\alpha+\eta'_\alpha\int_0^t dt\right)\int_0^\pi d\theta\,[\sin\{(2j+1)(\theta\mp\alpha)-(2l\theta\mp\gamma_\alpha)\}$$
$$-\sin\{(2j_{,}-1)(\theta\mp\alpha)-(2l\theta\mp\gamma_\alpha)\}]\frac{\cos tr_\theta}{\sin(\theta\mp\alpha)}$$
$$=\frac{1}{\pi}\left(\eta_\alpha+\eta'_\alpha\int_0^t dt\right)\int_0^\pi d\theta\,\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\cos\{(j+j_{,})(\theta-\alpha)-(2l\theta-\gamma_\alpha)\}\cos tr_\theta,$$

as the effect, at the time t, of an initial state in which all the particles from $P_{j_{,}}$ to P_j inclusive are disturbed according to the simple mode (I)″, so as to have

$$y_{l,0}=\eta_\alpha\cos(2l\alpha-\gamma_\alpha),\quad y'_{l,0}=\eta'_\alpha\cos(2l\alpha-\gamma_\alpha),$$

and all the other particles are originally undisturbed.

In the particular case when $\gamma_\alpha=\frac{\pi}{2}$, the recent expression for $y_{l,t}$ reduces itself to (29) or (31), in articles **52**, **54**; the initial conditions being then

$$y_{l,0}=\eta_\alpha\sin 2l\alpha,\quad y'_{l,0}=\eta'_\alpha\sin 2l\alpha,$$

if l be $>j_{,}-1$ but $<j+1$, & $y_{l,0}=0$, $y'_{l,0}=0$, for all other integer values of l. By assuming also the relations $y_{-l,0}=-y_{l,0}$, $y'_{-l,0}=-y'_{l,0}$, we can pass from the case of a doubly infinite to that of a singly infinite system. And by consideration of limits, the cases of a doubly infinite and of a doubly finite system may be connected so as to deduce each from the other. The consideration of limits shows also that if $j_{,}=-\infty$, so that, in the doubly infinite system, the particle P_j & all behind it are initially disturbed according to the law $\frac{y_{l,0}}{\eta_\alpha}=\frac{y'_{l,0}}{\eta'_\alpha}=\cos(2l\alpha-\gamma_\alpha)$, while all beyond it have neither initial displacement nor velocity, the state of the system at the time t is expressed as follows (at least if α be between 0 & π, or more generally if $\sin\alpha$ be different from 0):

$$y_{l,t}=\tfrac{1}{2}\cos(2l\alpha-\gamma_\alpha)\left(\eta_\alpha+\eta'_\alpha\int_0^t dt\right)\cos tr_\alpha$$
$$-\frac{1}{2\pi}\left(\eta_\alpha+\eta'_\alpha\int_0^t dt\right)\int_0^\pi d\theta\sin\{2l\theta-\gamma_\alpha-(2j+1)(\theta-\alpha)\}\frac{\cos tr_\theta}{\sin(\theta-\alpha)}.$$

By making $\gamma_\alpha=\frac{\pi}{2}$, this reduces itself to the formula (28) of the **50**th article.

(V)″. The general formula for the doubly indefinite system may be thus written:

$$y_{l,t}=\frac{1}{\pi}\Sigma_{(j)-\infty}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^\pi d\theta\cos(2l\theta-2j\theta-tr_\theta);\quad(r_\theta=2a\sin\theta).$$

This most general mode of motion of this system may therefore be considered as the resultant of an infinite number of component motions, of which each separately corresponds to the continual and uniform transmission of phase in one of two opposite directions. Nor is it necessary now to compound or conjugate two opposite transmissions of this sort, in order to obtain a particular integral; we may employ either singly, & shall still obtain thereby a possible permanent mode of motion of the system.

67. The three preceding articles, **64**, **65**, **66**, contain a recapitulation of the chief results obtained in the earlier articles of this manuscript. A few remarks may however be usefully made here, before passing to the solution of Problem III. In particular, it seems useful to observe that the particular integrals hitherto considered correspond either to *oscillating* or to *travelling sinusoids*. The *oscillating* are those which have fixed nodes, but oscillating venters; the *travelling* are those which have neither nodes nor venters fixed, but which, instead of oscillating & thereby *changing form*, *change place* by moving uniformly & continually either in the positive or in the negative direction. When any particle is fixed, this condition of fixity obliges us either to suppose a node to be fixed thereat, & therefore the sinusoid to oscillate, or else two oppositely travelling but otherwise similar sinusoids to be always conjugated together. But when the system is doubly infinite, this conjugation is not necessary & we may suppose a doubly indefinite sinusoid to travel continually in one direction, without being accompanied by any other travelling in the direction opposite. Even if the system be finite, we may suppose the sinusoids, whether oscillating or travelling, to be infinite. Finally, in the case of a finite system & finite sinusoid, we may suppose the number of venters to exceed the number of particles; but this will lead to no essentially new law of arrangement or vibration of the particles themselves. Thus, in the case of a single vibrating particle, we may treat that particle as a 3rd, 5th, … venter; but its motion will be the same as when it was treated as the 1st.

68. Returning now to Problem III, article **63**, we are to suppose that the initial states of some finite number of successive particles correspond to some one *travelling sinusoid*, while the other particles are initially undisturbed; & are to investigate the consequences of this supposition.

69. For the case of a finite system, we have found (see page 492)

$$y_{l,t} = \frac{1}{n+1} \Sigma_{(j)\,1}^{n} \Sigma_{(i)\,1}^{2n+1} \sin 2j\alpha_i \,(y_{j,0} \sin + r_i^{-1} y'_{j,0} \cos)\,(2l\alpha_i - tr_i),$$

in which $\alpha_i = \dfrac{i\pi}{2n+2}$ and $r_i = 2a \sin \alpha_i$. And we are now to suppose that for certain successive values of j, namely from $j_{,}$ to j, we have

$$y_{j,0} = B_k \sin(2j\alpha_k + \beta_k), \quad y'_{j,0} = -r_k B_k \cos(2j\alpha_k + \beta_k),$$

k being an integer which is less or greater than $n+1$, according as the initial sinusoid is travelling forward or backward.

Instead of $\Sigma_{(i)\,1}^{2n+1} \sin 2j\alpha_i \sin(2l\alpha_i - tr_i)$ we may write

$$\Sigma_{(i)\,1}^{2n+1} \sin 2j\alpha_i \sin 2l\alpha_i \cos tr_i, \quad \text{\& we have} \quad 0 = \Sigma_{(i)\,1}^{2n+1} \cos 2j\alpha_i \sin 2l\alpha_i \cos tr_i;$$

we may therefore change, under the signs of summation, the product

$$\sin 2j\alpha_i \sin(2j\alpha_k + \beta_k) \quad \text{to} \quad \cos\{2j(\alpha_i - \alpha_k) - \beta_k\},$$

& in like manner

$$-\sin 2j\alpha_i \cos(2j\alpha_k + \beta_k) \quad \text{to} \quad -\sin\{2j(\alpha_i - \alpha_k) - \beta_k\} = \cos\left(2j(\alpha_i - \alpha_k) - \beta_k + \frac{\pi}{2}\right);$$

therefore

$$y_{l,t} = \frac{B_k}{n+1} \Sigma_{(i)\,1}^{2n+1} \sin 2l\alpha_i \frac{\sin(j - j_{,} + 1)(\alpha_i - \alpha_k)}{\sin(\alpha_i - \alpha_k)}$$
$$\times \left\{\cos\{(j + j_{,})(\alpha_i - \alpha_k) - \beta_k\} \cos tr_i - \frac{r_k}{r_i} \sin\{(j + j_{,})(\alpha_i - \alpha_k) - \beta_k\} \sin tr_i\right\}.$$

70. This is the general expression for $y_{l,t}$ in the present question; if $j, = 1$, that is, if all the first j particles are originally disturbed in the way supposed, we have

$$y_{l,t}=\frac{B_k}{n+1}\Sigma_{(i)1}^{2n+1}\frac{\sin 2l\alpha_i\left(\sin\{(2j+1)(\alpha_i-\alpha_k)-\beta_k\}\cos tr_i+\frac{r_k}{r_i}\cos\{(2j+1)(\alpha_i-\alpha_k)-\beta_k\}\sin tr_i\right)}{2\sin(\alpha_i-\alpha_k)}$$

$$+\frac{B_k}{n+1}\Sigma_{(i)1}^{2n+1}\frac{\sin 2l\alpha_i\left(\sin\beta_k\cos tr_i-\frac{r_k}{r_i}\cos\beta_k\sin tr_i\right)}{2\tan(\alpha_i-\alpha_k)}.$$

Making both $j, = 1$ and $j = n$, that is, supposing all the n particles to be originally disturbed in the way already mentioned, we find

$$\frac{\sin(2j+1)(\alpha_i-\alpha_k)}{\sin(\alpha_i-\alpha_k)}=\frac{\sin\{(i-k)\pi-(\alpha_i-\alpha_k)\}}{\sin(\alpha_i-\alpha_k)}=\frac{\sin(i-k)\pi}{\tan(\alpha_i-\alpha_k)}-\cos(i-k)\pi;$$

the first part vanishes or is equal to $2n+2$, according as i is different from or equal to k; & the second part disappears in the summation; in like manner

$$\frac{\cos(2j+1)(\alpha_i-\alpha_k)}{\sin(\alpha_i-\alpha_k)}=\frac{\cos(i-k)\pi}{\tan(\alpha_i-\alpha_k)}+\sin(i-k)\pi=\frac{\cos(i-k)\pi}{\tan(\alpha_i-\alpha_k)};$$

therefore the expression for the state of the system at the time t is

$$y_{l,t}=B_k\sin 2l\alpha_k\cos(tr_k-\beta_k)$$

$$+\frac{B_k}{2n+2}\Sigma_{(i)1}^{2n+1}\sin 2l\alpha_i\operatorname{cotan}(\alpha_i-\alpha_k)\operatorname{vers}(i-k)\pi\left(\sin\beta_k\cos tr_i-\frac{r_k}{r_i}\cos\beta_k\sin tr_i\right).$$

In this expression the first part corresponds to a possible and permanent mode of simple vibration; & the second part must correspond to an initial state in which the displacements and velocities of all the particles are represented by the formulae

$$y_{j,0}=B_k\cos 2j\alpha_k\sin\beta_k,\quad y'_{j,0}=-r_kB_k\cos 2j\alpha_k\cos\beta_k.$$

Accordingly, the effect of such an initial state of any single particle P_j is

$$y_{l,t}=\frac{B_k}{n+1}\Sigma_{(i)1}^{2n+1}\sin 2l\alpha_i\sin 2j(\alpha_i-\alpha_k)(\sin\beta_k\cos tr_i-r_kr_i^{-1}\cos\beta_k\sin tr_i);$$

and

$$\Sigma_{(j)1}^{n}\sin 2j(\alpha_i-\alpha_k)=\frac{\cos(\alpha_i-\alpha_k)-\cos(2n+1)(\alpha_i-\alpha_k)}{2\sin(\alpha_i-\alpha_k)}=\frac{\operatorname{vers}(i-k)\pi}{2\tan(\alpha_i-\alpha_k)}.$$

This last expression becomes $=0$ when $i-k$ is any even integer (zero included); if therefore k be odd, of the form $2\kappa-1$, we must take i even, of the form 2ι, & we have, as the corresponding value of the formula of the last article,

$$y_{l,t}=B_{2\kappa-1}\sin 2l\alpha_{2\kappa-1}\cos(tr_{2\kappa-1}-\beta_{2\kappa-1})$$

$$+\frac{B_{2\kappa-1}}{n+1}\Sigma_{(\iota)1}^{n}\sin 2l\alpha_{2\iota}\operatorname{cotan}(\alpha_{2\iota}-\alpha_{2\kappa-1})\left(\sin\beta_{2\kappa-1}\cos tr_{2\iota}-\frac{r_{2\kappa-1}}{r_{2\iota}}\cos\beta_{2\kappa-1}\sin tr_{2\iota}\right);$$

but if k be of the form 2κ, we must take i of the form $2\iota-1$, and

$$y_{l,t}=B_{2\kappa}\sin 2l\alpha_{2\kappa}\cos(tr_{2\kappa}-\beta_{2\kappa})+\frac{B_{2\kappa}}{n+1}\Sigma_{(\iota)1}^{n+1}\sin 2l\alpha_{2\iota-1}\operatorname{cotan}(\alpha_{2\iota-1}-\alpha_{2\kappa})$$

$$:\left(\sin\beta_{2\kappa}\cos tr_{2\iota-1}-\frac{r_{2\kappa}}{r_{2\iota-1}}\cos\beta_{2\kappa}\sin tr_{2\iota-1}\right).$$

71. For example, if $n=2$ and $k=1$, so that the system contains only two moveable particles and that these are originally in the state of the travelling sinusoid

$$y_{j,dt}=B_1\sin(2j\alpha_1+\beta_1-r_1 dt),$$

in which $\alpha_1=\frac{\pi}{6}$ and $r_1=a$; so that

$$y_{j,0}=B_1\sin\left(\frac{j\pi}{3}+\beta_1\right),\quad y'_{j,0}=-r_1B_1\cos\left(\frac{j\pi}{3}+\beta_1\right);$$

then the parts proportional to $\sin\frac{j\pi}{3}$, namely the initial partial displacements

$$y_{j,0}=B_1\cos\beta_1\sin\frac{j\pi}{3}$$

& the initial partial velocities

$$y'_{j,0}=aB_1\sin\beta_1\sin\frac{j\pi}{3},$$

will produce the permanent partial vibration corresponding to a fixed sinusoid with one venter, and represented by the partial formula

$$y_{l,t}=B_1\sin\frac{l\pi}{3}\cos(at-\beta_1);$$

and the parts proportional to $\cos\frac{j\pi}{3}$, namely the initial partial displacements

$$y_{j,0}=B_1\sin\beta_1\cos\frac{j\pi}{3},$$

or more fully

$$y_{1,0}=\tfrac{1}{2}B_1\sin\beta_1,\quad y_{2,0}=-\tfrac{1}{2}B_1\sin\beta_1,$$

& the initial partial velocities

$$y'_{j,0}=-aB_1\cos\beta_1\cos\frac{j\pi}{3},$$

or more fully

$$y'_{1,0}=-\frac{a}{2}B_1\cos\beta_1,\quad y'_{2,0}=\frac{a}{2}B_1\cos\beta_1,$$

will produce another partial vibration at the time t, which may be thus represented,

$$y_{l,t}=\tfrac{1}{3}B_1.\Sigma^2_{(\iota)1}\sin\frac{2l\iota\pi}{3}\cot\!\operatorname{an}\left(\frac{\iota\pi}{3}-\frac{\pi}{6}\right)(\sin\beta_1\cos tr_{2\iota}-ar_{2\iota}^{-1}\cos\beta_1\sin tr_{2\iota}),$$

in which

$$r_{2\iota}=2a\sin\frac{\iota\pi}{3}=a\sqrt{3};$$

also

$$\operatorname{cotan}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)-\operatorname{cotan}\left(\frac{2\pi}{3}-\frac{\pi}{6}\right)=\operatorname{cotan}\frac{\pi}{6}-\operatorname{cotan}\frac{\pi}{2}=\sqrt{3};$$

$$\because\quad y_{l,t}=\frac{B_1}{\sqrt{3}}\sin\frac{2l\pi}{3}\left(\sin\beta_1\cos at\sqrt{3}-\frac{1}{\sqrt{3}}\cos\beta_1\sin at\sqrt{3}\right).$$

Accordingly this also is a possible permanent mode of vibration of the system of two particles, & satisfies the initial conditions.

72. Again, if $n=2$ & if the initial conditions be

$$y_{j,0}=B_2\sin\beta_2\cos\frac{2j\pi}{3},\quad y'_{j,0}=-r_2B_2\cos\beta_2\cos\frac{2j\pi}{3},\quad (r_2=a\sqrt{3}),$$

we find by the formula for k even

$$y_{l,t}=\frac{B_2}{3}\Sigma_{(\iota)1}^{3}\sin\frac{l(2\iota-1)\pi}{3}\operatorname{cotan}\frac{(2\iota-3)\pi}{6}\left(\sin\beta_2\cos tr_{2\iota-1}-\frac{a\sqrt{3}}{r_{2\iota-1}}\cos\beta_2\sin tr_{2\iota-1}\right).$$

Also

$$\operatorname{cotan}\left(-\frac{\pi}{6}\right)-\operatorname{cotan}\frac{\pi}{2}=-\sqrt{3};\quad \because\ y_{l,t}=-\frac{B_2}{\sqrt{3}}\sin\frac{l\pi}{3}(\sin\beta_2\cos at-\sqrt{3}\cos\beta_2\sin at).$$

Accordingly this gives

$$y_{1,0}=-\frac{B_2}{2}\sin\beta_2=y_{2,0};\quad y'_{1,0}=\frac{a\sqrt{3}}{2}B_2\cos\beta_2=y'_{2,0}.$$

73. Next let the system be infinite in one direction, so that the condition $y_{0,t}=0$ but not the condition $y_{n+1,t}=0$ is to be attended to. We have now to suppose that for all values of j, from $j_{,}$ to j, the initial state is represented by the formulae

$$y_{j,0}=B_\alpha\sin(2j\alpha+\beta_\alpha),\quad y'_{j,0}=-r_\alpha B_\alpha\cos(2j\alpha+\beta_\alpha);$$

but, because we already know the effect of the initial state

$$y_{j,0}=B_\alpha\cos\beta_\alpha\sin 2j\alpha,\quad y'_{j,0}=r_\alpha B_\alpha\sin\beta_\alpha\sin 2j\alpha,\quad \text{(see page 493)},$$

it is sufficient now to calculate the effect of

$$y_{j,0}=B_\alpha\sin\beta_\alpha\cos 2j\alpha,\quad y'_{j,0}=-r_\alpha B_\alpha\cos\beta_\alpha\cos 2j\alpha.$$

Using for this purpose the formula

$$y_{l,t}=\frac{2}{\pi}\Sigma_{(j)1}^{\infty}\int_0^\pi d\theta\sin 2j\theta\sin 2l\theta\,(y_{j,0}\cos tr_\theta+r_\theta^{-1}y'_{j,0}\sin tr_\theta),$$

& changing under the sign of summation $\sin 2j\theta\cos 2j\alpha$ to $\sin 2j(\theta-\alpha)$, we get, as the new part of the final state of the system,

$$y_{l,t}=\frac{2B_\alpha}{\pi}\int_0^\pi d\theta\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin(j+j_{,})(\theta-\alpha)\sin 2l\theta\left(\sin\beta_\alpha\cos tr_\theta-\frac{r_\alpha}{r_\theta}\cos\beta_\alpha\sin tr_\theta\right).$$

The old part was (see page 493)

$$y_{l,t}=\frac{2B_\alpha}{\pi}\int_0^\pi d\theta\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\cos(j+j_{,})(\theta-\alpha)\sin 2l\theta\left(\cos\beta_\alpha\cos tr_\theta+\frac{r_\alpha}{r_\theta}\sin\beta_\alpha\sin tr_\theta\right);$$

therefore the sum of these two parts, or the solution of the question proposed in the present article, is

$$y_{l,t}=\frac{2B_\alpha}{\pi}\int_0^\pi d\theta\frac{\sin(j-j_{,}+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin 2l\theta$$

$$\times[\cos\{(j+j_{,})(\theta-\alpha)-\beta_\alpha\}\cos tr_\theta-r_\alpha r_\theta^{-1}\sin\{(j+j_{,})(\theta-\alpha)-\beta_\alpha\}\sin tr_\theta].$$

It might have been deduced from that of article **69** by changing B_i to B_α, α_i to θ, α_k to α, β_k to β_α, r_i to r_θ, r_k to r_α, and $\frac{1}{n+1}\Sigma_{(i)1}^{2n+1}$ to $\frac{2}{\pi}\int_0^\pi d\theta$.

It may also be thus written:

$$y_{l,t}=\frac{B_\alpha}{\pi}\int_0^\pi d\theta\,\frac{\sin 2l\theta}{\sin(\theta-\alpha)}\Big\{[\sin\{(2j+1)(\theta-\alpha)-\beta_\alpha\}-\sin\{(2j,-1)(\theta-\alpha)-\beta_\alpha\}]\cos tr_\theta$$
$$+\frac{r_\alpha}{r_\theta}[\cos\{(2j+1)(\theta-\alpha)-\beta_\alpha\}-\cos\{(2j,-1)(\theta-\alpha)-\beta_\alpha\}]\sin tr_\theta\Big\}=\Phi(2j+1)-\Phi(2j,-1).$$

74. The function Φ introduced at the end of the last article is such that

$$\Phi(1)=\frac{B_\alpha}{\pi}\int_0^\pi d\theta\,\frac{\sin 2l\theta}{\tan(\theta-\alpha)}\{-\sin\beta_\alpha\cos tr_\theta+r_\alpha r_\theta^{-1}\cos\beta_\alpha\sin tr_\theta\};$$

this, therefore, with its sign changed, is to be added to $\Phi(2j+1)$, in order to obtain the effect of the initial disturbance of the first j particles of this singly indefinite system. On the other hand, if we seek the effect of the initial disturbance of the $j,$^th^ and all following particles, we are to suppose $j=\infty$, & to calculate $\Phi(\infty)$; which is

$$\Phi(\infty)=B_\alpha\sin 2l\alpha(\cos\beta_\alpha\cos tr_\alpha+\sin\beta_\alpha\sin tr_\alpha)=B_\alpha\sin 2l\alpha\cos(tr_\alpha-\beta_\alpha).$$

We have therefore, for the effect of an initial disturbance of the kind supposed, but extending to the whole system, the expression

$$y_{l,t}=\Phi(\infty)-\Phi(1)=B_\alpha\sin 2l\alpha\cos(tr_\alpha-\beta_\alpha)+\frac{B_\alpha}{\pi}\int_0 d\theta\,\frac{\sin 2l\theta}{\tan(\theta-\alpha)}$$
$$\times(\sin\beta_\alpha\cos tr_\theta-r_\alpha r_\theta^{-1}\cos\beta_\alpha\sin tr_\theta);$$

& the second part of this expression, namely $-\Phi(1)$, must be the effect of that part of the initial state of the whole system which is represented by the formula $y_{j,dt}=B_\alpha\cos 2j\alpha\sin(\beta_\alpha-r_\alpha dt)$, or by $y_{j,0}=B_\alpha\cos 2j\alpha\sin\beta_\alpha$, $y'_{j,0}=-r_\alpha B_\alpha\cos 2j\alpha\cos\beta_\alpha$. Accordingly it is easy to verify and rededuce this result by making $j,=1$ & $j=\infty$ in the formula given near the middle of the preceding page. We may also easily deduce the present result, as the limit of either of those given at the end of article **70** for the case of a finite system.

75. As an example we may take the case $\alpha=\frac{\pi}{2}$, in which the initial state of the system corresponds to a travelling sinusoid of the form $y_{j,dt}=B\sin(j\pi+\beta-2adt)$, so that

$$y_{j,0}=B\sin\beta(-1)^j,\quad y'_{j,0}=2aB\cos\beta(-1)^{j+1}.$$

In this case, the permanent part, free from the sign of integration, disappears, & we find

$$y_{l,t}=-\frac{B}{\pi}\int_0^\pi d\theta\sin 2l\theta\tan\theta\left(\sin\beta\cos tr_\theta-\frac{2a}{r_\theta}\cos\beta\sin tr_\theta\right).$$

Accordingly

$$\int_0^\pi d\theta\sin 2\theta\tan\theta=\pi,\quad\text{and}\quad\int_0^\pi d\theta\{\sin 2(l+1)\theta+\sin 2l\theta\}\tan\theta=0,$$

if l, being integer, is >0; so that

$$-\frac{1}{\pi}\int_0^\pi d\theta\sin 2l\theta\tan\theta=(-1)^l,$$

if l is >0, and the initial conditions are satisfied. These initial conditions correspond to an alternate arrangement of the particles both in displacement and in velocity; & as long as l is much greater than t, the arrangement ought to remain nearly alternate; that is, the above expression ought to give, nearly,

$$y_{l,t}=B\sin(l\pi+\beta-2at)=B\cos l\pi\sin(\beta-2at),$$

if l be much greater than t. Accordingly this result is obtained by integrating from $\frac{\pi}{2}-\delta\theta$ to $\frac{\pi}{2}+\delta\theta$. On the contrary, if t be much greater than l, we have nearly $y_{l,t}=0$. But the consequences of supposing t large, or the state of a system after a very long time, shall be the object of a full examination hereafter.

76. Finally if the system be indefinite in both directions, and if for some set of successive particles, $P_{j_,}, \dots P_j$, the initial state is represented by the formula

$$y_{j,dt}=B\sin(\beta+2j\alpha-r_\alpha dt),$$

that is, more fully, by

$$y_{j,0}=B\sin(\beta+2j\alpha),\quad y'_{j,0}=-r_\alpha B\cos(\beta+2j\alpha),$$

we may resolve this travelling sinusoid into two fixed sinusoids, namely,

$$1^{\text{st}},\quad y_{j,0}=B\sin\beta\cos 2j\alpha,\quad y'_{j,0}=-r_\alpha B\cos\beta\cos 2j\alpha,$$

and

$$2^{\text{nd}},\quad y_{j,0}=B\cos\beta\sin 2j\alpha,\quad y'_{j,0}=\ \ r_\alpha B\sin\beta\sin 2j\alpha.$$

The effect of the first is, by page 496, (making $\gamma_\alpha=0$),

$$y_{l,t}=\frac{B}{\pi}\int_0^\pi d\theta\,\frac{\sin(j-j_,+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\cos\{(j+j_,)(\theta-\alpha)-2l\theta\}(\sin\beta\cos tr_\theta-r_\alpha r_\theta^{-1}\cos\beta\sin tr_\theta);$$

and the effect of the 2^{nd} part is, by the same page, $\left(\text{making } \gamma_\alpha=\frac{\pi}{2},\right)$

$$y_{l,t}=-\frac{B}{\pi}\int_0^\pi d\theta\,\frac{\sin(j-j_,+1)(\theta-\alpha)}{\sin(\theta-\alpha)}\sin\{(j+j_,)(\theta-\alpha)-2l\theta\}(\cos\beta\cos tr_\theta+r_\alpha r_\theta^{-1}\sin\beta\sin tr_\theta);$$

therefore the whole effect, or the solution of the present problem, is

$$y_{l,t}=\frac{B}{\pi}\int_0^\pi d\theta\,\frac{\sin(j-j_,+1)(\theta-\alpha)}{\sin(\theta-\alpha)}$$
$$\times[\sin\{\beta+2l\theta-(j+j_,)(\theta-\alpha)\}\cos tr_\theta-r_\alpha r_\theta^{-1}\cos\{\beta+2l\theta-(j+j_,)(\theta-\alpha)\}\sin tr_\theta].$$

It may also be thus written,

$$y_{l,t}=\frac{B}{2\pi}\int_0^\pi\frac{d\theta}{\sin(\theta-\alpha)}[\{\cos((2j+1)(\theta-\alpha)-(2l\theta+\beta))-\cos((2j_,-1)(\theta-\alpha)-(2l\theta+\beta))\}\cos tr_\theta$$
$$-r_\alpha r_\theta^{-1}\{\sin((2j+1)(\theta-\alpha)-(2l\theta+\beta))-\sin((2j_,-1)(\theta-\alpha)-(2l\theta+\beta))\}\sin tr_\theta]$$
$$=\Psi(2j+1)-\Psi(2j_,-1).$$

77. In this expression the function Ψ is such that

$$-\Psi(-\infty)=\frac{B}{2}\{\sin(2l\alpha+\beta)\cos tr_\alpha-\cos(2l\alpha+\beta)\sin tr_\alpha\}=\tfrac{1}{2}B\sin(2l\alpha+\beta-tr_\alpha);$$

if therefore all the particles as far as P_j inclusive are originally disturbed in the way supposed in the last article, we have, at the time t, a state which is thus expressed,

$$y_{l,t} = \tfrac{1}{2}B\sin(\beta + 2l\alpha - tr_\alpha) + \frac{B}{2\pi}\int_0^\pi \frac{d\theta}{\sin(\theta-\alpha)}[\cos\{(2j+1)(\theta-\alpha) - (2l\theta+\beta)\}\cos tr_\theta$$
$$- r_\alpha r_\theta^{-1}\sin\{(2j+1)(\theta-\alpha) - (2l\theta+\beta)\}\sin tr_\theta].$$

The part involving the sign $\int$ must therefore express the effect of an initial state in which all the particles as far as P_j inclusive are agitated according to the formula

$$y_{l,dt} = +\tfrac{1}{2}B\sin(\beta + 2l\alpha - r_\alpha dt),$$

& all the particles beyond P_j according to

$$y_{l,dt} = -\tfrac{1}{2}B\sin(\beta + 2l\alpha - r_\alpha dt).$$

We ought therefore to have

$$\pm\pi\sin(\beta + 2l\alpha) = \int_0 \frac{d\theta}{\sin(\theta-\alpha)}\cos\{(2j+1)(\theta-\alpha) - (2l\theta+\beta)\},$$
$$\pm\pi\cos(\beta + 2l\alpha) = \int_0^\pi \frac{d\theta}{\sin(\theta-\alpha)}\sin\{(2j+1)(\theta-\alpha) - (2l\theta+\beta)\},$$

the upper signs to be taken if l be not greater than j: that is, we ought to have

$$\pm\pi = \int_0^\pi \frac{d\theta}{\sin(\theta-\alpha)}\sin(2j-2l+1)(\theta-\alpha),$$

according as l is $\ngtr$ or $> j$; and

$$0 = \int_0^\pi \frac{d\theta}{\sin(\theta-\alpha)}\cos(2j-2l+1)(\theta-\alpha).$$

And it is easy to prove, in fact, that these equations are true. (See article **51**.)

Problem IV.

78. It is now required to determine the approximate or limiting forms to which the solution of the foregoing problem tends, when the system is numerous & the time elapsed is large.

79. Beginning with the case when the system extends indefinitely in both directions, and when all the particles as far as P_j inclusive are originally agitated according to the formula

$$y_{j,dt} = B\sin(\beta + 2j\alpha - r_\alpha dt), \quad r_\alpha = 2a\sin\alpha, \quad \alpha > 0, \ < \pi,$$

but all beyond P_j are originally undisturbed, we have to discuss the formula of article **77**, on the supposition that t is very great. In this manner we obtain, approximately, attending only to values of θ nearly equal to α,

$$y_{l,t} = \tfrac{1}{2}B\sin(\beta + 2l\alpha - tr_\alpha) + \frac{1}{2\pi}B\int_{\alpha-\delta\alpha}^{\alpha+\delta\alpha}\frac{d\theta}{\theta-\alpha}\cos\{(2j+1)(\theta-\alpha) - (2l\theta+\beta) + tr_\theta\}$$
$$= \tfrac{1}{2}B\sin(\beta + 2l\alpha - tr_\alpha)\left\{1 + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\theta}{\theta-\alpha}\sin(2j+1-2l+tr'_\alpha)(\theta-\alpha)\right\},$$

in which $r'_\alpha = 2a\cos\alpha$. If therefore l be considerably less than $j + \frac{1}{2} + at\cos\alpha$, then we have nearly $y_{l,t} = B\sin(\beta + 2l\alpha - tr_\alpha)$; but if l be considerably greater than $j + \frac{1}{2} + at\cos\alpha$, then, nearly, $y_{l,t} = 0$. And these conclusions hold good, whether t be large or small, & even for negative values of t; we may therefore consider the initial state of the system as having been and as continuing to be dynamically propagated, forwards or backwards according as $\cos\alpha$ is $>$ or <0, and with a velocity $=a\cos\alpha$.

80. Let us consider particularly the case $\alpha=\frac{\pi}{2}$, for which this velocity vanishes. The rigorous formula of article **77** becomes in this case

$$y_{l,t}=\tfrac{1}{2}B\,(-1)^l\sin(\beta-2at)-\frac{1}{2\pi}B\,(-1)^j\int_0^\pi d\theta\sec\theta$$
$$\times\left\{\sin\{(2j+1-2l)\,\theta-\beta\}\cos(2at\sin\theta)+\cos\{(2j+1-2l)\,\theta-\beta\}\frac{\sin(2at\sin\theta)}{\sin\theta}\right\};$$

which may also be rigorously thus expressed:

$$y_{l,t}=\tfrac{1}{2}(-1)^l B\sin(\beta-2at)-\frac{1}{2\pi}(-1)^j B\int_0^\pi d\theta\,\frac{1+\sin\theta}{\sin 2\theta}\sin\{(2j+1-2l)\,\theta-\beta+2at\sin\theta\}$$
$$+\frac{1}{2\pi}(-1)^j B\int_0^\pi d\theta\,\frac{1-\sin\theta}{\sin 2\theta}\sin\{(2j+1-2l)\,\theta-\beta-2at\sin\theta\}.$$

Now, while $\Theta_\theta=(2j+1-2l)\,\theta-\beta-2at\sin\theta$ receives a small but finite increment, θ in general receives, nearly, the increment $\frac{\Delta\Theta_\theta}{2j+1-2l-2at\cos\theta}=\Delta\theta$; if then l be considerably different from $j+\frac{1}{2}-at\cos\theta$, the factor $\sin\{(2j+1-2l)\,\theta-\beta-2at\sin\theta\}$ will fluctuate often between its extreme values, ± 1, while the other factor $\frac{1-\sin\theta}{\sin 2\theta}$ will vary little, unless θ be nearly $=0$ or π; thus, in calculating the 2nd definite integral, we may in general attend only to these particular values of θ. But for these values, we must combine the corresponding parts of the 1st definite integral, and to do this we may write the 2nd integral as follows:

$$-\frac{1}{2\pi}(-)^j B\int_0^\pi d\theta\,\frac{1-\sin\theta}{\sin 2\theta}\sin\{(2j+1-2l)\,\theta+\beta+2at\sin\theta\};$$

the whole expression for $y_{l,t}$ may therefore rigorously be thus written,*

$$y_{l,t}=\tfrac{1}{2}(-1)^l B\sin(\beta-2at)-\frac{1}{\pi}(-1)^j B\cos\beta\int_0^\pi\frac{d\theta}{\sin 2\theta}\sin\{(2j+1-2l)\,\theta+2at\sin\theta\}$$
$$+\frac{1}{2\pi}(-1)^j B\sin\beta\int_0^\pi\frac{d\theta}{\cos\theta}\cos\{(2j+1-2l)\,\theta+2at\sin\theta\};$$

so that if l be considerably greater or less than both $j+\frac{1}{2}+at$ and $j+\frac{1}{2}-at$, the sum of the parts corresponding to θ nearly $=0$ and θ nearly $=\pi$ is insensible; but if l be considerably greater than $j+\frac{1}{2}-at$ and at the same time considerably less than $j+\frac{1}{2}+at$, (t being large and positive,) then

---

* [Accepting Hamilton's method of treating the integrals, it is a question of finding the value of the integral

$$-\frac{1}{\pi}(-1)^j B\cos\beta\int\frac{d\theta}{\sin 2\theta}\sin\{(2j+1-2l)\,\theta+2at\sin\theta\}$$

between the limits 0, ϵ and $\pi-\eta$, π, where ϵ, η are small and $2j+1-2l+2at$, $2j+1-2l-2at$ are large positive or negative numbers. We get then, easily,

$$-\frac{1}{2\pi}(-1)^j B\cos\beta\int_0^\epsilon\frac{d\theta}{\theta}\sin(2j+1-2l+2at)\,\theta,$$
$$+\frac{1}{2\pi}(-1)^j B\cos\beta\int_0^\eta\frac{d\theta}{\theta}\sin(2j+1-2l-2at)\,\theta.$$

If ϵ, η are such that $|(2j+1-2l+2at)\,\epsilon|$, $|(2j+1-2l-2at)\,\eta|$ are large, the results follow as above.]

this sum of parts is sensibly $= -\frac{1}{2}(-1)^j B\cos\beta$; & consequently, a disturbance or displacement, represented thus and due to the initial velocities, spreads with two equal and opposite velocities $\pm a$ on the two sides of the particle P_j which terminated the initial disturbance, or rather on both sides of the point $j+\frac{1}{2}$; and this constant amount of resultant displacement is $=\frac{1}{4}ay'_{j,0}$. If we have exactly $l=j+\frac{1}{2}+at$, & if t be large, we have to consider

$$\int_0^\pi \frac{d\theta}{\sin 2\theta}\sin\{2at(\theta-\sin\theta)\},\quad \int_0^\pi \frac{d\theta}{\cos\theta}\cos\{2at(\theta-\sin\theta)\};$$

of which two integrals the second may be neglected, so far as depends on values of θ near to 0 or π, but the first gives, for the parts depending on those values,

$$\tfrac{1}{2}\int_0^\infty \frac{d\theta}{\theta}\sin\left(\frac{at\theta^3}{3}\right)=\frac{\pi}{12},\quad\text{and}\quad -\tfrac{1}{2}\int_{-\infty}^0 \frac{d\theta}{\theta}\sin(4at\theta)=-\frac{\pi}{4},$$

$2at$ being here an odd integer; so that the sum is $-\frac{\pi}{6}$, and the resultant displacement is

$$-\tfrac{1}{6}(-1)^j B\cos\beta=\tfrac{1}{12}ay'_{j,0}.$$

If l exactly equals $j+\frac{1}{2}-at$, $t>0$, then $2at$ is still a large odd integer and the parts considered are

$$-\frac{1}{\pi}(-1)^j B\cos\beta\left(\int_0^{\delta\theta}+\int_{\pi-\delta\theta}^{\pi}\right)\frac{d\theta}{\sin 2\theta}\sin\{2at(\theta+\sin\theta)\}=-\frac{1}{\pi}(-1)^j B\cos\beta\left(\frac{\pi}{4}-\frac{\pi}{12}\right),$$

giving still the same sum

$$-\tfrac{1}{6}(-1)^j B\cos\beta=\tfrac{1}{12}ay'_{j,0}.$$

And it seems likely that if l be the nearest integer either to $j+\frac{1}{2}+at$ or to $j+\frac{1}{2}-at$, when t is large, we shall still have nearly this same displacement $\frac{a}{12}y'_{j,0}$ as the part of the general expression which corresponds to values of θ near to 0 and π. As to the values of θ near $\frac{\pi}{2}$, we may use the 1st definite integral in the second formula of the present article, which, for this purpose, may be put under the form

$$+\frac{1}{2\pi}(-1)^j B\int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} d\theta\,\frac{\sin\{(2j+1-2l)\left(\frac{\pi}{2}+\theta-\frac{\pi}{2}\right)-\beta+2at-2at\operatorname{vers}\theta-\frac{\pi}{2}\}}{\theta-\frac{\pi}{2}}$$

$$=\frac{1}{2\pi}(-1)^l B\int_{\frac{\pi}{2}-}^{\frac{\pi}{2}+}\frac{d\theta}{\theta-\frac{\pi}{2}}\cos\left\{2at-\beta+(2j+1-2l)\left(\theta-\frac{\pi}{2}\right)-2at\operatorname{vers}\left(\theta-\frac{\pi}{2}\right)\right\}$$

$$=\frac{1}{\pi}(-1)^l B\int_0^\infty\frac{d\theta_{,}}{\theta_{,}}\sin(\beta-2at\cos\theta_{,})\sin(2j+1-2l)\theta_{,}.$$

If $2j+1-2l$ be >0 and large, and if we put $(2j+1-2l)\theta_{,}=\Theta_{,}$, we may suppose $\Theta_{,}$ l enough to allow of our changing, with a sufficient approximation, this integral to the forn

$$\frac{1}{\pi}(-1)^l B\int_0^{\Theta_{,}}\frac{d\Theta_{,}}{\Theta_{,}}\sin\Theta_{,}\sin\left(\beta-2at\cos\frac{\Theta_{,}}{2j+1-2l}\right),$$

while $2at\operatorname{vers}\theta_{,}=at\left(2\sin\frac{\theta_{,}}{2}\right)^2$ is very small, being nearly $=\frac{at\Theta_{,}^2}{(2j+1-2l)^2}$; and then the inte becomes $\frac{1}{2}(-1)^l B \sin(\beta - 2at)$. In like manner if $2j+1-2l$ be large but negative, so that $\frac{at}{(2j+1-2l)^2}$ is still extremely small, this integral becomes $-\frac{1}{2}(-1)^l B \sin(\beta - 2at)$.

Hence, adding the term free from the sign $\int$ at the beginning of page 504, we find the following results, as consequences of the initial state expressed by the formula

$$y_{l,dt} = (-1)^l B \sin(\beta - 2a\,dt) \quad \text{for } l \ngtr j;$$

(*a*)... If l be much greater than $j + \frac{1}{2} + at$, $y_{l,t} = 0$;

(*b*)... If l be much less (algebraically) than $j + \frac{1}{2} - at$,

$$y_{l,t} = (-1)^l B \sin(\beta - 2at);$$

(*c*)... If l be much less than $j + \frac{1}{2} + at$ but much greater than $j + \frac{1}{2}$, and if $\frac{2l-2j-1}{\sqrt{at}}$ be much greater than a certain large number $\Theta_{,}$, then

$$y_{l,t} = -\tfrac{1}{2}(-1)^j B \cos\beta;$$

(*d*)... If l be much greater than $j + \frac{1}{2} - at$ but much less than $j + \frac{1}{2}$, and if $\frac{2j+1-2l}{\sqrt{at}}$ be much greater than the same large number $\Theta_{,}$, then

$$y_{l,t} = -\tfrac{1}{2}(-1)^j B \cos\beta + (-1)^l B \sin(\beta - 2at);$$

but peculiar calculations are required near the critical values $l = j + \frac{1}{2}$, $l = j + \frac{1}{2} \pm at$. Thus, if $l = j$, & if we wish to calculate the part depending on values of θ near $\frac{\pi}{2}$, we have, by the last page, for this part, the expression (if t be large)

$$\frac{\pm}{\pi}(-1)^j B \int_0 d\theta_{,} \frac{\sin\theta_{,}}{\theta_{,}} \sin(\beta - 2at\cos\theta_{,});$$

which is insensible.

And generally if $2l - 2j - 1$ be small (whether positive or negative) in comparison with $\sqrt{at}$, so that $2at \operatorname{vers}\theta_{,}$ may attain a considerable value while $(2l-2j-1)\theta_{,}$ is very small, we have then, by the last page, to consider the part*

$$\frac{B}{\pi}(-1)^l \frac{2j+1-2l}{\sqrt{at}} \int_0^\infty d\theta_{\prime\prime} \sin(\beta - 2at + \theta_{\prime\prime}^2) = \tfrac{1}{2}B(-1)^l \frac{2j+1-2l}{\sqrt{a\pi t}} \sin\left(\beta - 2at + \frac{\pi}{4}\right);$$

which corresponds to a vibration, but with diminished amplitude, and with a *change of phase*; to which is to be added the constant displacement,

$$-\tfrac{1}{2}(-1)^j B \cos\beta, \quad \& \text{ also } \quad \tfrac{1}{2}(-1)^l B \sin(\beta - 2at).$$

---

* [The integral is of the form

$$-\frac{1}{\pi}(-1)^l B \int_0^{\Theta_{,}} \frac{d\theta}{\theta} \sin\theta \sin\left(\beta - 2at + \frac{at\theta^2}{(2j+1-2l)^2}\right);$$

where since $\Theta_{,}$ is small we may put $\theta = \sin\theta$. Putting $\theta_{\prime\prime}^2 = \frac{at\theta^2}{2j+1-2l}$, we get the required form.]

81. The corresponding constant part has not yet been calculated in the problem of article **79**. To do so, we must resume the formula of article **77**, attending to the factor r_θ^{-1} & to the values of θ which are near to 0 & π.

The part depending on these values of θ is

$$\frac{B}{4\pi}\int_0^\infty \frac{d\theta}{\theta}[\cos\{(2j+1)(\theta-\alpha)-(2l\theta+\beta)-2at\sin\theta\}-\cos\{(2j+1)(\theta-\alpha)-(2l\theta+\beta)+2at\sin\theta\}]$$

$$-\frac{B}{4\pi}\int_0^\infty \frac{d\theta}{\theta}[\cos\{(2j+1)(\theta+\alpha)-(2l\theta-\beta)-2at\sin\theta\}-\cos\{(2j+1)(\theta+\alpha)-(2l\theta-\beta)+2at\sin\theta\}]$$

$$=\frac{B}{2\pi}\sin\{(2j+1)\alpha+\beta\}\int_0^\infty \frac{d\theta}{\theta}[\sin\{(2j+1-2l)\theta-2at\sin\theta\}-\sin\{(2j+1-2l)\theta+2at\sin\theta\}];$$

it vanishes or is insensible if $j+\frac{1}{2}-l-at$ and $j+\frac{1}{2}-l+at$ are both large and have the same sign, that is, if l be much greater than $j+\frac{1}{2}+at$ or much less than $j+\frac{1}{2}-at$; but if l be much less than $j+\frac{1}{2}+at$ and yet much greater than $j+\frac{1}{2}-at$, so that $2j+1-2l-2at$ is large and negative while $2j+1-2l+2at$ is large and positive, then the above part becomes, nearly,

$$-\tfrac{1}{2}B\sin\{(2j+1)\alpha+\beta\}.$$

When $\alpha=\frac{\pi}{2}$, this reduces itself to the value found in article **80**, namely, $-\frac{1}{2}(-1)^j B\cos\beta$. In general it may be represented by $-\frac{1}{2}y_{j+\frac{1}{2},0}$, if the initial formula $y_{j,0}=B\sin(2j\alpha+\beta)$ be conceived to extend as far as the point $j+\frac{1}{2}$, or to the middle point between the particles P_j and P_{j+1}. It may also be thus written, $\frac{a}{4}y'_{j,0}-\frac{\cos\alpha}{2}y_{j,0}$.

82. If it happen that $j+\frac{1}{2}-at$ is an integer and if we take l equal hereto, so that $2at=2j+1-2l=$ a large positive odd integer number, the formula of the last article conducts us to calculate the integrals

$$\int_0^\infty \frac{d\theta}{\theta}\sin\{2at(\theta-\sin\theta)\},\quad -\int_0^\infty \frac{d\theta}{\theta}\sin\{2at(\theta+\sin\theta)\};$$

which are $\frac{\pi}{6}$, $-\frac{\pi}{2}$; their sum is therefore $=-\frac{\pi}{3}$, and the corresponding displacement is

$$-\frac{B}{6}\sin\{(2j+1)\alpha+\beta\}.$$

And the same result is obtained by supposing $l=j+\frac{1}{2}+at$. At these critical positions, the constant displacement is therefore only one third part of the value which it has for particles nearer to P_j.

83. If $2j+1-2l-2at$ is only small in comparison with $\sqrt[3]{at}$ but not exactly $=0$, we may still reduce the integral

$$\int_0^\infty \frac{d\theta}{\theta}\sin\{(2j+1-2l)\theta+2at\sin\theta\}$$

to $\frac{\pi}{2}$, but the integral

$$\int_0^\infty \frac{d\theta}{\theta}\sin\{(2j+1-2l)\theta-2at\sin\theta\}$$

assumes the form

$$\int_0^\infty \frac{d\theta}{\theta}\sin\{(2j+1-2l-2at)\theta+2at(\theta-\sin\theta)\},$$

in which the second part within the brackets now predominates; $2at(\theta - \sin\theta)$ being able now to attain a considerable magnitude, while not only θ is small but also $(2j + 1 - 2l - 2at)\,\theta$, this latter being nearly equal to

$$\frac{2j + 1 - 2l - 2at}{\sqrt[3]{\left(\frac{at}{3}\right)}}\sqrt[3]{2at(\theta - \sin\theta)}.$$

Also
$$\int_0^\infty \frac{d\theta}{\theta}\sin\{2at(\theta - \sin\theta)\} \text{ is } = \frac{\pi}{6},$$

at being very large; we have therefore only to calculate $\dfrac{2j+1-2l-2at}{\sqrt[3]{(\frac{1}{3}at)}}\displaystyle\int_0^\infty d\theta \cos(\theta^3)$. Now (see 1st Blank Book of the present year, 1839, page 65, left hand & the references there made)*,

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \int_0^\infty (ax + \sqrt{-1}bx)^{n-1}e^{-(ax+\sqrt{-1}bx)}d(ax + \sqrt{-1}bx), \quad (\text{if } a > 0),$$
$$= (a + \sqrt{-1}b)^n \int_0^\infty x^{n-1}e^{-ax}(\cos bx - \sqrt{-1}\sin bx)\,dx;$$

therefore

$$\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = r^{-n}\cos nv\,\Gamma(n), \quad \text{and} \quad \int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = r^{-n}\sin nv\,\Gamma(n),$$

Γ being the celebrated function tabulated by Legendre, and r, v being connected with a, b by the relations $a = r\cos v$ (> 0), $b = r\sin v$. These theorems hold however near a may be to 0; they hold even at that limit, and thereby give $\left(\text{making } v = \frac{\pi}{2},\ a = 0,\ r = b\right)$

$$\int_0^\infty x^{n-1}\cos bx\,dx = b^{-n}\cos\frac{n\pi}{2}\Gamma(n), \quad \int_0^\infty x^{n-1}\sin bx\,dx = b^{-n}\sin\frac{n\pi}{2}\Gamma(n).$$

Making $b = 1$ and $x = \theta^{\frac{1}{n}}$, these become

$$\int_0^\infty d\theta\cos(\theta^{\frac{1}{n}}) = n\cos\frac{n\pi}{2}\Gamma(n), \quad \int_0^\infty d\theta\sin(\theta^{\frac{1}{n}}) = n\sin\frac{n\pi}{2}\Gamma(n);$$

or, changing n to $\dfrac{1}{m}$, and observing that $n\,\Gamma(n) = \Gamma(n+1)$,

$$\int_0^\infty d\theta\cos(\theta^m) = \cos\frac{\pi}{2m}\Gamma\left(1 + \frac{1}{m}\right), \quad \int_0^\infty d\theta\sin(\theta^m) = \sin\frac{\pi}{2m}\Gamma\left(1 + \frac{1}{m}\right).$$

For example, $\displaystyle\int_0^\infty d\theta\cos\theta = 0$, $\displaystyle\int_0^\infty d\theta\sin\theta = 1$; (to be integrated as limiting results;)

$$\int_0^\infty d\theta\cos(\theta^2) = \sqrt{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \quad \int_0^\infty d\theta\sin(\theta^2) = \sqrt{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}};$$

and finally, returning to the present question, $\displaystyle\int_0^\infty d\theta\cos(\theta^3) = \frac{\sqrt{3}}{2}\Gamma\left(\frac{4}{3}\right)$; in which, by Legendre's Table, at the end of the 2nd Part of his Exercises, we have $\Gamma(\frac{4}{3}) = 10^{\bar{1},9508414} = 0,892979$. Thus, in the question of the present article, we have, because $\Gamma(\frac{4}{3}) = \frac{1}{3}\Gamma(\frac{1}{3})$,

$$\int_0^\infty \frac{d\theta}{\theta}\sin\{(2j + 1 - 2l)\theta - 2at\sin\theta\} = \frac{\pi}{6} + \frac{2j + 1 - 2l - 2at}{2\sqrt[3]{at\sqrt{3}}}\Gamma\left(\frac{1}{3}\right);$$

---

* [There is no trace of this book among the manuscripts.]

and the expression at the beginning of article **81** becomes

$$-B\sin\{(2j+1)\alpha+\beta\}\left\{\frac{1}{6}-\frac{2j+1-2l-2at}{4\pi\sqrt[3]{at}\sqrt{3}}\Gamma\left(\frac{1}{3}\right)\right\}.$$

84. In like manner, if $2j+1-2l+2at$ be small in comparison with $\sqrt[3]{at}$, the same expression becomes

$$-B\sin\{(2j+1)\alpha+\beta\}\left\{\frac{1}{6}+\frac{2j+1-2l+2at}{4\pi\sqrt[3]{at}\sqrt{3}}\Gamma\left(\frac{1}{3}\right)\right\};$$

so that if l be as much less than $j+\frac{1}{2}+at$, in this last result, as it was greater than $j+\frac{1}{2}-at$, in the result immediately preceding, or vice versà, these two results (of the present & the former articles) will coincide; or in other words the amount of disturbance, as distinct from vibration, increases very nearly according to the same law as we advance inward from both extremities towards the middle of its extent $=2at$, for the greater part of which extent it is nearly constant, but is reduced to one third of this constant amount at each extremity.

85. If l be nearly $=j+\frac{1}{2}+at\cos\alpha$, the integral of article **79** will take another form. In this case, because

$$\begin{aligned}(2j+1)(\theta-\alpha)-(2l\theta+\beta)+2at\sin\theta&=2at\sin\alpha-(2l\alpha+\beta)\\&\quad+(2j+1-2l+2at\cos\alpha)(\theta-\alpha)+2at\{\sin\theta-\sin\alpha-(\theta-\alpha)\cos\alpha\},\end{aligned}$$

and

$$2at\{\sin\theta-\sin\alpha-(\theta-\alpha)\cos\alpha\}=-at(\theta-\alpha)^2\sin\alpha,$$

nearly ($\alpha>0,\ <\pi$), we may write

$$\begin{aligned}\frac{B}{2\pi}\int_{\alpha-\delta\alpha}^{\alpha+\delta\alpha}\frac{d\theta}{\theta-\alpha}&\cos\{(2j+1)(\theta-\alpha)-(2l\theta+\beta)+2at\sin\theta\}\\&=\frac{B}{\pi}\frac{2j+1-2l+2at\cos\alpha}{\sqrt{at\sin\alpha}}\int_0^\infty d\theta\sin(2l\alpha+\beta-2at\sin\alpha+\theta^2)\\&=\frac{B}{2}\frac{2j+1-2l+2at\cos\alpha}{\pi at\sin\alpha}\sin\left(2l\alpha+\beta+\frac{\pi}{4}-2at\sin\alpha\right);\end{aligned}$$

the *change of phase* presenting itself still, as at the end of article **80**. But we must add the constant part and also the part free from the sign ∫; & thus we find that if l be nearly $=j+\frac{1}{2}+at\cos\alpha$, we have, nearly,*

$$\begin{aligned}y_{l,t}=\tfrac{1}{2}B\sin(\beta+2l\alpha-2at\sin\alpha)&-\tfrac{1}{2}B\sin(\beta+2j\alpha+\alpha)\\&+\tfrac{1}{2}B\frac{2j+1-2l+2at\cos\alpha}{\sqrt{\pi at\sin\alpha}}\sin\left(\beta+2l\alpha+\frac{\pi}{4}-2at\sin\alpha\right).\end{aligned}$$

86. We see, then, that although a disturbance, distinct from vibration, spreads, with two equal but opposite velocities, $\pm a$, and with a certain constant amount $=\frac{a}{4}y'_{j,0}-\frac{\cos\alpha}{2}y_{j,0}$, in both directions, from the point intermediate between the particles P_j and P_{j+1}, accompanied by two terminal diffusions, which are similar to each other, and are nearly proportional in longitudinal extent to the cube-root of the time t elapsed from the original state of the system;

---

* [The general term in the asymptotic expansion of the Bessel Function was first given, without proof, by Hamilton, *R.I.A. Trans.* Vol. XIX (1843), p. 313. See Watson, *Theory of Bessel Functions*, p. 12.]

there is also a real spreading or *dynamical propagation* of the initial mode of vibration, preserving the same constants of amplitude and phase, and accompanied by a terminal diffusion which is nearly proportional, in longitudinal extent, to the square-root of the time t; and the velocity of this forward spreading of the vibration is represented by $a \cos \alpha$, if $\alpha > 0$, $< \frac{\pi}{2}$. If $\alpha = \frac{\pi}{2}$, there is only terminal diffusion and spreading of a constant disturbance, but no proper propagation of vibration. And if $\alpha > \frac{\pi}{2}$, $< \pi$, there is a backward propagation of vibration, or an uniform rate of abandonment of particles originally occupied by that vibration, the negative velocity of this propagation being still represented by $a \cos \alpha$.

87. It is remarkable that this *velocity of propagation*, $a \cos \alpha$, is the algebraical *sum of all the velocities of transmission of phase*,

$$a \sin \alpha + \frac{a \sin \alpha}{\alpha - \pi} + \frac{a \sin \alpha}{\alpha + \pi} + \frac{a \sin \alpha}{\alpha - 2\pi} + \frac{a \sin \alpha}{\alpha + 2\pi} + \&c. = a \cos \alpha,$$

these several velocities corresponding to the several ways in which the phase may be expressed, namely

$$\beta + 2l\alpha - 2at \sin \alpha, \quad \beta + 2l(\alpha \mp \pi) - 2at \sin \alpha, \quad \beta + 2l(\alpha \mp 2\pi) - 2at \sin \alpha, \quad \&c.$$

In fact

$$\sin \alpha = \alpha \left(1 - \frac{\alpha}{\pi}\right)\left(1 + \frac{\alpha}{\pi}\right)\left(1 - \frac{\alpha}{2\pi}\right)\left(1 + \frac{\alpha}{2\pi}\right) \ldots;$$

therefore

$$\cos \alpha = \frac{d \sin \alpha}{d\alpha} = \frac{\sin \alpha}{\alpha} - \frac{1}{\pi} \frac{\sin \alpha}{1 - \frac{\alpha}{\pi}} + \frac{1}{\pi} \frac{\sin \alpha}{1 + \frac{\alpha}{\pi}} - \frac{1}{2\pi} \frac{\sin \alpha}{1 - \frac{\alpha}{2\pi}} + \frac{1}{2\pi} \frac{\sin \alpha}{1 + \frac{\alpha}{2\pi}} - \&c.$$

For example, if $\alpha = \frac{\pi}{3}$, the series $\frac{a \sin \alpha}{\alpha} + \&c.$ becomes

$$\frac{3a\sqrt{3}}{2\pi}\{1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{8} + \&c.\},$$

which may also be thus written

$$\frac{a}{\alpha}\left(\frac{\sin 2\alpha}{1} + \frac{\sin 4\alpha}{2} + \frac{\sin 6\alpha}{3} + \frac{\sin 8\alpha}{4} + \&c.\right) = \frac{a(\pi - 2\alpha)}{2\alpha} = \frac{a}{2} = a \cos \frac{\pi}{3}.$$

But it is not obvious what dynamical interpretation ought to be put upon this theorem of summation, as applied to the present question; or in other words, it is not clear, à priori, *why* the actual velocity of propagation of vibration *ought* to be the sum of all the possible velocities of transmission of phase.

88. It is evident that the solution of the case of Problem IV, proposed for consideration in article **79**, includes the solution of that other case of the same Problem, in which the initial disturbance is confined to a limited number of successive particles; since this finite number may be regarded as the difference of two infinities. Thus, for this latter case also, analogous results hold good; & we have still a propagation of vibration in one direction & with one velocity expressed still by $a \cos \alpha$. It seems then that even a single undulation tends to propagate itself with this velocity.

To illustrate this subject, let us consider the following Problem.

Problem V.

89. A single particle P_0 of a doubly indefinite system being compelled to vibrate according to a known law, it is required to determine the motion of the system which will ensue in consequence of this vibration of this particle.

90. Suppose, first, that the particle P_0 is obliged to remain fixed from the time 0 to the time t, so that $y_{0,t}$ is, during this interval of time, constant & $=y_{0,0}$. We may now consider ourselves as falling back on the case of a singly indefinite system, and may employ a modification of the formula (10) of article **17**, namely the following, in which $l > 0$,

$$y_{l,t}=\frac{4}{\pi}\,\Sigma_{(j)1}^{\infty}\left(y_{j,0}-y_{0,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos(2at\sin\theta)+y_{0,0}.$$

In fact this reduces itself to $y_{l,0}$ when $t=0$; it gives also an expression for $y'_{l,t}$ which reduces itself to $y'_{l,0}$ when $t=0$; & it gives $y''_{l,t}=a^2(y_{l+1,t}-2y_{l,t}+y_{l-1,t})$; if l be any integer >0. Had we suppressed the terms proportional to $y_{0,0}$, we should have had $y''_{1,t}=a^2(y_{2,t}-2y_{1,t})$, instead of having $y''_{1,t}=a^2(y_{2,t}-2y_{1,t}+y_{0,0})$. The part proportional to $y_{0,0}$ is

$$y_{0,0}\left\{1-\frac{4}{\pi}\,\Sigma_{(j)1}^{\infty}\int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos(2at\sin\theta)\right\};$$

it is the effect of the displacement $y_{0,0}$ of P_0, continued forcibly constant throughout the interval of time t, the system being supposed to extend only in the positive direction.

Instead of $\Sigma_{(j)1}^{\infty}\sin 2j\theta$, we may write $\lim\limits_{j=\infty}\Sigma_{(j)1}^{j}\sin 2j\theta$, that is,

$$\lim_{j=\infty}\frac{\cos\theta-\cos(2j+1)\theta}{2\sin\theta}\left(=\lim_{j=\infty}\frac{\sin j\theta\sin(j\theta+\theta)}{\sin\theta}\right);$$

& at the limit we may reduce this to the term $\dfrac{\cos\theta}{2\sin\theta}=\frac{1}{2}\cot\theta$, on account of the integration $\int$.*

Thus, the solution of the question of the present article may be expressed as follows,

$$y_{l,t}=\frac{4}{\pi}\,\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos(2at\sin\theta)$$
$$+\,y_{0,0}\left\{1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin 2l\theta}{\tan\theta}\cos(2at\sin\theta)\right\};$$

the first line being only the old expression (10) for the effect of an arbitrary initial state of a singly indefinite system, the particle P_0 being fixed at the origin of coordinates; and the second line being that new part which results from the fixing of that particle P_0 in the displaced position $y_{0,0}$ during the time t.

91. Imagine next, that after being thus displaced during an interval of time $=\tau$, the particle P_0 is suddenly removed to a new position $y_{0,\tau}$ & is retained there during some subsequent interval $\Delta\tau$; & let us inquire what will be the state of the system at the end of the time $\tau+\Delta\tau$. The immediate effect of the displacement $y_{0,\tau}$, continued during the time $\Delta\tau$, is

---

* [The terms which vanish are $\operatorname*{Lt}\limits_{j\to\infty}\Sigma_{(s)-l}^{l}\,J_{2j+1+2s}(2at)$.]

$$y_{0,\tau}\left\{1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin 2l\theta}{\tan\theta}\cos\left(2a\,\Delta\tau\sin\theta\right)\right\};$$

but we have to consider also the effect of the initial displacements & velocities $y_{l,0}$, $y'_{l,0}$, expressed still by the first line of the formula at the end of the previous page, or by the old expression (10), in which $t=\tau+\Delta\tau$; we have also to consider the effect of the new displacements and velocities,

$$y_{j,\tau}=y_{0,0}\left\{1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin 2j\theta}{\tan\theta}\cos\left(2a\tau\sin\theta\right)\right\},$$

$$y'_{j,\tau}=\frac{4a}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}} d\theta\cos\theta\sin 2j\theta\sin\left(2a\tau\sin\theta\right),$$

which result, at the end of the interval τ, from the first fixed displacement $y_{0,0}$. The effect of these is

$$\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,\tau}+y'_{j,\tau}\int_0^{\Delta\tau}d\Delta\tau\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos\left(2a\,\Delta\tau\sin\theta\right);$$

in which the part of $y_{j,\tau}$ independent of j, namely the term $y_{0,0}$, gives, as its part of the effect,

$$\frac{2}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}} d\theta\,\frac{\sin 2l\theta}{\tan\theta}\cos\left(2a\,\Delta\tau\sin\theta\right);$$

the element $-\dfrac{2d\alpha}{\pi}y_{0,0}\dfrac{\sin 2j\alpha}{\tan\alpha}\cos\left(2a\tau\sin\alpha\right)$, in the expression for $y_{j,\tau}$, produces the effect

$$-\frac{2d\alpha}{\pi}y_{0,0}\frac{\sin 2l\alpha}{\tan\alpha}\cos\left(2a\tau\sin\alpha\right)\cos\left(2a\,\Delta\tau\sin\alpha\right);$$

and the element $\dfrac{4a\,d\alpha}{\pi}y_{0,0}\cos\alpha\sin 2j\alpha\sin\left(2a\tau\sin\alpha\right)$, in the expression for $y'_{j,\tau}$, produces the effect

$$\frac{2d\alpha}{\pi}y_{0,0}\frac{\sin 2l\alpha}{\tan\alpha}\sin\left(2a\tau\sin\alpha\right)\sin\left(2a\,\Delta\tau\sin\alpha\right);$$

so that the joint effect of these two elements is $-\dfrac{2d\alpha}{\pi}y_{0,0}\dfrac{\sin 2l\alpha}{\tan\alpha}\cos\left(2at\sin\alpha\right)$, in which $t=\tau+\Delta\tau$. The state of the system at the end of the time t is therefore expressed by the formula*

$$\begin{aligned}y_{l,t}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)&\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos\left(2at\sin\theta\right)\\ &+y_{0,\tau}-\frac{2}{\pi}\left(y_{0,\tau}-y_{0,0}\right)\int_0^{\frac{\pi}{2}} d\theta\,\mathrm{cotan}\,\theta\sin 2l\theta\cos\left\{2a\left(t-\tau\right)\sin\theta\right\}\\ &-\frac{2}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}} d\theta\,\mathrm{cotan}\,\theta\sin 2l\theta\cos\left(2at\sin\theta\right).\end{aligned}$$

---

* [This could, of course, have been inferred from article **90** by taking the displacement $y_{0,0}$ from 0 to t and $y_{0,\tau}-y_{0,0}$ from τ to t.]

Accordingly this expression gives, if $l>0$,

$$y_{l,\tau}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^{\tau}dt\right)\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$+y_{0,0}\left\{1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\,\text{cotan}\,\theta\sin 2l\theta\cos(2a\tau\sin\theta)\right\},$$

$$y'_{l,\tau}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}\frac{d}{d\tau}+y'_{j,0}\right)\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2a\tau\sin\theta)$$

$$+\frac{4a}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}}d\theta\cos\theta\sin 2l\theta\sin(2a\tau\sin\theta),$$

so that it reproduces the known state of the system at the end of the first interval τ; it gives also, for any moment of the 2nd interval $\Delta\tau$, that is, for any value of t from τ to $\tau+\Delta\tau$, (l being still >0,)

$$y''_{l,t}=a^2(y_{l+1,t}-2y_{l,t}+y_{l-1,t}),$$

including the equation

$$y''_{1,t}=a^2(y_{2,t}-2y_{1,t}+y_{0,\tau}),$$

& therefore it satisfies the differential equations of vibration, on the hypothesis of $y_{0,t}$ being, throughout the whole of this second interval, constant & $=y_{0,\tau}$.

92. In the next place, if there be three successive displacements $y_{0,0}$, y_{0,t_1}, y_{0,t_2} lasting for the successive intervals τ_1, τ_2, τ_3, we must suppose that at the moment $t_2=\tau_1+\tau_2$ the displacements and velocities are represented thus:

$$y_{l,t_2}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^{t_2}dt_2\right)\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2at_2\sin\theta)$$

$$+y_{0,t_1}-\frac{2}{\pi}(y_{0,t_1}-y_{0,0})\int_0^{\frac{\pi}{2}}d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2a\tau_2\sin\theta)-\frac{2}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}}d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2at_2\sin\theta);$$

$$y'_{l,t_2}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}\frac{d}{dt_2}+y'_{j,0}\right)\int_0^{\frac{\pi}{2}}d\theta\sin 2j\theta\sin 2l\theta\cos(2at_2\sin\theta)$$

$$+\frac{4a}{\pi}(y_{0,t_1}-y_{0,0})\int_0^{\frac{\pi}{2}}d\theta\cos\theta\sin 2l\theta\sin(2a\tau_2\sin\theta)$$

$$+\frac{4a}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}}d\theta\cos\theta\sin 2l\theta\sin(2at_2\sin\theta);$$

and the function $y_{l,t}$, in which $l>0$ and $t=t_2+\tau_3$, is to satisfy the differential equations of the form $y''_{l,t}=a^2(y_{l+1,t}-2y_{l,t}+y_{l-1,t})$, including the equation $y''_{1,t}=a^2(y_{2,t}-2y_{1,t}+y_{0,t_2})$; the final effect is therefore expressed by the following formula:

$$y_{l,t}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$+y_{0,t_2}-\frac{2}{\pi}(y_{0,t_2}-y_{0,t_1})\int_0^{\frac{\pi}{2}} d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2a\tau_3\sin\theta)$$

$$-\frac{2}{\pi}(y_{0,t_1}-y_{0,0})\int_0^{\frac{\pi}{2}} d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2a\overline{\tau_3+\tau_2}\sin\theta)$$

$$-\frac{2}{\pi}y_{0,0}\int_0 d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2a\overline{\tau_3+\tau_2+\tau_1}\sin\theta);$$

which may also be thus written,

$$y_{l,t}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$+\frac{2}{\pi}y_{0,t_2}\int_0^{\frac{\pi}{2}} d\theta\frac{\sin 2l\theta}{\tan\theta}\{1-\cos(2a\tau_3\sin\theta)\}$$

$$+\frac{2}{\pi}y_{0,t_1}\int_0^{\frac{\pi}{2}} d\theta\frac{\sin 2l\theta}{\tan\theta}\{\cos(2a\tau_3\sin\theta)-\cos(2a\overline{\tau_3+\tau_2}\sin\theta)\}$$

$$+\frac{2}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}} d\theta\frac{\sin 2l\theta}{\tan\theta}\{\cos(2a\overline{\tau_3+\tau_2}\sin\theta)-\cos(2a\overline{\tau_3+\tau_2+\tau_1}\sin\theta)\}.$$

93. It is easy to see that this law continues; and that it gives, as the solution of Problem V in article **89**, the expression

$$y_{l,t}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$+\frac{4a}{\pi}\int_0^t d\tau\, y_{0,\tau}\int_0^{\frac{\pi}{2}} d\theta\cos\theta\sin 2l\theta\sin\{2a(t-\tau)\sin\theta\}.$$

Accordingly this expression gives

$$y'_{l,t}=\frac{4}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}\frac{d}{dt}+y'_{j,0}\right)\int_0^{\frac{\pi}{2}} d\theta\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$+\frac{8a^2}{\pi}\int_0^t d\tau\, y_{0,\tau}\int_0^{\frac{\pi}{2}} d\theta\sin\theta\cos\theta\sin 2l\theta\cos\{2a(t-\tau)\sin\theta\},$$

$$y''_{l,t}=-\frac{16a^2}{\pi}\Sigma_{(j)1}^{\infty}\left(y_{j,0}+y'_{j,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\sin\theta^2\sin 2j\theta\sin 2l\theta\cos(2at\sin\theta)$$

$$-\frac{16a^2}{\pi}\int_0^t d\tau\, y_{0,\tau}\int_0^{\frac{\pi}{2}} d\theta\sin\theta^2\cos\theta\sin 2l\theta\sin\{2a(t-\tau)\sin\theta\}$$

$$+\frac{4a^2}{\pi}y_{0,t}\int_0^{\frac{\pi}{2}} d\theta\sin 2\theta\sin 2l\theta,$$

so that $y_{l,t}$ and $y'_{l,t}$ reduce themselves when $t=0$ to the given initial values, and also the differential equations of the form $y''_{l,t}=a^2(y_{l+1,t}-2y_{l,t}+y_{l-1,t})$ are satisfied if $l>0$, including the equation $y''_{1,t}=a^2(y_{2,t}-2y_{1,t}+y_{0,t})$, in which the function $y_{0,t}$ is arbitrary. As a verification we may observe that if this function reduce itself to the constant $y_{0,0}$, we have

$$\int_0^t d\tau\, y_{0,\tau}\sin\{2a(t-\tau)\sin\theta\}=y_{0,0}\frac{1-\cos(2at\sin\theta)}{2a\sin\theta},$$

so that if the initial displacements and velocities $y_{l,0}$ and $y'_{l,0}$ vanish, we have

$$y_{l,t}=\frac{2}{\pi}y_{0,0}\int_0^{\frac{\pi}{2}}d\theta\frac{\sin 2l\theta}{\tan\theta}\operatorname{vers}(2at\sin\theta)=y_{0,0}\left\{1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\frac{\sin 2l\theta}{\tan\theta}\cos(2at\sin\theta)\right\},$$

as found in article **90**.

94. The effect of the vibration $y_{0,t}$ of the particle P_0 being thus found to be

$$\frac{4a}{\pi}\int_0^t d\tau\, y_{0,\tau}\int_0^{\frac{\pi}{2}}d\theta\cos\theta\sin 2l\theta\sin\{2a(t-\tau)\sin\theta\},$$

let us suppose that $y_{0,\tau}=-\eta\sin(2a\tau\sin\alpha-\beta)$, α being >0, $<\frac{\pi}{2}$.

Multiplying this by $2\sin\{2a(t-\tau)\sin\theta\}$, we get

$$\eta\cos\{2at\sin\theta-\beta-2a\tau(\sin\theta-\sin\alpha)\}-\eta\cos\{2at\sin\theta+\beta-2a\tau(\sin\theta+\sin\alpha)\};$$

and multiplying again by $2\sin 2l\theta$, we get

$$\eta\sin\{2at\sin\theta-\beta+2l\theta-2a\tau(\sin\theta-\sin\alpha)\}-\eta\sin\{2at\sin\theta+\beta+2l\theta-2a\tau(\sin\theta+\sin\alpha)\}$$
$$-\eta\sin\{2at\sin\theta-\beta-2l\theta-2a\tau(\sin\theta-\sin\alpha)\}+\eta\sin\{2at\sin\theta+\beta-2l\theta-2a\tau(\sin\theta+\sin\alpha)\},$$

which is to be multiplied by $\frac{a}{\pi}\cos\theta$, & integrated relatively to τ from 0 to t and relatively to θ from 0 to $\frac{\pi}{2}$. In this manner we obtain*

---

* [The expression for $y_{l,t}$ may be written

$$\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\{\cos\theta\sin 2l\theta\sin(2at\sin\theta-\beta)-\cos\alpha\sin 2l\alpha\sin(2at\sin\alpha-\beta)\}}{\sin\theta-\sin\alpha}$$
$$-\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta(\cos\theta\sin 2l\theta-\cos\alpha\sin 2l\alpha)\sin(2at\sin\alpha-\beta)}{\sin\theta-\sin\alpha}.$$

The first integral may, obviously, be written

$$\operatorname*{Lt}_{\epsilon\to 0}\frac{\eta}{\pi}\left(\int_{-\frac{\pi}{2}}^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{\frac{\pi}{2}}\right)\frac{d\theta}{\sin\theta-\sin\alpha}\{\qquad\}$$

or

$$\operatorname*{Lt}_{\to 0}\frac{\eta}{\pi}\left(\int_{-\frac{\pi}{2}}^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{\frac{\pi}{2}}\right)\frac{d\theta}{\sin\theta-\sin\alpha}\cos\theta\sin 2l\theta\sin(2at\sin\theta-\beta),$$

$$\frac{\eta}{2\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\cos\theta}{\sin\theta-\sin\alpha}\{\cos(2at\sin\alpha-\beta+2l\theta)-\cos(2at\sin\theta-\beta+2l\theta)$$
$$-\cos(2at\sin\alpha-\beta-2l\theta)+\cos(2at\sin\theta-\beta-2l\theta)\}$$
$$-\frac{\eta}{2\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\cos\theta}{\sin\theta+\sin\alpha}\{\cos(-2at\sin\alpha+\beta+2l\theta)-\cos(2at\sin\theta+\beta+2l\theta)$$
$$-\cos(-2at\sin\alpha+\beta-2l\theta)+\cos(2at\sin\theta+\beta-2l\theta)\}$$
$$=\frac{\eta}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta}{\sin\theta-\sin\alpha}\{\cos(2at\sin\alpha-\beta+2l\theta)-\cos(2at\sin\theta-\beta+2l\theta)$$
$$-\cos(2at\sin\alpha-\beta-2l\theta)+\cos(2at\sin\theta-\beta-2l\theta)\}$$
$$=\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{\sin\theta-\sin\alpha}\{\sin(2at\sin\theta-\beta)-\sin(2at\sin\alpha-\beta)\}$$
$$=\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta\sin(2at\sin\theta-\beta)}{\sin\theta-\sin\alpha}-\eta\cos 2l\alpha\sin(2at\sin\alpha-\beta),$$

observing that by article **39** we have, if l be any integer >0,

$$\cos 2l\alpha = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin 2\theta\sin 2l\theta}{\cos 2\alpha-\cos 2\theta}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin\theta\cos\theta\sin 2l\theta}{\sin\theta^2-\sin\alpha^2}$$
$$=\frac{1}{\pi}\int_0^{2}d\theta\cos\theta\sin 2l\theta\left\{\frac{1}{\sin(\theta-\alpha)}+\frac{1}{\sin(\theta+\alpha)}\right.$$
$$=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{\sin(\theta-\alpha)}.$$

As a verification, the effect calculated in the present article ought to vanish when l becomes infinite, t remaining finite. Now

$$\sin 2l\theta\sin(2at\sin\theta-\beta)=\tfrac{1}{2}\cos(2l\theta+\beta-2at\sin\theta)-\tfrac{1}{2}\cos(2l\theta-\beta+2at\sin\theta);$$

& the first of these two terms gives, at the limit, $-\frac{\eta}{2}\sin(2l\alpha+\beta-2at\sin\alpha)$, while the second gives $+\frac{\eta}{2}\sin(2l\alpha-\beta+2at\sin\alpha)$; their sum gives therefore $+\eta\cos 2l\alpha\sin(2at\sin\alpha-\beta)$, as it ought to do.

95. We find therefore that in the singly indefinite system, extending only in the positive direction, the effect of the vibration $y_{0,t}=\eta\sin(-2at\sin\alpha+\beta)$ $\left(\text{in which } \alpha>0,\ <\frac{\pi}{2}\right)$ of the first

because $$\underset{\epsilon\to 0}{\mathrm{Lt}}\left(\int_{-\frac{\pi}{2}}^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{\frac{\pi}{2}}\right)\frac{d\theta}{\sin\theta-\sin\alpha}=0.$$

The second line gives $$-\eta\cos 2l\alpha\sin(2at\sin\alpha-\beta).$$

We thus arrive at the final form given for $y_{l,t}$, the integral to be interpreted as this special Cauchy value. The case $\sin 2l\alpha=0$ is exceptional. Here the Cauchy value is unnecessary.]

particle P_0 is represented by the formula

$$y_{l,t}=\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta\sin(2at\sin\theta-\beta)}{\sin\theta-\sin\alpha}-\eta\cos 2l\alpha\sin(2at\sin\alpha-\beta)$$

$$=\frac{\eta}{2\pi}\int_{-1}^{+1}\frac{d(\sin\theta)}{\sin\theta-\sin\alpha}\{\cos(2l\theta+\beta-2at\sin\theta)-\cos(2l\theta-\beta+2at\sin\theta)\}$$

$$-\eta\cos 2l\alpha\sin(2at\sin\alpha-\beta).$$

If l be much larger than $at\cos\alpha$, this effect is insensible; but if, on the contrary, $at\cos\alpha$ be much larger than l, the effect is nearly represented thus,

$$y_{l,t}=\eta\{\tfrac{1}{2}\sin(2l\alpha+\beta-2at\sin\alpha)+\tfrac{1}{2}\sin(2l\alpha-\beta+2at\sin\alpha)-\cos 2l\alpha\sin(2at\sin\alpha-\beta)\}$$

$$=\eta\{\sin 2l\alpha\cos(2at\sin\alpha-\beta)-\cos 2l\alpha\sin(2at\sin\alpha-\beta)\}$$

$$=\eta\sin(2l\alpha+\beta-2at\sin\alpha).$$

Thus it is true, in a certain sense, that even the vibration of a single particle P_0, with a periodic time $=\frac{\pi}{a}\operatorname{cosec}\alpha$, $\left(\text{in which } \alpha \text{ is any real arc} > 0, \text{ but } < \frac{\pi}{2}\right)$, produces vibration, with the same periodic time, in all the other particles; the transmission of phase having a velocity $=\frac{a\sin\alpha}{\alpha}$, but the propagation of vibration having a somewhat less velocity, namely $a\cos\alpha$.

96. If $\alpha=\frac{\pi}{2}$, so that the vibration of P_0 is

$$y_{0,t}=\eta\sin(\beta-2at),$$

then the same analysis shows that the effect of this continued vibration of this single particle is

$$y_{l,t}=\frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{1-\sin\theta}\{\sin(\beta-2at\sin\theta)-\sin(\beta-2at)\};$$

now

$$-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{1-\sin\theta}=-2\int_0^{\frac{\pi}{2}}d\theta\tan\theta\sin 2l\theta=\pi(-1)^l;$$

therefore in the present case,

$$y_{l,t}=\eta\left\{(-1)^l\sin(\beta-2at)+\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta\sin(\beta-2at\sin\theta)}{1-\sin\theta}\right\}.$$

Changing θ to $\frac{\pi}{2}-\theta$, this becomes

$$y_{l,t}=\eta(-1)^l\left\{\sin(\beta-2at)-\frac{1}{\pi}\int_0^{\pi}\frac{d\theta\sin\theta\sin 2l\theta}{1-\cos\theta}\sin(\beta-2at\cos\theta)\right\}$$

$$=\eta(-1)^l\left\{\sin(\beta-2at)-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin 4l\theta}{\tan\theta}\sin(\beta-2at\cos 2\theta)\right\}.$$

This is insensible, if l be much greater than $\sqrt{at}$; but if, on the contrary, l be much less than $\sqrt{at}$, it becomes, making $2\sqrt{at}\sin\theta = \theta_,$,

$$\eta\,(-1)^l\left\{\sin(\beta-2at) - \frac{4l}{\pi\sqrt{at}}\int_0^\infty d\theta_,\sin(\beta-2at+\theta_,^2)\right\}$$

$$=\eta\,(-1)^l\left\{\sin(\beta-2at) - \frac{2l}{\sqrt{\pi at}}\sin\left(\beta-2at+\frac{\pi}{4}\right)\right.$$

97. We might suppose $y_{0,t} = \eta\sin(\beta-2aAt)$, $A > 1$; & the analysis of article **94** would then give

$$y_{l,t} = \frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{A-\sin\theta}\{\sin(\beta-2at\sin\theta)-\sin(\beta-2aAt)\};$$

but whether l be large or small in comparison with t, if both be very great, this function will become insensible, because the denominator $A-\sin\theta$ cannot now vanish. Thus a vibration of shorter periodic time than the minimum $\frac{\pi}{a}$ cannot sensibly propagate itself far.

However, when t is very great, we have nearly

$$y_{l,t} = \frac{\eta}{\pi}\sin(\beta-2aAt)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{\sin\theta-A};$$

in which*

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2l\theta}{\sin\theta-A} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin 2\theta\sin 2l\theta}{\sin\theta^2-A^2} = -f_l$$

= a function such that

$$f_{l+1}+f_{l-1}-2f_l+4A^2f_l = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sin 2\theta\sin 2l\theta = 0 \text{ or } 1,$$

according as l, being > 0, is > 1 or $= 1$. Also

$$f_0 = 1;\quad f_1 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{(\sin 2\theta)^2\,d\theta}{A^2-\sin\theta^2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta\cos\theta\sin 2\theta}{\pi(A-\sin\theta)},$$

in which

$$\sin 2\theta\cos\theta = 2\,(\overline{\sin\theta-A}+A)\,(\overline{1-A^2}+\overline{A^2-\sin\theta^2}),$$

therefore

$$f_1 = \frac{4A^2(1-A^2)}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{A^2-\sin\theta^2}-1+2A^2;$$

also

$$A^2(A^2-\sin\theta^2)^{-1} = 1+A^{-2}\sin\theta^2+A^{-4}\sin\theta^4+\&c.,$$

---

* [If $A = \cosh\alpha$, $\cos\theta/(A-\sin\theta) = \Sigma_{(n)1}^{\infty}e^{-n\alpha}\sin n\left(\frac{\pi}{2}-\theta\right)$.]

and

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin\theta^{2n}\,d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\theta^{2n}\,d\theta = 2^{-2n}\,[2n]^n\,[0]^{-n} = \frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots(2n)} = [-\tfrac{1}{2}]^n\,[0]^{-n}\,(-1)^n;$$

therefore

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-A^{-2}\sin\theta^2} = (1-A^{-2})^{-\frac{1}{2}},$$

and

$$f_1 = \frac{2(1-A^2)}{\sqrt{1-A^{-2}}} - 1 + 2A^2 = -2A\sqrt{A^2-1} - 1 + 2A^2 = (A-\sqrt{A^2-1})^2;$$

$$f_2 = 1 + 2(1-2A^2)f_1 = -(A-\sqrt{A^2-1})^4;$$

and generally

$$f_l = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta\sin 2\theta\sin 2l\theta}{A^2-\sin\theta^2} = (-1)^{l+1}(A-\sqrt{A^2-1})^{2l},$$

l being any integer >0.

Hence in the present question, if t be great, we have

$$y_{l,t} = \eta\,(-1)^l\,(A+\sqrt{A^2-1})^{-2l}\sin(\beta-2aAt).$$

The amplitudes therefore, in this case, decrease in geometrical proportion, being proportional to the power of the fraction $-\dfrac{1}{(A+\sqrt{A^2-1})^2}$.

98. This result, on the analytic side, bears some analogy to the well-known theorems

$$\int_0^\infty \frac{\cos qx\,dq}{1+q^2} = \frac{\pi}{2}e^{-x}, \quad \int_0^\infty \frac{q\sin qx\,dq}{1+q^2} = \frac{\pi}{2}e^{-x},$$

in which x is any real quantity >0, & of which the former includes the latter and may be proved by observing that if we put

$$X_x = \int_0^\infty \frac{\cos qx\,dq}{1+q^2}, \text{ we have } X_x'' = -\int_0^\infty \frac{q^2\cos qx\,dq}{1+q^2} = X_x - \int_0^\infty \cos qx\,dq,$$

which gives $X_x'' = X_x$ if x be >0; therefore $X_x = ae^x + be^{-x} = be^{-x}$ because $X_\infty = 0$; and although the differential equation $X_x'' = X_x$ does not hold good for the particular value $x=0$, yet the coefficient b must be $=$ the limit to which $e^x X_x$ tends as x decreases to 0, & must therefore be equal to $\dfrac{\pi}{2}$.

Laplace, in the Analytic Theory of Probabilities, Article **26**, deduces the theorem from the consideration of a double definite integral, as follows:

$$X_r = \int_0^\infty \frac{dx\cos rx}{1+x^2} = \int_0^\infty dx\int_0^\infty dy\,2ye^{-y^2(1+x^2)}\cos rx = \sqrt{\pi}\int_0^\infty dy\,e^{-y^2-\frac{r^2}{4y^2}}$$

$$= \sqrt{\pi}e^{-r}\int_0^\infty dy\,e^{-\left(y-\frac{r}{2y}\right)^2} = \frac{\sqrt{\pi}}{2}e^{-r}\int_{-\infty}^\infty dz\,e^{-z^2}\left(1+\frac{z}{\sqrt{z^2+2r}}\right) = \frac{\pi}{2e^r}.$$

And he remarks that by pursuing a similar analysis, the following theorems may be deduced:

$$\int_{-\infty}^{\infty} dx \frac{(a+bx)\cos rx}{m+2nx+x^2} = \left(\frac{a-bn}{\sqrt{m-}} \cos rn + b \sin rn\right) \pi e^{-r\sqrt{m-n^2}}$$

$$\int_{-\infty} dx \frac{(a+bx)\sin rx}{m+2nx+x^2} = \left(b \cos rn - \frac{a-bn}{} \sin rn\right) \pi e^{-r}$$

and ultimately the values of the definite integrals

$$\int_{-\infty}^{\infty} dx \frac{M}{N} \cos rx, \quad \int_{-\infty}^{\infty} dx \frac{M}{N} \sin rx,$$

in which M and N are rational and integral polynomes, such that the degree of M (relatively to x) is less than that of N, and the roots of $N = 0$ are all imaginary. The two last theorems may thus be written,

$$\int_{-\infty}^{\infty} dx \frac{(a+x)\cos rx}{m^2+x^2} = \frac{\pi a}{m} e^{-rm}, \quad \int_{-\infty}^{\infty} dx \frac{(a+x)\sin rx}{m^2+x^2} = \pi e^{-}$$

& under this form they follow easily from those first cited in this article.

99. In the foregoing investigations, we have, expressly or tacitly, employed often the principle that if any finite function of a real variable be multiplied by the sine or cosine of an infinitely great multiple of that variable, & integrated within any finite limits, (or even, in most cases, between negative and positive infinity,) the result is evanescent; & therefore that if the function be not constantly finite, we need attend only to those values of the variable which differ infinitely little from the values which make the function infinite.[*] And in some cases of a constantly finite function, such as $F(\theta) = \frac{\cos\theta}{\sin\theta - A}$, $(A > 1)$, in article **97**, or $F(q) = \frac{1}{1+q^2}$ in article **98**, we have been able to assign the law according to which the integral tends to become infinitely small as the multiplier of the variable under the sign sine or cosine in the rapidly fluctuating factor tends to become infinitely great; namely, in these cases, the exponential law expressed by the formulae

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{\sin\theta - A} \sin 2l\theta = \pi(-1)^l (A + \sqrt{A^2-1})^{-2l},$$

A being any real quantity > 1, and l being any integer number > 1; and

$$\int_{-\infty}^{\infty} dq \frac{1}{1+q^2} \cos qx = \pi e^{-qx},$$

x being any real quantity $\nless 0$.

It seems that in most cases of this last kind, that is, in most cases of a constantly finite function $F(q)$ multiplied by the cosine of a large multiple qx of the variable q & integrated from $-\infty$ to $+\infty$, $(F(\mp\infty)$ being supposed $= 0)$, we must have, nearly, the expression[†]

$$\int_{-\infty}^{\infty} dq\, F(q) \cos qx = \frac{2}{x} \Sigma_{(n)-\infty}^{\infty} (-1)^n F\left(\frac{n\pi}{x}\right);$$

x being large.

---

[*] [This is Hamilton's principle of fluctuation, which he later developed at length in a memoir on fluctuating functions. *Trans. R.I.A.* (1843) XIX, pp. 264–321.]

[†] $\left[\sum_{(n)} \int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} F(q) \cos qx\, dq.\right]$

It will however be probably in general more exact to substitute, in this expression, instead of the middle ordinate $F\left(\frac{n\pi}{x}\right)$ of the curve in which q is abscissa and $F(q)$ ordinate, taken between the limits $q = \frac{(n \mp \frac{1}{2})\pi}{x}$, the average ordinate of this curve between these limits, namely $\frac{x}{\pi}\int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} dq\, F(q)$; & thus we find

$$\int_{-\infty}^{\infty} dq\, F(q)\cos qx = \frac{2}{\pi}\Sigma_{(n)-\infty}^{\infty}(-1)^n \int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} dq\, F(q);$$

x being large. It will be useful to test these formulae by some examples.

100. Let $F(q) = \cos q$; then $2\cos q \cos qx = \cos(qx+q) + \cos(qx-q)$; & since this function F does not vanish when $q=\infty$, we shall take finite limits of integration, such as 0 and π, & employ the approximate formula

$$\int_0^{\pi} dq\, F(q)\cos qx = \frac{1}{x}\left\{F(0) + 2\Sigma_{(n)1}^{x-1}(-1)^n F\left(\frac{n\pi}{x}\right) + F(\pi)\cos(x\pi)\right\},$$

x being now some large and positive integer. Applying this to the case $F(q) = \cos q$, the first member vanishes, & the 2nd member ought to be found to be nearly $=0$. We ought therefore to have, nearly, $\frac{2}{x}\Sigma_{(n)1}^{x-1}(-1)^n \cos\frac{n\pi}{x} = \frac{\cos x\pi - 1}{x}$, if x be a large positive integer. Now

$$2\cos\frac{\pi}{2x}\left(-\cos\frac{\pi}{x} + \cos\frac{2\pi}{x} - \cos\frac{3\pi}{x} + \ldots + \cos m\pi \cos\frac{m\pi}{x}\right) = -\cos\frac{\pi}{2x} + \cos m\pi \cos\left(\frac{m\pi}{x} + \frac{\pi}{2x}\right);$$

making therefore $m = x-1$, we have

$$-\cos\frac{\pi}{2x} + \cos(x\pi - \pi)\cos\left(\pi - \frac{\pi}{2x}\right) = (\cos x\pi - 1)\cos\frac{\pi}{2x};$$

in this case therefore the theorem is rigorously true. If we had employed *average* instead of *middle* ordinates, we should have had

$$\int_0^{\pi} dq\, F(q)\cos qx = \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2x}} + \Sigma_{(n)1}^{x-1}(-1)^n \int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} + \cos x\pi \int_{\pi-\frac{\pi}{2x}}^{\pi}\right\} dq\, F(q);$$

x being still a large and positive integer. Making $F(q) = \cos q$, we ought to find nearly

$$0 = \frac{2}{\pi}\sin\frac{\pi}{2x}\left\{1 + 2\Sigma_{(n)1}^{x-1}(-1)^n \cos\frac{n\pi}{x} - \cos x\pi\right\};$$

& in fact this also is rigorously true. But we cannot expect to find so perfect an agreement, in general, between the equated expressions.

101. Let $F(q) = e^{-q^2}$; then

$$\int_{-\infty}^{\infty} dq\, F(q)\cos qx = \sqrt{\pi}e^{-\frac{x^2}{4}};$$

is this then nearly $=\frac{2}{x}\Sigma_{(n)-\infty}^{\infty}(-1)^n e^{-\frac{n^2\pi^2}{x^2}}$? In general, we have, *rigorously*,

$$\Sigma_{(n)-\infty}^{\infty}(-1)^n F(n\alpha)=\underset{s=\infty}{\mathrm{Lt}}\int_{-\infty}^{\infty}dt\,F(t\alpha)\frac{\sin 2st\pi}{\sin t\pi}$$

$$=2\int_{-\infty}^{\infty}dt\,F(t\alpha)(\cos t\pi+\cos 3t\pi+\cos 5t\pi+\text{\&c.}).$$

We ought therefore to have, nearly, in general, if x be large,

$$\int_{-\infty}^{\infty}dq\,F(q)\cos qx=\frac{4}{x}\int_{-\infty}^{\infty}dt\,F\left(\frac{t\pi}{x}\right)(\cos t\pi+\cos 3t\pi+\cos 5t\pi+\text{\&c.})$$

$$=\frac{4}{\pi}\int_{-\infty}^{\infty}dq\,F(q)(\cos qx+\cos 3qx+\cos 5qx+\text{\&c.}).$$

But in the case $F(q)=e^{-q^2}$, this would give

$$\sqrt{\pi}e^{-\frac{x^2}{4}}=\frac{4}{\sqrt{\pi}}\left(e^{-\frac{x^2}{4}}+e^{-\frac{9x^2}{4}}+e^{-\frac{25x^2}{4}}+\text{\&c.}\right);$$

& the second member is greater than the first, if x be large, in the ratio nearly of 4 to π. There can be no doubt but that this arises chiefly from the circumstance that the definite integral

$$\int_{-\frac{\pi}{2x}}^{\frac{\pi}{2x}}dq\cos qx=\frac{2}{x}\ \text{ is greater than }\ \int_{-\frac{\pi}{2x}}^{\frac{\pi}{2x}}dq\,e^{-q^2}\cos qx;$$

and generally from the inequality of

$$\int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}}dq\,e^{-q^2}\cos qx\ \text{ and }\ e^{-\frac{n^2\pi^2}{x^2}}\int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}}dq\cos qx=(-1)^n\frac{2}{x}e^{-\frac{n^2\pi^2}{x^2}};$$

such inequalities though small being numerous & giving an accumulated result, which bears a sensible ratio ($4-\pi$ to π) to the small value of the integral in question, when x is very great. To allow for these inequalities, at least nearly, we might make

$$e^{-q^2}=e^{-\frac{n^2\pi^2}{x^2}}e^{-\left(q^2-\frac{n^2\pi^2}{x^2}\right)}=e^{-\frac{n^2\pi^2}{x^2}}.$$

103. Returning to the investigation of article **97**, & to the first expression there given for $y_{l,t}$, that is for the transversal displacement of the particle P_l at the time t, l being any integer >0, and the particle P_0 being obliged to vibrate according to the law $y_{0,t}=\eta\sin(\beta-2aAt)$, in which $A>1$, while $y_{l,0}=0$, $y'_{l,0}=0$; we see that, in order to develope this expression according to the powers of t, it is sufficient to calculate generally

$$f_{l,i}=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\theta^i\cos\theta\sin 2l\theta}{A-\sin\theta}=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\frac{(A+\sin\theta)\sin\theta^i\cos\theta\sin 2l\theta}{A^2-\sin\theta^2}$$

or simply

$$f_{l,2i}=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}d\theta\frac{\sin\theta^{2i}\sin 2\theta\sin 2l\theta}{A^2-\sin\theta^2}$$

since $f_{l,2i+1} = Af_{l,2i}$. And we have already found that

$$f_l = f_{l,0} = (-1)^{l+1}(A - \sqrt{A^2-1})^{2l},$$

if $l > 0$; $f_{0,0} = 0$. Now

$$A^{2i}f_l - f_{l,2i} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta\,\{A^{2i-2} + A^{2i-4}\sin\theta^2 + \ldots + \sin\theta^{2i-2}\}\sin 2\theta \sin 2l\theta;$$

we are therefore to calculate

$$g_{l,k} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sin\theta^{2k} \sin 2\theta \sin 2l\theta;$$

and it is convenient to suppose at first $l > 1$, reserving the case $l = 1$ for separate study.

$$\sin\theta^{2k} = (-\tfrac{1}{4})^k (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})^{2k} = (-\tfrac{1}{4})^k [2k]^{2k} \Sigma_{(n)\,0}^{2k} e^{2(k-n)\theta\sqrt{-1}} [0]^{-(2k-n)} [0]^{-n} (-1)^n *$$
$$= (\tfrac{1}{4})^k [2k]^{2k} \Sigma_{(n)-k}^{k} [0]^{-(k+n)} [0]^{-(k-n)} (-1)^n \cos 2n\theta;$$

hence, coefficient of $\cos(2l\theta - 2\theta)$, if $l > 1$, is $2(\frac{1}{4})^k [2k]^{2k} (-1)^{l-1} [0]^{-(k+l-1)} [0]^{-(k-l+1)}$; and coefficient of $\cos(2l\theta + 2\theta)$ is $2(\frac{1}{4})^k [2k]^{2k} (-1)^{l+1} [0]^{-(k+l+1)} [0]^{-(k-l-1)}$; therefore $g_{l,k} = \frac{1}{8}$th of (former − latter) coefficient, if $l > 1$,

$$= (\tfrac{1}{4})^{k+1} [2k]^{2k} (-1)^l [0]^{-(k+l+1)} [0]^{-(k-l+1)} \{[k-l+1]^2 - [k+l+1]^2\}$$
$$= (-1)^{l+1} [2k+1]^{2k+1} (\tfrac{1}{2})^{2k+1} l\, [0]^{-(k+l+1)} [0]^{-(k-l+1)} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sin\theta^{2k} \sin 2\theta \sin 2l\theta,$$

if $l > 1$. And

$$g_{1,k} = (\tfrac{1}{4})^{k+1} [2k]^{2k} \{([0]^{-k})^2 - [0]^{-(k+2)} [0]^{-(k-2)}\}$$
$$= [2k+1]^{2k+1} (\tfrac{1}{2})^{2k+1} [0]^{-k} [0]^{-(k+2)} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sin\theta^{2k} \sin 2\theta^2;$$

so that the formula just now found for $g_{l,k}$ holds even when $l = 1$.

104. Hence in the expression of article **97**,

$$y_{l,t} = \frac{\eta}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{\cos\theta \sin 2l\theta}{A - \sin\theta} \{\sin(\beta - 2at\sin\theta) - \sin(\beta - 2aAt)\}$$

$$= \frac{\eta}{\pi}\int_0^{\frac{\pi}{2}} d\theta\, \frac{\sin 2\theta \sin 2l\theta}{A^2 - \sin\theta^2} \left\{\sin\beta\,\{\cos(2at\sin\theta) - \cos(2aAt)\} \right.$$
$$\left. + \cos\beta \left(\sin(2aAt) - \frac{A}{\sin\theta}\sin(2at\sin\theta)\right)\right\}$$

$$= \frac{\eta}{\pi}\int_0^{\frac{\pi}{2}} d\theta\, \frac{\sin 2\theta \sin 2l\theta}{A^2 - \sin\theta^2} \Sigma_{(i)\,1}^{\infty} (A^{2i} - \sin\theta^{2i})(2at)^{2i} [0]^{-2i} (-1)^i \left\{-\sin\beta + \frac{2aAt}{2i+1}\cos\beta\right.$$

$$= \eta\, \Sigma_{(i)\,1}^{\infty} (-4a^2t^2)^i [0]^{-2i} \left\{\frac{2aAt}{2i+1}\cos\beta - \sin\beta\right\} \Sigma_{(k)\,0}^{i-1} A^{2i-2k-2} g_{l,k}$$

$$= \eta\, \Sigma_{(i)\,1}^{\infty} (-4a^2A^2t^2)^i [0]^{-2i} \left\{\frac{2aAt}{2i+1}\cos\beta - \sin\beta\right\} A^{-2} \Sigma_{(k)\,0}^{i-1} A^{-2k} g_{l,k},$$

* [$[n]^m = n!/(n-m)!$; $[0]^{-m} = 1/(m!)$; this is Vandermonde's notation. See Vol. I, p. 468.]

we have $g_{l,0} = (-1)^{l+1}\frac{l}{2}[0]^{-(1+l)}[0]^{-(1-l)}$, $\because g_{1,0} = \frac{1}{4}$, and $g_{l,0} = 0$ if $l > 1$; so that the part of $y_{l,t}$ which involves t^2 and t^3 vanishes for all the particles beyond P_1, but becomes, for that particle,

$$\frac{\eta a^2 t^2}{2}\left(\sin\beta - \frac{2aAt}{3}\cos\beta\right).$$

In fact, the differential equations are, in the present question,

$$y''_{1,t} = a^2\{\eta\sin(\beta - 2aAt) - 2y_{1,t} + y_{2,t}\} \quad \text{and} \quad y''_{l,t} = a^2(y_{l-1,t} - 2y_{l,t} + y_{l+1,t}),$$

if $l > 1$; also $y_{l,0} = y'_{l,0} = 0$, if $l > 0$; and these conditions are satisfied if we neglect t^4 in $y_{l,t}$, or t^2 in $y''_{l,t}$, & suppose $y_{1,t} = \eta a^2\left(\frac{t^2}{2}\sin\beta - \frac{aAt^3}{3}\cos\beta\right)$, $y_{2,t} =$ &c. $= 0$.

For the parts involving t^4 and t^5, we are to suppose $i = 2$, & to calculate

$$\Sigma^1_{(k)\,0} A^{-2k} g_{l,k} = g_{l,0} + A^{-2} g_{l,1},$$

in which we already know $g_{l,0}$ and in which $g_{l,1} = (-1)^{l+1}\frac{3l}{4}[0]^{-(2+l)}[0]^{-(2-l)}$, so that

$$g_{3,1} = g_{4,1} = \text{\&c.} = 0, \quad \text{and} \quad g_{1,1} = \tfrac{3}{4}[0]^{-3} = \tfrac{1}{8}, \quad g_{2,1} = -\tfrac{3}{2}[0]^{-4} = -\tfrac{1}{16};$$

these parts therefore vanish for all particles beyond P_2; they are, for P_1,

$$\tfrac{2}{3}\eta a^4 A^2 t^4\left(\frac{2aAt}{5}\cos\beta - \sin\beta\right)\left(\frac{1}{4} + \frac{A^{-2}}{8}\right) = -\frac{\eta a^4 t^4}{12}\left(\sin\beta - \frac{2aAt}{\;}\cos\beta\right)(1 + 2A^2),$$

and, for P_2,

$$\frac{\eta a^4 t^4}{24}\left(\sin\beta - \frac{2aAt}{\;}\cos\beta\right).$$

Accordingly, if we suppose

$$y_{1,t} = \eta\sin\beta\left\{\frac{a^2t^2}{2} - \frac{a^4t^4}{12}(1 + 2A^2)\right\} + \eta A\cos\beta\left\{-\frac{a^3t^3}{3} + \frac{a^5t^5}{30}(1 + 2A^2)\right\},$$

$$y_{2,t} = \eta\sin\beta\frac{a^4t^4}{24} - \eta A\cos\beta\frac{a^5t^5}{60}, \quad y_{3,t} = \text{\&c.} = 0,$$

and neglect t^4 in $y''_{l,t}$, we shall have

$$y''_{3,t} = \text{\&c.} = 0; \quad y''_{2,t} = a^2 y_{1,t};$$

$$y''_{1,t} + 2a^2 y_{1,t} = a^2\eta\{\sin\beta(1 - 2a^2A^2t^2) + A\cos\beta(-2at + \tfrac{4}{3}a^3A^2t^3)\} = a^2\eta\sin(\beta - 2aAt);$$

so that the differential equations are as well satisfied as they ought to be.

105. No power of t being neglected, the displacement of the particle P_1 is

$$y_{1,t} = \eta A^{-2}\Sigma^{\infty}_{(i)\,1}[0]^{-2i}(-4a^2A^2t^2)^i\left\{\frac{2aAt}{2i+1}\cos\beta - \sin\beta\right\}\Sigma^{i-1}_{(k)\,0}A^{-2k}g_{1,k};$$

in which

$$g_{1,k} = 2^{-(2k+1)}[2k+1]^{2k+1}[0]^{-k}[0]^{-(k+2)}.$$

We may order $y_{1,t}$ according to the ascending powers of A; & the coefficient of A^0 is

$$-\sin\beta\,.\,\eta\,\Sigma^{\infty}_{(i)\,1}[0]^{-2i}(-4a^2t^2)^i g_{1,i-1} = \eta\sin\beta\,\Sigma^{\infty}_{(i)\,1}[0]^{-i}[0]^{-(i+1)}(at)^{2i}(-1)^{i+1};$$

which latter series can be summed, if we can sum $1 - \left(\frac{x}{1}\right)^2 + \left(\frac{x^2}{1\,.\,2}\right)^2 - \left(\frac{x^3}{1\,.\,2\,.\,3}\right)^2 +$ &c. Now this last sum $=\frac{1}{\pi}\int_0^\pi dr\cos(2x\sin r)$, because $\frac{2^{2i}}{\pi}[0]^{-2i}\int_0^\pi dr\sin r^{2i}=([0]^{-i})^2$, (see page 523); therefore, differentiating relatively to x,

$$\frac{1}{\pi}\int_0^\pi dr\sin r\sin(2x\sin r)=x-\frac{x^3}{1.1.2}+\frac{x^5}{1.2.1.2.3}-\&c.;$$

$\because$ coefficient of A^0 in $y_{1,t}$ is $\left\{1-\frac{1}{\pi x}\int_0^\pi dr\sin r\sin(2x\sin r)\right\}\eta\sin\beta$, if $x=at$. In fact

$$\int_0^\pi dr\sin r\sin(2x\sin r)=-\Delta\{\cos r\sin(2x\sin r)\}+2x\int_0^\pi dr\cos r^2\cos(2x\sin r);$$

the coefficient of A^0 in $y_{1,t}$ is therefore

$$\eta\sin\beta\left\{1-\frac{2}{\pi}\int_0^\pi dr\cos r^2\cos(2at\sin r)\right\};$$

& accordingly if we make $A=0$, $l=1$, in the expression of **97** or **104** for $y_{l,t}$, we get

$$y_{1,t}=\frac{2\eta\sin\beta}{\pi}\int_0^{\frac{\pi}{2}}d\theta\,2\cos\theta^2\{1-\cos(2at\sin\theta)\}=\frac{\eta\sin\beta}{\pi}\int_0^\pi d\theta\ \&c.$$

But it is probably improper, or at least disadvantageous, to develope according to ascending powers of A, when A is >1.

106. It may be convenient to make $A-\sqrt{A^2-1}=B$, and therefore

$$A+\sqrt{A^2-1}=\frac{1}{B},\quad A=\tfrac{1}{2}(B+B^{-1})=\frac{1+B^2}{2B},$$

$$\begin{aligned}y_{l,t}&=\frac{(-1)^l\eta}{\pi}\int_0^\pi d\theta\,\frac{2B\sin\theta\sin 2l\theta}{1-2B\cos\theta+B^2}\{\sin(\beta-2aAt)-\sin(\beta-2at\cos\theta)\}\\&=\frac{2(-1)^l\eta}{\pi}\Sigma_{(n)1}^{\infty}\int_0^\pi d\theta B^n\sin n\theta\sin 2l\theta\{\sin(\beta-2aAt)-\sin(\beta-2at\cos\theta)\}\\&=\eta(-B^2)^l\sin(\beta-2aAt)+\frac{\eta(-1)^l}{\pi}\Sigma_{(n)1}^{\infty}B^n\int_0^\pi\{\cos(2l\theta+n\theta)-\cos(2l\theta-n\theta)\}\\&\qquad\times\sin(\beta-2at\cos\theta)\,d\theta.\end{aligned}$$

Accordingly it is evident that this last expression for $y_{l,t}$ satisfies the equation in mixed differences $y''_{l,t}=a^2(y_{l+1,t}-2y_{l,t}+y_{l-1\,t})$, because $B^2+B^{-2}+2=4A^2$; also $y_{0,t}=\eta\sin(\beta-2atA)$, & $y_{l,0}=y'_{l,0}=0$ if $l>0$.

We may therefore write

$$\begin{aligned}y_{l,t}&=\eta(-B^2)^l\sin(\beta-2aAt)\\&+\frac{2}{\pi}\eta(-1)^l\sin\beta\,\Sigma_{(n)1}^{\infty}B^{2n}\int_0^{\frac{\pi}{2}}d\theta\{\cos(2l\theta+2n\theta)-\cos(2l\theta-2n\theta)\}\cos(2at\cos\theta)\\&-\frac{2}{\pi}\eta(-1)^l\cos\beta\,\Sigma_{(n)1}^{\infty}B^{2n-1}\int_0^{\frac{\pi}{2}}d\theta\{\cos(2l\theta+2n\theta-\theta)-\cos(2l\theta-2n\theta+\theta)\}\sin(2at\cos\theta).\end{aligned}$$

And we see that it remains to calculate the values of the definite integrals

$$\frac{2}{\pi}\int_0^1 dx\frac{x^{2m}\cos(2atx)}{\sqrt{1-x^2}},\quad \frac{2}{\pi}\int_0^1 dx\frac{x^{2m+1}\sin(2atx)}{\sqrt{1-x^2}}$$

or simply of the former. We may even deduce all, by differentiation, from the function

$$\frac{2}{\pi}\int_0^1 dx \frac{\cos(2atx)}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos(2at\cos\theta) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos(2at\sin\theta)$$

$$= 1 - \left(\frac{at}{1}\right)^2 + \left(\frac{a^2t^2}{1\,.\,2}\right)^2 - \left(\frac{a^3t^3}{1\,.\,2\,.\,3}\right)^2 + \&\text{c.}$$

Or we may consider the question as being now to determine, at least approximately & for integer values of m much larger than t, the definite integrals

$$\int_0^{\frac{\pi}{2}} d\theta \cos 2m\theta \cos(2at\cos\theta) = M_m,$$

$$\int_0^{\frac{\pi}{2}} d\theta \cos(2m\theta - \theta)\sin(2at\cos\theta) = N_m.$$

107. Adopting this last view, & integrating by parts, so as to develope according to descending powers of m, we find

$$\int_0^{\theta} d\theta \cos 2m\theta \cos(2at\cos\theta) = \frac{\sin 2m\theta}{2m}\cos(2at\cos\theta) - \frac{at}{m}\int_0^{\theta} d\theta \sin\theta \sin 2m\theta \sin(2at\cos\theta),$$

$$\int_0^{\theta} d\theta \cos(2m\theta - \theta)\cos(2at\cos\theta)$$

$$= \frac{\sin(2m\theta-\theta)}{2m-1}\sin(2at\cos\theta) + \frac{2at}{2m-1}\int_0^{\theta} d\theta \sin\theta\sin(2m\theta-\theta)\cos(2at\cos\theta).$$

If then we employ the symbols M_m and N_m as defined in the last article, we have*

$$M_m = \frac{at}{2m}(N_{m+1} - N_m), \quad N_m = \frac{at}{2m-1}(M_{m-1} - M_m);$$

$$\because \quad N_{m+1} = \frac{at}{2m+1}(M_m - M_{m+1}), \quad M_m = \frac{a^2t^2}{2m(4m^2-1)}\{(2m-1)(M_m - M_{m+1})$$

$$-(2m+1)(M_{m-1} - M_m)\};$$

$$0 = M_m \left\{1 - \frac{2a^2t^2}{4m^2-1}\right\} + \frac{a^2t^2}{2m}\left(\frac{M_{m+1}}{2m+1} + \frac{M_{m-1}}{2m-1}\right);$$

$\because$ nearly, $\frac{2m}{at}$ being large, $M_{m+1} - 2M_m + M_{m-1} = -\frac{4m^2}{a^2t^2}M_m$.

[Manuscript ends.]

[$M_m = (-)^m J_{2m}(2at)$; $N_m = (-)^m J_{2m-1}(2at)$.]

XX.

PROPAGATION OF MOTION IN ELASTIC MEDIUM—DISCRETE MOLECULES*

[1839.]

[Note Book 53.]

[CONTENTS

PAGE
Statement of problem. . . . 527
Properties of operator ∇_{τ}. . . . 530
∇_{τ} and $\frac{d}{d\tau}$ not commutative. . . . 533
$(\nabla_{\tau}^{-2}-1)^{-\frac{1}{2}}\tau^{i}$. . . . 535
Expression for $\eta_{0,\tau}$. . . . 538
The operator I_{τ}, $\eta_{n,\tau}=(\tan\delta)^{n}\tan 2\delta.\beta$. . . . 542
$\tan\delta_{\tau}.\phi_{\tau}$ as an integral. . . . 543
$\eta_{x,t}$ expressed by means of operator D_{t} . . . 544
$\eta_{x,t}$ expressed as an integral. . . . 548
Discussion of asymptotic value of this integral. . . . 554
Airy's integral. . . . 563
Detailed description of wave motion. . . . 563
Extension of original problem. . . . 567
Two-dimensional waves. . . . 568
Three-dimensional waves in dispersive medium. . . . 569
Velocity of propagation $\frac{ds^{\backprime}}{dk^{\backprime}}$.] 572

[*Statement of problem.*]

(Jan. 9th, 1839.)

Let there be proposed the equation in mixed differences:

$$\frac{d^{2}}{dt^{2}}\eta_{x,t}=a^{2}(\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t}); \tag{1}$$

in which a is constant, real & positive. One integral of this equation is

$$\eta_{x,t}=\text{vers}\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_{,}}-t\right)\right\}, \tag{2}$$

λ, $a_{,}$ being constants, and a being connected with λ, $a_{,}$ by the relation

$$a^{2}=\left(\frac{\pi}{\lambda}\right)^{2}\left(\sin\frac{\pi}{\lambda a_{,}}\right)^{-2}; \tag{3}$$

---

* [For a synopsis of this paper see pp. 576, 577 and also Letter to Herschel, p. 599.]

because the equation (2) gives

$$\frac{d^2}{dt^2}\eta_{x,t}=\left(\frac{2\pi}{\lambda}\right)^2\cos\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_{,}}-t\right)\right\},$$

and

$$\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t}=2\operatorname{vers}\frac{2\pi}{\lambda a_{,}}\cos\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_{,}}-t\right)\right\}.$$

So that we may suppose

$$\frac{\pi}{\lambda a}=\sin\frac{\pi}{\lambda a_{,}}, \qquad (3)'$$

and a and $a_{,}$ are nearly equal, if $\frac{\pi}{\lambda a}$ be small.

We may put $$\tau=ta\sqrt{2};$$
(so that if τ now represent the time, the total attraction is equal to the distance from the position of equilibrium); & then the original equation (1) becomes

$$\left(\frac{d^2}{d\tau^2}+1\right)\eta_{x,\tau}=\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau}); \qquad (1)'$$

while the particular integral (2) becomes

$$\eta_{x,\tau}=\operatorname{vers}\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_{,}}-\frac{\tau}{a\sqrt{2}}\right)\right\}; \qquad (2)'$$

the relation (3) or (3)′ still holding good. We may also put, for abridgment,

$$\mu=\frac{2\pi}{\lambda a_{.}},\quad m=\frac{\pi\sqrt{2}}{\lambda a}=\sqrt{\operatorname{vers}\mu}; \qquad (3)''$$

& then the particular integral (2)′ becomes

$$\eta_{x,\tau}=\operatorname{vers}(\mu x-m\tau). \qquad (2)''$$

That is, (1)′ has (2)″ for a particular integral, if the relation (3)″ hold good.

The particular integral (2)″ gives by differentiation

$$\eta'_{x,\tau}=\frac{d}{d\tau}\eta_{x,\tau}=-m\sin(\mu x-m\tau); \qquad (2)'''$$

$$\eta''_{x,\tau}=\left(\frac{d}{d\tau}\right)^2\eta_{x,\tau}=m^2\cos(\mu x-m\tau); \qquad (2)^{\text{iv}}$$

also

$$\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau})=1-\cos(\mu x-m\tau)\cos\mu=\eta_{x,\tau}+\operatorname{vers}\mu\cos(\mu x-m\tau); \qquad (2)^{\text{v}}$$

& accordingly these give, by (3)″,

$$\eta''_{x,\tau}+\eta_{x,\tau}=\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau}). \qquad (1)'$$

Now let it be supposed that when $\tau=0$ the values of $\eta_{x,\tau}$ & $\frac{d}{d\tau}\eta_{x,\tau}$ vanish for $x=0$ & for all positive values of x, & are consistent with the equations (2)″ (2)‴ for all negative values of x; so that we have

$$\eta_{x,0}=0,\quad \eta'_{x,0}=0,\quad \text{if } x\not<0, \qquad (4)_1$$

but

$$\eta_{x,0}=\operatorname{vers}\mu x,\quad \eta'_{x,0}=-m\sin\mu x,\quad \text{if } x\not>0; \qquad (4)_2$$

and let it be required to determine generally the function $\eta_{x,\tau}$, x being at present restricted to integer values.

This question corresponds to the propagation of vibration in an elastic medium, & must illustrate the spreading of such vibration from one part of the medium to another, while the said vibration remains, or may remain, (approximately) confined between 2 parallel and indefinite planes which move (& why move? Sept. 30th/57) with one common velocity. It ought to turn out that if the velocity of propagation $\frac{m}{\mu}$ is positive (τ here being taken to represent the time), so that $\eta'_{x,0}$ is positive for small negative values of x, then $\eta_{x,\tau}$ will take sensibly the value (2)″ as soon as $\frac{m\tau}{\mu}$ is sensibly greater than x; but that if $\frac{m}{\mu}$ be supposed negative, then $\eta_{x,\tau}$ will sensibly vanish when $-\frac{m\tau}{\mu}$ is sensibly $> -x$, the value of $\eta'_{x,0}$ being in this case negative for small negative values of x. And the *physical explanation* is doubtless of this kind: μ being in both cases supposed for simplicity to be such that $\frac{\pi}{2\mu}$ is a large positive integer number, $\nu = \frac{\pi}{2\mu} = \frac{\lambda a,}{4}$, we have in both cases positive displacements $\eta_{x,0}$ for all the values $x = -1$, $x = -2$, … $x = -(4\nu - 1)$, the greatest being that which corresponds to $x = -2\nu = -\frac{\pi}{\mu}$; but in the first case, in which m is > 0, the displacements corresponding to $x = -1$, $x = -2$, … $x = -(2\nu - 1)$ are all *increasing*, while in the 2nd case, in which $m < 0$, the same displacements are all *decreasing*, at the same original moment $\tau = 0$; it is clear then that in the first case the system is departing more and more from equilibrium, while in the 2nd case it is approaching more and more to equilibrium, at the moment $\tau = 0$, & within the extent included between the limits $x = 0$ and $x = -2\nu = -\frac{\pi}{\mu}$; and that therefore the originally quiescent part on the positive side of the limit $x = 0$ must be much more disturbed in the first case than in the second. But it is interesting to investigate the law & quantity of this inequality of disturbance, by resolving, at least approximately, the algebraic problem proposed in the preceding page.

Perhaps the *theorem of Fourier** might help to resolve this problem; but I prefer at present to proceed in a way of my own, as follows.

Let
$$y_{-x,\tau} = \eta_{x,\tau} - \text{vers}\,(\mu x - m\tau); \tag{5}$$

---

* [Hamilton appends the following note. "Fourier's theorem is that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\cos(\alpha x - \alpha x')\,f(x')\,d\alpha\right)dx',$$

or, as it may be better stated,

$$f(x) = \frac{1}{2\pi}\underset{h=0}{\Lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x - \alpha x')\,d\alpha\right)f(x')\,dx'.$$

In fact, as I have remarked in an old book (Oct. 1835), we have

$$\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x - \alpha x')\,d\alpha = \sqrt{\frac{\pi}{h}}\,e^{-\frac{(x'-x)^2}{4h}};$$

∵ multiplying this by $f(x')\,dx'$ and changing x' to $x + 2t\sqrt{h}$, we have

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x - \alpha x')\,d\alpha\right)f(x')\,dx' = 2\sqrt{\pi}\int_{-\infty}^{\infty}e^{-t^2}f(x + 2t\sqrt{h})\,dt,$$

of which the limit relatively to $h = 0$ is $2\pi f(x)$."]

& let us introduce, instead of the old system of functions $\eta_{-n,\tau}$, $\eta_{n-1,\tau}$ (n being >0), a new system, $y_{n,\tau}$, $\eta_{n-1,\tau}$, of which each vanishes, as also does its 1st differential coefficient relatively to τ, at the moment $\tau=0$. We shall then have

$$y''_{-x,\tau}+y_{-x,\tau}=\tfrac{1}{2}y_{-x-1,\tau}+\tfrac{1}{2}y_{-x+1,\tau}; \tag{1}'''$$

$$\because\quad\begin{cases} y''_{n+1,\tau}+y_{n+1,\tau}=\tfrac{1}{2}(y_{n+2,\tau}+y_{n,\tau}); & (1)'''_1\\ y''_{1,\tau}+y_{1,\tau}=\tfrac{1}{2}(y_{2,\tau}+\eta_{0,\tau}-\text{vers}\,m\tau); & (1)'''_2\\ \eta''_{0,\tau}+\eta_{0,\tau}=\tfrac{1}{2}(\eta_{1,\tau}+y_{1,\tau}+\text{vers}\,\overline{\mu+m\tau}); & (1)'''_3\\ \eta''_{n,\tau}+\eta_{n,\tau}=\tfrac{1}{2}(\eta_{n+1,\tau}+\eta_{n-1,\tau}); & (1)'''_4 \end{cases}$$

n being, in each of these 4 last formulae, an integer >0. Also

$$0=y_{n,0}=y'_{n,0}=\eta_{n-1,0}=\eta'_{n-1,0};\quad n \text{ being still } >0. \tag{4}'$$

We have also

$$\begin{aligned}\text{vers}\,(\mu+m\tau)&=1-\cos\mu\cos m\tau+\sin\mu\sin m\tau\\ &=1-(1-m^2)\cos m\tau+m\sqrt{2-m^2}\sin m\tau;\end{aligned} \tag{3}''$$

so that m may be considered as the only arbitrary constant of the problem & may be supposed to be small and positive: attending only at first to the case of forward propagation.

[*Properties of operator* ∇_τ.]

Let the notation

$$\nabla_\tau f_\tau=\phi_\tau \tag{6}$$

imply that

$$\phi''_\tau+\phi_\tau=f_\tau; \tag{6}'$$

& that

$$\phi_0=0,\quad \phi'_0=0. \qquad (6)'',\ (6)'''$$

Then if we put

$$\tfrac{1}{2}\{1-(1-m^2)\cos m\tau\}=\alpha_\tau, \tag{7}$$

we shall have

$$\tfrac{1}{2}\,\text{vers}\,m\tau=\nabla_\tau\alpha_\tau; \tag{7}'$$

& putting also

$$\tfrac{1}{2}m\sqrt{2-m^2}\sin m\tau=\beta_\tau, \tag{7}''$$

we have the equations

$$\begin{cases} y_{n+1,\tau}=\tfrac{1}{2}\nabla_\tau(y_{n+2,\tau}+y_{n,\tau}); & (8)_1\\ y_{1,\tau}=\tfrac{1}{2}\nabla_\tau(y_{2,\tau}+\eta_{0,\tau})-\nabla^2_\tau\alpha_\tau; & (8)_2\\ \eta_{0,\tau}=\tfrac{1}{2}\nabla_\tau(\eta_{1,\tau}+y_{1,\tau})+\nabla_\tau(\alpha_\tau+\beta_\tau); & (8)_3\\ \eta_{n,\tau}=\tfrac{1}{2}\nabla_\tau(\eta_{n+1,\tau}+\eta_{n-1,\tau}); & (8)_4 \end{cases}$$

n being still an integer >0. Make

$$\eta_{0,\tau}-\nabla_\tau\alpha_\tau=H_\tau; \tag{9}$$

then

$$\begin{cases} y_{n+1,\tau}=\tfrac{1}{2}\nabla_\tau(y_{n+2,\tau}+y_{n,\tau}); \quad \eta_{n+1,\tau}=\tfrac{1}{2}\nabla_\tau(\eta_{n+2,\tau}+\eta_{n,\tau}); & (9)_1,\ (9)_5\\ y_{1,\tau}=\tfrac{1}{2}\nabla_\tau(y_{2,\tau}+H_\tau-\nabla_\tau\alpha_\tau); \quad \eta_{1,\tau}=\tfrac{1}{2}\nabla_\tau(\eta_{2,\tau}+H_\tau+\nabla_\tau\alpha_\tau); & (9)_2,\ (9)_4\\ H_\tau=\tfrac{1}{2}\nabla_\tau(\eta_{1,\tau}+y_{1,\tau}+2\beta_\tau); & (9)_3 \end{cases}$$

& these would give*

$$\left.\begin{aligned} H_\tau=0,\quad y_{n,\tau}&=-\eta_{n,\tau},\\ \eta_{0,\tau}=\nabla_\tau\alpha_\tau&=\tfrac{1}{2}\,\text{vers}\,m\tau, \end{aligned}\right\}\text{ if } \beta_\tau \text{ were } =0. \tag{10}$$

---

* [If the term β_τ is absent, H_τ, $y_{1,\tau}+\eta_{1,\tau}$, $y_{2,\tau}+\eta_{2,\tau}$, ... represent the transverse displacements of points P_1, P_2, P_3, ... of a stretched string having equal masses at P_1, P_2, P_3, ... and $P_0P_1=P_1P_2=$etc. If P_0 is always fixed and if the others are initially at rest in the equilibrium position, then there are no solutions except

$$H_\tau=0,\ y_{1,\tau}+\eta_{1,\tau}=0,\ \text{etc.}]$$

It results from the investigations of Oct. 1835, that the same supposition ($\beta_\tau = 0$) would give generally*

$$-y_{n,\tau} = \eta_{n,\tau} = \frac{n}{2^n}\Sigma_{(k)0}^{\infty} \frac{[n+2k-1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{n+2k+1}\alpha_\tau}{2^{2k}}, \qquad (10)'$$

the square brackets denoting factorials; so that, more fully,

$$-y_{n,\tau} = \eta_{n,\tau} = 2^{-n}\left\{\nabla_\tau^{n+1}\alpha_\tau + n\,2^{-2}\nabla_\tau^{n+3}\alpha_\tau + \frac{n\,(n+3)}{1\,.\,2}2^{-4}\nabla_\tau^{n+5}\alpha_\tau\right.$$
$$\left. + \frac{n\,(n+4)\,(n+5)}{1\,.\,2\,.\,3}2^{-6}\nabla_\tau^{n+7}\alpha_\tau + \frac{n\,(n+5)\,(n+6)\,(n+7)}{1\,.\,2\,.\,3\,.\,4}2^{-8}\nabla_\tau^{n+9}\alpha_\tau + \&c.\right\}, \qquad (10)''$$

& in particular (developing as far as $\nabla_\tau^{10}\alpha_\tau$ inclusive)

$$\left\{\begin{array}{ll}
-y_{1,\tau} = \eta_{1,\tau} = \frac{1}{2}\nabla_\tau^2\alpha_\tau + \frac{1}{8}\nabla_\tau^4\alpha_\tau + \frac{1}{16}\nabla_\tau^6\alpha_\tau + \frac{5}{128}\nabla_\tau^8\alpha_\tau + \frac{7}{256}\nabla_\tau^{10}\alpha_\tau + \&c.; & (10)_1'' \\
-y_{2,\tau} = \eta_{2,\tau} = \frac{1}{4}\nabla_\tau^3\alpha_\tau + \frac{1}{8}\nabla_\tau^5\alpha_\tau + \frac{5}{64}\nabla_\tau^7\alpha_\tau + \frac{7}{128}\nabla_\tau^9\alpha_\tau + \&c.; & (10)_2'' \\
-y_{3,\tau} = \eta_{3,\tau} = \frac{1}{8}\nabla_\tau^4\alpha_\tau + \frac{3}{32}\nabla_\tau^6\alpha_\tau + \frac{9}{128}\nabla_\tau^8\alpha_\tau + \frac{7}{128}\nabla_\tau^{10}\alpha_\tau + \&c.; & (10)_3'' \\
-y_{4,\tau} = \eta_{4,\tau} = \frac{1}{16}\nabla_\tau^5\alpha_\tau + \frac{1}{16}\nabla_\tau^7\alpha_\tau + \frac{7}{128}\nabla_\tau^9\alpha_\tau + \&c.; & (10)_4'' \\
-y_{5,\tau} = \eta_{5,\tau} = \frac{1}{32}\nabla_\tau^6\alpha_\tau + \frac{5}{128}\nabla_\tau^8\alpha_\tau + \frac{5}{128}\nabla_\tau^{10}\alpha_\tau + \&c.; & (10)_5'' \\
-y_{6,\tau} = \eta_{6,\tau} = \frac{1}{64}\nabla_\tau^7\alpha_\tau + \frac{3}{128}\nabla_\tau^9\alpha_\tau + \&c.; & (10)_6'' \\
-y_{7,\tau} = \eta_{7,\tau} = \frac{1}{128}\nabla_\tau^8\alpha_\tau + \frac{7}{512}\nabla_\tau^{10}\alpha_\tau + \&c.; & (10)_7'' \\
-y_{8,\tau} = \eta_{8,\tau} = \frac{1}{256}\nabla_\tau^9\alpha_\tau + \&c.; & (10)_8'' \\
-y_{9,\tau} = \eta_{9,\tau} = \frac{1}{512}\nabla_\tau^{10}\alpha_\tau + \&c. & (10)_9''
\end{array}\right.$$

In fact these developments verify, to the same degree of accuracy, the conditions

$$\eta_{1,\tau} - \tfrac{1}{2}\nabla_\tau\eta_{2,\tau} = \tfrac{1}{2}\nabla_\tau^2\alpha_\tau; \quad \eta_{2,\tau} - \tfrac{1}{2}\nabla_\tau(\eta_{1,\tau}+\eta_{3,\tau}) = 0; \quad \eta_{3,\tau} - \tfrac{1}{2}\nabla_\tau(\eta_{2,\tau}+\eta_{4,\tau}) = 0; \quad \&c.$$

And, taking the whole series $(10)'$, we have

$$\eta_{1,\tau} = \Sigma_{(k)0}^{\infty}\frac{[2k]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+2}\alpha_\tau}{2^{2k+1}} = \tfrac{1}{2}\nabla_\tau^2\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{[2k+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+4}\alpha_\tau}{2^{2k+2}}; \qquad (10)_1'$$

$$\eta_{2,\tau} = 2\,\Sigma_{(k)0}^{\infty}\frac{[2k+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+3}\alpha_\tau}{2^{2k+2}} = \tfrac{1}{4}\nabla_\tau^3\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{[2k+3]^{k}}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+5}\alpha_\tau}{2^{2k+3}}; \qquad (10)_2'$$

$$\eta_{3,\tau} = 3\,\Sigma_{k(0)}^{\infty}\frac{[2k+2]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+4}\alpha_\tau}{2^{2k+3}} = \qquad (10)_3'$$

$$\eta_{n+1,\tau} = (n+1)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+n+2}\alpha_\tau}{2^{2k+n+1}}$$
$$= \frac{1}{2^{n+1}}\nabla_\tau^{n+2}\alpha_\tau + (n+1)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+2]^{k}}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+n+4}\alpha_\tau}{2^{2k+n+3}}; \qquad (10)_{n+1}'$$

$$\eta_{n+2,\tau} = (n+2)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+n+3}\alpha_\tau}{2^{2k+n+2}}; \qquad (10)_{n+2}'$$

$$\eta_{n,\tau} = \frac{1}{2^n}\nabla_\tau^{n+1}\alpha_\tau + n\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+1]^{k}}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+n+3}\alpha_\tau}{2^{2k+n+2}}; \qquad (10)_n'$$

and since

$$n\,[2k+n+1]^k + (n+2)\,[2k+n+1]^{k-1}(k+1) = \{n\,(k+n+2)+(n+2)\,(k+1)\}\,[2k+n+1]^{k-1}$$
$$= (n+1)\,(n+2k+2)\,[2k+n+1]^{k-1} = (n+1)\,[2k+n+2]^k,$$

we have, as we ought,

$$\eta_{n+1,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n,\tau}+\eta_{n+1,\tau}); \quad \eta_{1,\tau} = \tfrac{1}{2}\nabla_\tau(\nabla_\tau\alpha_\tau + \eta_{2,\tau}).$$

---

* [See top of page 532.]

[Here follow some investigations concerning finite expressions for series like (10)′. In some pencilled notes added later Hamilton points out that (10)′ is equal to $(1-\sqrt{1-\nabla_\tau^2})^n \nabla_\tau^{-n+1}\alpha_\tau$. It is easy to see that this expression satisfies $(8)_4$ and, when $n=0$, is equal to $\nabla_\tau \alpha_\tau$. Moreover from the definition of the operator ∇_τ, since (10)′ contains only positive powers of ∇_τ, $\eta_{n,\tau}$ and $\eta'_{n,\tau}$ are zero when $\tau=0$.]

(Jan. 10th, 1839.)

In the next place let us consider the system of equations

$$\left.\begin{aligned} \alpha_\tau = 0, \quad y_{n,\tau} = \eta_{n,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n+1,\tau}+\eta_{n-1,\tau}), \qquad (n>0),\\ \eta_{0,\tau} = \nabla_\tau \eta_{1,\tau} + \nabla_\tau \beta_\tau. \end{aligned}\right\} \tag{11}$$

...[Here $\eta_{n,\tau}$ is obtained by ordinary methods. Its value however could be inferred from (10)′]....

Hence

$$\eta_{n,\tau} = 2^{-n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}[n+2k]^k[0]^{-k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau.$$

Hence the equations (8) give, when both α_τ and β_τ are retained, the expressions

$$\eta_{0,\tau} = \nabla_\tau\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{1\,.\,3\,.\,5\dots(2k-1)}{2\,.\,4\,.\,6\dots 2k}\nabla_\tau^{2k+1}\beta_\tau; \tag*{$(12)_1$}$$

$$\eta_{n,\tau} = \frac{n}{2^n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}\frac{[n+2k-1]^{k-1}}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\left(\beta_\tau+\alpha_\tau+\frac{2k}{n}\beta_\tau\right); \tag*{$(12)_2$}$$

$$y_{n,\tau} = \frac{n}{2^n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}\frac{[n+2k-1]^{k-1}}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\left(\beta_\tau-\alpha_\tau+\frac{2k}{n}\beta_\tau\right). \tag*{$(12)_3$}$$

The expression for $\eta_{0,\tau}$ may be concisely represented thus:

$$\eta_{0,\tau} = \nabla_\tau\alpha_\tau + \nabla_\tau(1-\nabla_\tau^2)^{-\frac{1}{2}}\beta_\tau = \nabla_\tau\alpha_\tau + (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau. \tag{13}$$

But

$$\nabla_\tau = \left\{1+\left(\frac{d}{d\tau}\right)^2\right\}^{-1}$$

$$\nabla_\tau^{-2}-1 = \left\{1+\left(\frac{d}{d\tau}\right)^2\right\}^2 - 1 = \left(\frac{d}{d\tau}\right)^2\left\{2+\left(\frac{d}{d\tau}\right)^2\right\};$$

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}} = \left(\frac{d}{d\tau}\right)^{-1}\left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}} = \int_0^\tau \left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}} d\tau. \tag{14}$$

We have

$$\begin{aligned} \eta_{0,\tau} &= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau\left\{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}}\sin m\tau\,d\tau \\ &= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau \Sigma_{(k)0}^{\infty}\frac{[-\frac{1}{2}]^k\,2^{-k}}{[k]^k}\left(\frac{d}{d\tau}\right)^{2k}\sin m\tau\,d\tau \\ &= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau \Sigma_{(k)0}^{\infty}\frac{[-\frac{1}{2}]^k}{[k]^k}\left(-\frac{m^2}{2}\right)^k\sin m\tau\,d\tau \\ &= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}(1-\tfrac{1}{2}m^2)^{\frac{1}{2}}(1-\tfrac{1}{2}m^2)^{-\frac{1}{2}}\int_0^\tau \sin m\tau\,d\tau = \operatorname{vers} m\tau. \end{aligned} \tag{15}$$

We seem then to be conducted to the equation

$$\eta_{0,\tau} = \operatorname{vers} m\tau \tag*{$(15)_1$}$$

as the exact sum of the series $(12)_1$. On the other hand we should have had exactly

$$\eta_{0,\tau} = 0, \tag*{$(15)_2$}$$

if we had changed the sign of the radical $\sqrt{2-m^2}$ in β_τ.

Perhaps also we should be able to prove by some similar reasoning that the part depending on β_τ should be considered as negative for negative values of τ, & $\because$ that for such values $\eta_{0,\tau}$ vanishes, the radical $\sqrt{2-m^2}$ or the ratio m/μ, which expresses the velocity of propagation, being still taken with the positive sign.

On the whole it is perhaps possible to prove by the foregoing investigation or by some one very similar that $\eta_{0,\tau}$ may become exactly $= \operatorname{vers} m\tau$ for all positive values of τ after having been exactly $= 0$ for all negative values thereof. And $\eta''_{0,0}$ is the mean of the two values 0 and m^2. Can we apply a similar process to the expression for $\eta_{n,\tau}$?

$$\left[\nabla_\tau \textit{ and } \frac{d}{d\tau} \textit{ not commutative.}\right]$$

The part depending on β_τ in $\eta_{1,\tau}$ is

$$\tfrac{1}{2}\nabla_\tau^2 \Sigma_{(k)\,0}^{\infty} \frac{[2k+1]^k}{[k]^k} (\tfrac{1}{2}\nabla_\tau)^{2k} \beta_\tau.$$

Here

$$[2k+1]^k = (2k+1)\,2k\,(2k-1)\ldots(k+2) = \frac{[2k+1]^{2k+1}}{[k+1]^{k+1}}$$

$$= \frac{2^k(2k+1)(2k-1)\ldots 3\,.\,1}{k+1} = \frac{2^{2k+1}(-1)^{k+1}[-\frac{1}{2}]^{k+1}}{k+1};$$

$\because$ the part depending on β_τ in $\eta_{1,\tau}$ is

$$\Sigma_{(k)\,0}^{\infty} \frac{[-\frac{1}{2}]^{k+1}}{[k+1]^{k+1}} (-\nabla_\tau^2)^{k+1} \beta_\tau = \{-1 + (1-\nabla_\tau^2)^{-\frac{1}{2}}\}\,\beta_\tau;$$

which accordingly agrees with the equation (11),

$$\eta_{0,\tau} = \nabla_\tau(\eta_{1,\tau} + \beta_\tau),$$

α_τ being supposed to vanish. This equation gives also

$$\eta_{1,\tau} = -\beta_\tau + \eta''_{0,\tau} + \eta_{0,\tau},$$

in which $\eta_{0,\tau}$ is to be made only $\frac{1}{2}\operatorname{vers} m\tau$, because we have supposed α_τ to vanish; thus

$$\eta''_{0,\tau} + \eta_{0,\tau} = \frac{1}{2} - \frac{1-m^2}{2}\cos m\tau = \frac{1-\cos\mu\cos m\tau}{2}; \quad \& \quad -\beta_\tau = -\frac{\sin\mu\sin m\tau}{2};$$

it seems $\because$ that $\eta_{1,\tau} = \dfrac{\operatorname{vers}(m\tau-\mu)}{2}$, so far as it depends on β_τ alone. Yet this value of $\eta_{1,\tau}$ does not vanish with τ, though it ought to contain ∇_τ as a factor. Perhaps the development of $\eta_{1,\tau}$ diverges when $\tau < \mu/m$. But we must observe that the equation $\eta''_{0,\tau} + \eta_{0,\tau} = \eta_{1,\tau} + \beta_\tau$ requires that $\eta''_{0,0}$ should vanish, which the foregoing formulae do not make it do.

We can certainly conceive that $\eta_{0,\tau}$, $\eta_{n,\tau}$ are functions such that $\eta_{0,0}=\eta_{n,0}=\eta'_{0,0}=\eta'_{n,0}=0$, and that

$$\eta''_{0,\tau}+\eta_{0,\tau}-\eta_{1,\tau}=\beta_\tau=\tfrac{1}{2}\sin\mu\sin m\tau,$$

$$\eta''_{n,\tau}+\eta_{n,\tau}=\tfrac{1}{2}(\eta_{n+1,\tau}+\eta_{n-1,\tau}); \quad (m^2=\operatorname{vers}\mu,\ n>0);$$

but will these functions $\eta_{0,\tau}$, $\eta_{n,\tau}$ be convergently expressed by the series

$$\eta_{0,\tau}=\nabla_\tau\,\Sigma_{(k)\,0}^{\infty}\frac{[2k]^k}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau,$$

$$\eta_{n,\tau}=\frac{\nabla_\tau^{n+1}}{2^n}\Sigma_{(k)\,0}^{\infty}\frac{[n+2k]^k}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau?$$

And when we have found thus that

$$\eta''_{0,\tau}+\eta_{0,\tau}=(1-\nabla_\tau^2)^{-\frac{1}{2}}\beta_\tau,$$

can we effect this operation $(1-\nabla_\tau^2)^{-\frac{1}{2}}$?

We just now made $\nabla_\tau(1-\nabla_\tau^2)^{-\frac{1}{2}}C_\tau=(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}C_\tau$ & transformed this to

$$\int_0^\tau\left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}}C_\tau d\tau=\frac{1}{\sqrt{2}}\int_0^\tau\frac{C_\tau d\tau}{\sqrt{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}},$$

but it is an obvious objection to this transformation that the $\frac{d}{d\tau}$ of the last expression does not *in general* vanish with τ. However this objection does not apply in the case where $C_\tau=\beta_\tau=\frac{1}{2}\sin\mu\sin m\tau$, because $C_0^{(2k+1)}=0$; but on the other hand we ought to have

$$(1-\nabla_\tau^2)^{-\frac{1}{2}}C_\tau=\frac{1}{\sqrt{2}}\left\{1+\left(\frac{d}{d\tau}\right)^2\right. \qquad \frac{C_\tau d\tau}{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}$$

& the $\frac{d}{d\tau}$ of $\dfrac{\beta_\tau}{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}$ does not vanish with τ.

The transformation therefore is erroneous, and although its elegance appears to point to some true process, intimately connected with it,* yet I believe it may be better to employ for the present a different method & one more analogous to that of my old investigations.

Let us then admit that, (β_τ being still $=\frac{1}{2}\sin\mu\sin m\tau$,)

$$\eta_{0,\tau}=\nabla_\tau\beta_\tau+\tfrac{1}{2}\nabla_\tau^3\beta_\tau+\tfrac{3}{8}\nabla_\tau^5\beta_\tau+\ \&c.$$

and endeavour to calculate finite trigonometrical expressions for at least some of the terms of this development. For this purpose we are to calculate if possible a general expression for $\nabla_\tau^k\sin m\tau$ or at least expressions for this function, for several moderate values of k.

* [The error arises from the fact that the operations $\frac{d}{d\tau}$ and ∇_τ are not commutative. Correct results are arrived at later because $\int_0^\tau d\tau$ and ∇_τ *are* commutative.]

If we were to stop at this degree of approximation, [the first three terms of the last expression for $\eta_{0,\tau}$,] we should have

$$\nabla_\tau \sin m\tau = \frac{\sin m\tau - m\sin\tau}{1-m^2},$$

$$\tfrac{1}{2}\nabla_\tau^3 \sin m\tau = \frac{\sin m\tau - m\sin\tau}{2(1-m^2)^3} - \frac{m(\sin\tau - \tau\cos\tau)}{4(1-m^2)^2} - \frac{m\{(3-\tau^2)\sin\tau - 3\tau\cos\tau\}}{16(1-m^2)},$$

$$\tfrac{3}{8}\nabla_\tau^5 \sin m\tau = \frac{3(\sin m\tau - m\sin\tau)}{8(1-m^2)^5} - \frac{3m(\sin\tau - \tau\cos\tau)}{16(1-m^2)^4} - \frac{3m\{(3-\tau^2)\sin\tau - 3\tau\cos\tau\}}{64(1-m^2)^3}$$
$$- \frac{m\{(15-6\tau^2)\sin\tau - (15-\tau^2)\tau\cos\tau\}}{128(1-m^2)^2}$$
$$- \frac{m\{(105-45\tau^2+\tau^4)\sin\tau - (105-10\tau^2)\tau\cos\tau\}}{1024(1-m^2)};^*$$

which 3 expressions indeed are rigorous: & the sum of these three expressions, multiplied by $\frac{1}{2}m\sqrt{2-m^2}$, will give an *accurate* expression for $(\nabla_\tau + \frac{1}{2}\nabla_\tau^3 + \frac{3}{8}\nabla_\tau^5)\beta_\tau$, & ∵ (probably) an *approximate* expression for that part of $\eta_{0,\tau}$ which depends on β_τ, & which is to be added to $\frac{1}{2}$ vers $m\tau$.

[Neglecting powers of m above the square, Hamilton works out numerically the value of this expression for

$$\tau = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ 4,\ 5;$$

e.g. for $\tau = 5$ he finds that $\eta_{0,5} = \dfrac{17{,}5169}{17{,}6776}\,\text{vers}\,5m = \left(1 - \dfrac{1}{110}\right)\text{vers}\,5m$ approximately.]

$[(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau^i.]$

(Jan. 12th, 1839.)

We found that
$$\eta_{0,\tau} = \nabla_\tau \alpha_\tau + (\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\beta_\tau,$$
in which
$$\nabla_\tau \alpha_\tau = \tfrac{1}{2}\,\text{vers}\,m\tau \quad \text{and} \quad \beta_\tau = \tfrac{1}{2}m\sqrt{2-m^2}\sin m\tau.$$
Now the operation $(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}$ may be transformed in the following way, which occurred to me late last night,

$$\nabla_\tau = \left\{1 + \left(\frac{d}{d\tau}\right)^2\right\}^{-1} = \left(\int_0^\tau d\tau\right)^2 \left\{1 + \left(\int_0^\tau d\tau\right)^2\right\}^{-1}$$
$$= \left(\int_0^\tau d\tau\right)^2 - \left(\int_0^\tau d\tau\right)^4 + \left(\int_0^\tau d\tau\right)^6 - \&\text{c}.$$

For example

$$\nabla_\tau \tau = \frac{\tau^3}{6} - \frac{\tau^5}{120} + \&\text{c.} = \tau - \sin\tau; \quad \nabla_\tau 1 = \frac{\tau^2}{2} - \frac{\tau^4}{24} + \&\text{c.} = \text{vers}\,\tau;$$

$$\nabla_\tau \tau^2 = \frac{\tau^4}{4.3} - \frac{\tau^6}{6.5.4.3} + \frac{\tau^8}{8.7.6.5.4.3} - \&\text{c.} = 2\left(\cos\tau - 1 + \frac{\tau^2}{2}\right);$$

$$\nabla_\tau \tau^3 = \frac{\tau^5}{5.4} - \frac{\tau^7}{7.6.5.4} + \frac{\tau^9}{9.8.7.6.5.4} - \&\text{c.} = 6\left(\sin\tau - \tau + \frac{\tau^3}{6}\right);$$

---

* [The law of formation is obvious, e.g.

$$(105 - 45\tau^2 + \tau^4)\sin\tau - (105 - 10\tau^2)\tau\cos\tau = \tau^8\left(\frac{d}{d\tau}\frac{1}{\tau}\right)^4 \sin\tau.]$$

& generally

$$\nabla_\tau \tau^{2k} = \frac{\tau^{2k+2}}{(2k+2)(2k+1)} - \frac{\tau^{2k+4}}{[2k+4]^4} + \&\text{c.} = [2k]^{2k}(-1)^k\left(1 - \frac{\tau^2}{2} + \frac{\tau^4}{24} - \ldots + \frac{(-\tau^2)^k}{[2k]^{2k}} - \cos\tau\right);$$

$$\nabla_\tau \tau^{2k+1} = [2k+1]^{2k+1}(-1)^k\left(\tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \ldots + \frac{(-1)^k\tau^{2k+1}}{[2k+1]^{2k+1}} - \sin\tau\right);$$

$$\nabla_\tau \sin m\tau = m(\tau - \sin\tau) + m^3\left(\tau - \frac{\tau^3}{6} - \sin\tau\right) + m^5\left(\tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \sin\tau\right) + \&\text{c.}$$

$$\frac{m}{1-m^2}(\tau - \sin\tau) - \frac{m^3\tau^3}{6(1-m^2)} + \frac{m^5\tau^5}{120(1-m^2)} - \&\text{c.} = \frac{\sin m\tau - m\sin\tau}{1-m^2}$$

$$\nabla_\tau \cos m\tau = 1 - \cos\tau + m^2\left(1 - \frac{\tau^2}{2} - \cos\tau\right) + m^4\left(1 - \frac{\tau^2}{2} + \frac{\tau^4}{24} - \cos\tau\right) + \&\text{c.} = \frac{\cos m\tau - \cos\tau}{1-m^2}.$$

Hence

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}} = \left(\int_0^\tau d\tau\right)^2\left\{\left(1 + \left(\int_0^\tau d\tau\right)^2\right)^2 - \left(\int_0^\tau d\tau\right)^4\right\}^{-\frac{1}{2}} = \left(\int_0^\tau d\tau\right)^2\left\{1 + 2\left(\int_0^\tau d\tau\right)^2\right\}^{-}$$

$$= \Sigma_{(k)\,0}^{\infty}[-\tfrac{1}{2}]^k[0]^{-k}2^k\left(\int_0^\tau d\tau\right)^{2k+2} = \Sigma_{(k)\,0}^{\infty}\frac{1\,.\,3\,.\,5\ldots(2k-1)}{1\,.\,2\,.\,3\ldots k}(-1)^k\left(\int_0^\tau d\tau\right)^{2k+2}.$$

Hence

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau^i = [i]^i\,\Sigma_{(k)\,0}^{\infty}[-\tfrac{1}{2}]^k[0]^{-k}2^k[0]^{-(2k+2+i)}\tau^{2k+2+i}$$

$$= [i]^i\left(\int_0^\tau d\tau\right)^{i+2}\Sigma_{(k)\,0}^{\infty}([0]^{-k})^2\left(-\frac{\tau^2}{2}\right)^k = \ldots$$

$$= \frac{2}{\pi}[i]^i\left(\int_0^\tau d\tau\right)^{i+2}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta)$$

and we may change the order of the signs of integration.

The expression which I deduced (in pencil, late last night,) from a result in Poisson's Memoir on Waves was

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{2}{3\pi}\left(\int_0^\tau d\tau\right)^3\int_0^1 da\,(1-a^2)^{\frac{3}{2}}\{(8 - 2a^2\tau^2)\cos(a\tau\sqrt{2}) - 7a\tau\sqrt{2}\sin(a\tau\sqrt{2})\}$$

$$= \frac{2}{3\pi}\left(\int_0^\tau d\tau\right)^3\int_0^{\frac{\pi}{2}} d\theta\sin\theta^4\{(8 - 2\tau^2\cos\theta^2)\cos(\tau\sqrt{2}\cos\theta) - 7\tau\sqrt{2}\cos\theta\sin(\tau\sqrt{2}\cos\theta)\}.$$

Here also I deduced

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{1}{3\pi\sqrt{2}}\int_0^1 da\,\frac{(1-a^2)^{\frac{3}{2}}}{a^3}\{(1 + 2a^2\tau^2)\sin(a\tau\sqrt{2}) - a\tau\sqrt{2}\cos(a\tau\sqrt{2})\}$$

$$\frac{\tau^2\sqrt{2}}{3\pi}\int_0^{\tau\sqrt{2}}\frac{dx}{x^3}\left(1 - \frac{x^2}{2\tau^2}\right)^{\frac{3}{2}}\{(1+x^2)\sin x - x\cos x\}\,dx.$$

This gives approximately, when τ is large,

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{\tau^2\sqrt{2}}{3\pi}\int_0^\infty\frac{dx}{x^3}\{(1+x^2)\sin x - x\cos x\} = \frac{\tau^2}{2\sqrt{2}};$$

because

$$\int_0 \frac{\sin x}{x}\,dx = \frac{\pi}{2}, \quad \text{and} \quad \int_0^\infty\frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}.$$

[Extract from pencil note.] (Poisson has shown in his Memoir on Waves, pages 168, 169, that

$$\frac{8}{3\pi}\int_0^1 (1-a^2)^{\frac{3}{2}}\cos(2a\sqrt{p})\,da=\frac{8P}{3\pi}=\frac{1}{1\,.\,2}-\frac{p}{2\,.\,3}+\frac{p^2}{3\,.\,4\,.\,(1\,.\,2)^2}-\frac{p^3}{4\,.\,5\,.\,(1\,.\,2\,.\,3)^2}+\&\text{c}.$$

Hence

$$\frac{8}{3\pi}\left(\frac{d}{dp}\right)^2(Pp^2)=1-p+\frac{p^2}{(1\,.\,2)^2}-\frac{p^3}{(1\,.\,2\,.\,3)^2}+\&\text{c}.$$

In this change p to $\frac{1}{2}\tau^2$, & perform the operation $\int_0^\tau d\tau$ three times; and we shall have my

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau.)$$

To use now the function given at the middle of the previous page, it is necessary to have an expression for $\left(\int_0^x dx\right)^r\cos x$. This

$$=\left(\int_0^x dx\right)^r\Sigma_{(k)0}^{\infty}\frac{(-x^2)^k}{[2k]^{2k}}$$

$$=\Sigma_{(k)0}^{\infty}\frac{(-1)^k}{[2k]^{2k}}\frac{x^{2k+r}}{[2k+r]^r}=\Sigma_{(k)0}^{\infty}\frac{(-1)^k\,x^{2k+r}}{[2k+r]^{2k+r}}.$$

Hence

$$\left(\int_0^x dx\right)^{2r}\cos x=(-1)^r\left(\cos x-1+\frac{x^2}{2}-\frac{x^4}{2\,.\,3\,.\,4}+\cdots-\frac{(-x^2)^{r-1}}{[2r-2]^{2r-2}}\right),$$

and

$$\left(\int_0^x dx\right)^{2r+1}\cos x=(-1)^r\left(\sin x-x+\frac{x^3}{2\,.\,3}-\frac{x^5}{2\,.\,3\,.\,4\,.\,5}+\cdots-\frac{(-x^2)^{r-1}x}{[2r-1]^{2r-1}}\right).$$

Hence

$$\left(\int_0^\tau d\tau\right)\cos n\tau=\frac{1}{n}\sin n\tau;\quad \left(\int_0^\tau d\tau\right)^2\cos n\tau=-\frac{1}{n^2}(\cos n\tau-1);$$

$$\left(\int_0^\tau d\tau\right)^3\cos n\tau=-\frac{1}{n^3}(\sin n\tau-n\tau);\ \&\text{c}.$$

Hence

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^i=\frac{1}{\pi}[i]^i\left(\int_0^\tau d\tau\right)^i\int_0^{\frac{\pi}{2}}d\theta\sec\theta^2\operatorname{vers}(\tau\sqrt{2}\cos\theta).$$

This relation gives

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau=\frac{1}{\pi\sqrt{2}}\int_0^{\frac{\pi}{2}}d\theta\sec\theta^3\{\tau\sqrt{2}\cos\theta-\sin(\tau\sqrt{2}\cos\theta)\};$$

which ought to turn out to be nearly $=\frac{\tau^2}{2\sqrt{2}}$, when τ is large.

Hence we ought to find

$$\lim_{\tau=\infty}\int_0^1 da\,(1-a^2)^{-\frac{1}{2}}\frac{\tau\sqrt{2}\,a-\sin(\tau\sqrt{2}\,a)}{\tau^2a^3}=\frac{\pi}{2};$$

that is,

$$\frac{\pi}{4}=\lim_{\tau=\infty}\int_0^{\tau\sqrt{2}}dx\left(1-\frac{x^2}{2\tau^2}\right)^{-\frac{1}{2}}\frac{x-\sin x}{x^3}=\int_0^\infty\frac{x-\sin x}{x^3}\,dx.$$

This will require $\left(\text{because we found } \int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}\right)$ that we should find

$$\int_0 \frac{\operatorname{vers} x}{x^2}\,dx = \frac{\pi}{2}.$$

And reciprocally, if we can establish this last proposition, we shall be able to infer the former.

$$\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \lim_{y=\infty}\int_0^\infty e^{-\frac{x}{y^3}}\frac{x-\sin x}{x^3}\,dx$$

$$= \lim_{y=\infty} y^{-2}\int_0^\infty e^{-z^2}\frac{yz-\sin yz}{z^3}\,dz = \lim_{y=\infty}\frac{1}{2y}\int_0^\infty e^{-z^2}z^{-2}\operatorname{vers}(yz)\,dz$$

$$= \tfrac{1}{2}\lim_{y=\infty}\int_0^\infty e^{-z^2}z^{-1}\sin(yz)\,dz = \lim_{y=\infty}\tfrac{1}{2}\int_0^\infty e^{-\frac{x^2}{y^2}}x^{-1}\sin x\,dx$$

$$= \tfrac{1}{2}\int_0 \frac{\sin x}{x}\,dx = \frac{\pi}{4}.$$

Hence, going back, we ought to have

$$\frac{\pi}{4} = \lim_{\tau=\infty}\int_0^{\tau\sqrt{2}} dx\left(1-\frac{x^2}{2\tau^2}\right)^{-\frac{1}{2}}\frac{x-\sin x}{x^3}$$

$$= \lim_{\tau=\infty}\int_0^1 da\,(1-a^2)^{-\frac{1}{2}}\frac{\tau\sqrt{2}a - \sin(\tau\sqrt{2}a)}{2\tau^2a^3}$$

$$= \lim_{\tau=\infty}\int_0^{\frac{\pi}{2}} d\theta\sec\theta^3\,\frac{\tau\sqrt{2}\cos\theta - \sin(\tau\sqrt{2}\cos\theta)}{2\tau^2}$$

$$= \frac{\pi}{\sqrt{2}}\times\lim_{\tau=\infty}\frac{1}{\tau^2}(\nabla_\tau^{-2}\quad 1)^{-\frac{1}{2}}\tau;$$

∵ this last limit $=\frac{1}{2\sqrt{2}}$, as before. Thus the difference $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau - \frac{\tau^2}{2\sqrt{2}}$ becomes an indefinitely small fraction of τ^2, when τ becomes indefinitely great. It is however conceivable that this difference may itself be constant or even may increase indefinitely. I think indeed that this will not be found to be the case: but it requires to be examined.

[*Expression for* $\eta_{0,\tau}$.]

(Jan. 18th, 1839.)

It seems probable that the part $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau$, of the expression on page 535 for $\eta_{0,\tau}$, bears ultimately the ratio of equality to the other part $\nabla_\tau\alpha_\tau$ of the same expression, when τ increases without limit; & ∵ that

$$\frac{1}{m^2}\frac{\operatorname{vers} m\tau}{\sqrt{2-m^2}} = (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{\sin m\tau}{m}\ \text{(ultimately)};$$

$$\sqrt{\tfrac{1}{2}}\,\Sigma_{(k_{\prime})0}^{\infty}[-\tfrac{1}{2}]^{k_{\prime}}[0]^{-k_{\prime}}\left(-\frac{m^2}{2}\right)^{k_{\prime}}.\Sigma_{(k_{\prime\prime})0}^{\infty}\frac{(-m^2\tau^2)^{k_{\prime\prime}}\tau^2}{[2k_{\prime\prime}+2]^{2k_{\prime\prime}+2}} = (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\,\Sigma_{(k)0}^{\infty}\frac{(-m^2\tau^2)^k\tau}{[2k+1]^{2k+1}}\ \text{(ultimately)};$$

$$\because\ \text{(ultimately)}\ \frac{(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^{2k+1}}{[2k+1]^{2k+1}} = \sqrt{\tfrac{1}{2}}\,\Sigma_{(k_{\prime})0}^{k}[-\tfrac{1}{2}]^{k_{\prime}}[0]^{-k_{\prime}}(\tfrac{1}{2})^{k_{\prime}}[0]^{-2(k-k_{\prime}+1)}\tau^{2(k-k_{\prime}+1)};$$

in which we know that the 1st member

$$= \frac{2}{\pi}\left(\int_0^\tau d\tau\right)^{2k+3}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta).$$

For example, making $k=0$, we have, as before,

$$\frac{2}{\pi}\left(\int_0^\tau d\tau\right)^3\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta)=\frac{\tau^2}{2\sqrt{2}},$$

τ tending to ∞; & if the same law hold for $k=1$, we shall have ultimately

$$\frac{2}{\pi}\left(\int_0^\tau d\tau\right)^5\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta)=\frac{\tau^4}{24\sqrt{2}}-\frac{\tau^2}{8\sqrt{2}}.$$

Changing τ to $t\sqrt{2}$, we ought ∵ to have, ultimately, as t tends to infinity,

$$\frac{2}{\pi}\left(\int_0^t dt\right)^3\int_0^{\frac{\pi}{2}} d\theta\cos(2t\cos\theta)=\frac{t^2}{4};\quad\left(-\frac{1}{16}\right)^*;$$

$$\frac{2}{\pi}\left(\int_0^t dt\right)^5\int_0^{\frac{\pi}{2}} d\theta\cos(2t\cos\theta)=\frac{t^4}{48}-\frac{t^2}{32}.\qquad -\frac{3}{256}\Big)^*.$$

(Jan. 19th, 1839.)

$$\frac{d}{d\,.\,m\tau}\Big)\;+1\Big\}\,(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau=\frac{1}{\pi}\int_0^{2} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta);\dagger$$

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)}{1-\frac{2}{\quad}\cos\theta^2}-\frac{\sqrt{2}}{m\pi}\sin m\tau\int^{\frac{\pi}{2}}\quad\frac{d\theta}{2\cos\theta^2}$$

$$=\frac{m}{\pi}\int_0^{\frac{\pi}{2}}\frac{m\sin(\tau\sqrt{2}\cos\theta)-\sqrt{2}\cos\theta\sin m\tau}{(m^2-2\cos\theta^2)\cos\theta}d\theta=\frac{m^2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\frac{m\sin n\tau-n\sin m\tau}{(m^2-n^2)\,mn}d\theta$$

$$=\frac{m^2\sqrt{2}}{\pi}\int_0^\tau\left(\int_0^{\frac{\pi}{2}}\frac{\cos n\tau-\cos m\tau}{m^2-n^2}d\theta\right)d\tau,\quad(\text{if } n=\sqrt{2}\cos\theta),$$

$$=\frac{m^2}{\pi}\int_0^\tau\left(\int_0^{\sqrt{2}}\frac{\cos n\tau-\cos m\tau}{m^2-n^2}\;\frac{dn}{\sqrt{1-\frac{n^2}{2}}}\right)d\tau;\quad\because\ \text{m}\qquad\frac{n}{m}=p,\ m\tau=v,$$

$$=\frac{1}{\pi}\int_0^{m\tau}\left(\int_0^{\frac{\sqrt{2}}{m}}\frac{\cos pv-\cos v}{1-p^2}\;\frac{dp}{\sqrt{1-\frac{1}{2}m^2p^2}}\right)dv.$$

---

* [The figures in round brackets are added in pencil in the manuscript (see page 551). Hamilton later on arrived at a complete form for the values of these integrals when t is large. He seemed to have known it about 1840 but after much numerical testing did not publish it until Nov. 1857, *Phil. Mag.* XIV, pp. 375–382. He gives the result as follows:

If $$f(t)=\frac{2}{\pi}\int_0 d\omega\cos(2t\cos\omega),$$

then $$\left(\int_0^t dt\right)^n f(t)-\left(-\int_t^\infty dt\right)^n f(t)=\Sigma_{(m)}\,2^{-2m-1}\,[-\tfrac{1}{2}]^m\,[0]^{-m}\,[0]^{-(n-2m-1)}\,t^{n-2m-1},$$

the summation to extend to all positive integral values of m such that $n-2m-1\geqslant 0$. The asymptotic expressions can be derived by integration of the asymptotic expression for $J_0(\tau\sqrt{2})$.]

† $$\left[\left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau=\left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\Sigma_{(k)\,0}^{\infty}\frac{(-m^2\tau^2)^k\,m\tau}{[2k+1]^{2k+1}}\right.$$

$$\left.=\frac{\sqrt{2}}{\pi}\left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}\Sigma_{(k)\,0}^{\infty}(-1)^k m^{2k+2}\left(\int_0^\tau d\tau\right)^{2k+3}\int_0^{\frac{\pi}{2}}d\theta\cos(\tau\sqrt{2}\cos\theta)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sec\theta\sin(\tau\sqrt{2}\cos\theta)\,d\theta.\right]$$

Hence, *rigorously*,

$$\eta_{0,\tau}=\tfrac{1}{2}\operatorname{vers} v+\frac{1}{\pi}\int_0^v\left(\int_1^{\frac{\sqrt{2}}{m}}\frac{\cos pv-\cos v}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,dp\right)dv,\quad \text{if } v=m\tau.$$

As m tends to 0, $v=m\tau$ remaining given, this expression tends to the limit

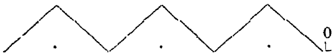
$$\eta_{0,\tau}=\tfrac{1}{2}\operatorname{vers} v+\frac{1}{\pi}\int_0^v\left(\int^{\infty}\frac{\cos pv-\cos v}{1-p^2}\,dp\right)dv;$$

I therefore think it *probable* that the integral $\int_0^v\left(\int_0^{\infty}\frac{\cos pv-\cos v}{1-p^2}\,dp\right)dv$ is *rigorously* $=\frac{\pi}{2}\operatorname{vers} v$.

If we take $m=$ its extreme value, namely $\sqrt{2}$, then the integral vanishes & the expression for $\eta_{0,\tau}$ reduces itself exactly to $\frac{1}{2}\operatorname{vers} v=\frac{1}{2}\operatorname{vers}(\tau\sqrt{2})$. In this case $\mu=\pi=\operatorname{vers}^{-1}m^2$; & the equations $(4)_2$ become

$$\eta_{x,0}=\operatorname{vers}\pi x,\quad \eta'_{x,0}=-\sqrt{2}\sin\pi x,$$

x being here any integer $\not> 0$; $\because$ $\eta'_{x,0}=0$; & $\eta_{x,0}=0$ or $=2$, according as $-x$ is even or odd. In this case we have initial displacements of the form



but no initial velocities; & the origin 0 receives only *half* the disturbance which it would do for the other extreme value of m, namely 0, corresponding to an infinitely long wave.* In fact, if we conceive a great but finite succession of initial displacements (without initial velocities) of the kind corresponding to the short wave lately mentioned, there is no reason why they should propagate themselves in one direction rather than in the opposite; they will therefore divide themselves between both.

(Jan. 26th, 1839.)

Is the integral $\int_0^{\infty}dp\,\frac{\sin pv-p\sin v}{p(1-p^2)}=\frac{\pi}{2}\operatorname{vers} v$?

In the 1st place we may consider it as = some finite quantity, and may treat this as a function of v; but when we add to this function its own 2nd differential coefficient we get

$$\left\{1+\left(\frac{d}{dv}\right)^2\right\}\int_0^{\infty}\frac{\sin pv-p\sin v}{p(1-p^2)}\,dp=\int_0^{\infty}\frac{\sin pv}{p}\,dp=-\frac{\pi}{2}\text{ or } 0 \text{ or } \frac{\pi}{2},$$

according as v is <0 or $=0$ or >0; therefore we have *rigorously*

$$\int_0^{\infty}\frac{\sin pv-p\sin v}{p(1-p^2)}\,dp=\pm\frac{\pi}{2}\operatorname{vers} v,$$

the upper or lower sign being taken according as v is $>$ or $<$ 0.

---

* [The wave-length is $2\pi/\mu$; $m\tau=v$, $m^2=\operatorname{vers}\mu$.]

*Hence the function** $\frac{1}{2}\operatorname{vers} v + \frac{1}{\pi}\int_0^\infty \frac{\sin pv - p\sin v}{p(1-p^2)}\,dp$ *is rigorously* $=0$ *when* v *is* <0 *and* $=\operatorname{vers} v$ *when* $v>0$. *And this particular case of discontinuity appears to me to be of great importance in the dynamics of light and darkness.*

The rigorous equation $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^i = [i]^i \left(\int_0^\tau d\tau\right)^i \gamma_\tau$, (page 537), in which

$$\gamma_\tau = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sec\theta^2 \operatorname{vers}(\tau\sqrt{2}\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\tau \int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)$$

$$= \int_0^\tau \gamma'_\tau\,d\tau, \text{ in which } \gamma'_\tau = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta),$$

combined with the general development

$$\beta_\tau = \beta_0 + \tau\beta'_0 + [0]^{-2}\tau^2\beta''_0 + \&\text{c}.$$

gives, generally,

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \beta_0\gamma_\tau + \beta'_0\int_0^\tau d\tau\,\gamma_\tau + \beta''_0\left(\int_0^\tau d\tau\right)^2\gamma_\tau + \&\text{c}.$$

$$= \underset{x=0}{\Sigma_{(h)\,0}^{\infty}}\left(\frac{d}{dx}\right)^k \beta_x \left(\int_0^\tau d\tau\right)^k \gamma_\tau = \underset{x=0}{\Sigma_{(k)\,0}^{\infty}}\left(\frac{d}{dx}\right)^k \beta_x\left(\int_0^\tau d\tau\right)^{k+1}\gamma'_\tau$$

$$= \frac{\int_0^\tau d\tau}{1-\underset{x=0}{\frac{d}{dx}}\int_0^\tau d\tau}\cdot\beta_x\gamma'_\tau;\quad \because\ \underset{x=0}{\left(\frac{d}{d\tau}-\frac{d}{dx}\right)}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \underset{x=0}{\beta_x\gamma'_\tau};$$

but, suppressing for the present the condition $x=0$, the partial differential equation of the 1st order

$$\left(\frac{d}{d\tau}-\frac{d}{dx}\right)\Phi_{x,\tau} = F_{x,\tau}$$

gives $$\Phi_{x,\tau} = \int_0^\tau F_{\tau_,-\tau,\tau}\,d\tau + \text{funct}\,(x+\tau),$$

$\tau_,$ being treated as a constant in effecting the 1st integration, but being afterwards made equal to $x+\tau$; if then $\Phi_{x,0}=0$ for all values of x, we have $\Phi_{x,\tau} = \int_0^\tau F_{\tau_,-\tau,\tau}\,d\tau$, & the same expression holds for the case $x=0$, except that then $\tau_,$ is to be made equal to τ after the integration; hence, generally,†

---

* [The results can be verified by taking the integrals

$$\int\frac{e^{ipv}-e^{iv}}{1-p^2}\,dv \quad\text{and}\quad \int\frac{e^{ipv}-pe^{iv}}{p(1-p^2)}\,dp,\quad (p>0)$$

over suitable contours.]

† [Changing the notation, we have to show that

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \int_0^\tau du\,\beta_{\tau-u}\gamma'_u,\quad \beta_{\tau-u} = \Sigma\frac{(\tau-u)^n}{n!}\beta_0^{(n)}.$$

Then making use of $\int_0^\tau(\tau-u)^n\gamma'_u\,du = n!\left(\int_0^\tau du\right)^{n+1}\gamma'_u$, we verify the required form. It may be noticed that $\frac{d^2\gamma_\tau}{d\tau^2} = 2J_0(\tau\sqrt{2})$.]

$$(\nabla_{\tau}^{-2}-1)^{-\frac{1}{2}}\beta_{\tau}=\underset{\tau_{,}=\tau}{\int_{0}^{\tau}}d\tau\,\beta_{\tau_{,}-\tau}\gamma'_{\tau}$$

$$=\frac{\sqrt{2}}{\pi}\underset{\tau_{,}=\tau}{\int_{0}^{\tau}}d\tau\int_{0}^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)\,\beta_{\tau_{,}}$$

$$=\frac{\sqrt{2}}{\pi}\underset{\tau_{,}=\tau}{\int_{0}^{\tau}}d\tau\int_{0}^{\sqrt{2}}\frac{dn}{n}\cdot\frac{\sin n\tau}{\sqrt{1-\frac{1}{2}n^{2}}}\beta_{\tau_{,}-\tau}.$$

[*The operator* I_{τ}.*]

Can we find any analogous expressions for $\eta_{1,\tau}$, $\eta_{2,\tau}$, &c., in so far as they depend on β_{τ}? For if so, we can easily get afterwards expressions for the parts which depend on α_{τ}.

Already we have found the law of the developments, according to ascending powers of ∇_{τ}; but it will be useful to investigate developments proceeding according to ascending powers of $\left(\int_{0}^{\tau}d\tau\right)^{2}$ instead. For this purpose it is convenient to employ, instead of the equations (11) of page 532, the following;

$$\eta_{0,\tau}=I_{\tau}^{2}(\beta_{\tau}+\eta_{1,\tau}-\eta_{0,\tau});\quad \eta_{1,\tau}=\tfrac{1}{2}I_{\tau}^{2}(\eta_{0,\tau}-2\eta_{1,\tau}+\eta_{2,\tau});\quad \&\text{c.};$$

in which $I_{\tau}=\int_{0}^{\tau}d\tau$. We may for the present omit the τ's & write simply

$$\eta_{0}=I^{2}(\beta+\eta_{1}-\eta_{0});\quad \eta_{1}=\tfrac{1}{2}I^{2}(\eta_{0}-2\eta_{1}+\eta_{2});\ \&\text{c.}$$

And it may be useful to investigate first from the equations

$$\eta_{1}=\tfrac{1}{2}I^{2}(\eta_{0}-2\eta_{1}+\eta_{2}),\quad \eta_{2}=\tfrac{1}{2}I^{2}(\eta_{1}-2\eta_{2}+\eta_{3}),\ \&\text{c.},$$

which are all similar to each other, developments of the forms

$$\eta_{n}=\Sigma_{(k)\,0}^{\infty}\,e_{n,k}I^{2(n+k)}\eta_{0};$$

or, more fully,

$$\eta_{1}=e_{1,0}I^{2}\eta_{0}+e_{1,1}I^{4}\eta_{0}+\&\text{c.},\quad \eta_{2}=e_{2,0}I^{4}\eta_{0}+\&\text{c.},\ \&\text{c.}$$

$$e_{1,0}=\tfrac{1}{2};\quad e_{1,1}+e_{1,0}=0;\quad e_{1,2}+e_{1,1}=\tfrac{1}{2}e_{2,0};\ \&\text{c.}$$

$$e_{2,0}=\tfrac{1}{2}e_{1,0};\quad e_{2,1}+e_{2,0}=\tfrac{1}{2}e_{1,1};\ \&\text{c.};\ \&\text{c.}$$

In general

$$e_{n,k}+e_{n,k-1}=\tfrac{1}{2}(e_{n-1,k}+e_{n+1,k-2});$$

negative indices being considered to make the resulting coefficient vanish, & the only value of $e_{0,k}$ which does not vanish being $e_{0,0}=1$.

Now make $e_{n,k}=f_{n+k,k}$; we shall have

$$f_{n+k,k}+f_{n+k-1,k-1}=\tfrac{1}{2}(f_{n+k-1,k}+f_{n+k-1,k-2}),$$

so that

$$f_{m+1,k}=\tfrac{1}{2}(f_{m,k}-2f_{m,k-1}+f_{m,k-2}).$$

---

* [This is Heaviside's operator. Cf. Jeffreys, *Operational Methods in Mathematical Physics* (1927); Poole, *Theory of Linear Differential Equations* (1936), Chap. II.]

Hence

$$\eta_n = \left.\frac{1+I^2-\sqrt{1+2I^2}}{I^2}\right)^n \eta_0 = \left(\frac{I^2}{1+I^2+\sqrt{1+2I^2}}\right. \eta_0;$$

and
$$\eta_0 = \frac{I^2\beta}{\sqrt{1+2I^2}}.$$

As a verification, these last expressions give

$$\frac{\eta_{n+1}-2\eta_n+\eta_{n-1}}{\eta_n} = \frac{1+I^2-\sqrt{1+2I^2}}{I^2} + \frac{1+I^2+\sqrt{1+2I^2}}{I^2} - 2 = \frac{2}{I^2};$$

$$\eta_n = \frac{I^2}{2}(\eta_{n+1}-2\eta_n+\eta_{n-1});$$

also
$$(1+I^2)\eta_0 - I^2\eta_1 = \sqrt{1+2I^2}\,\eta_0 = I^2\beta.$$

If we now make
$$\frac{I^2}{1+I^2} = \nabla,$$

$$1-\nabla = \frac{1}{1+I^2}, \quad \frac{1+I^2}{I^2} = \nabla^{-1}, \quad \frac{\sqrt{1+2I^2}}{I^2} = (\nabla^{-2}-1)^{\frac{1}{2}},$$

we shall have

$$\eta_0 = (\nabla^{-2}-1)^{-\frac{1}{2}}\beta, \quad \text{and} \quad \eta_n = \{\nabla^{-1}-(\nabla^{-2}-1)^{\frac{1}{2}}\}^n(\nabla^{-2}-1)^{-\frac{1}{2}}\beta.$$

If, by a further symbolic transformation, we were to make $\nabla = \sin 2\delta$, *we should have the expression*

$$\eta_n = (\tan\delta)^n \,.\, \tan 2\delta \,.\, \beta.$$

[*tan* $\delta_\tau . \phi_\tau$ *as an integral.*]

We have found that the operation here called tan 2δ, or more fully tan $2\delta_\tau$, is such that

$$\tan 2\delta_\tau \,.\, \beta_\tau = \frac{\sqrt{2}}{\pi}\int_0^\tau d\tau \int_0^{\frac{\pi}{2}} d\theta \sec\theta \sin(\tau\sqrt{2}\cos\theta)\,\beta_{\tau,-\tau}.$$
$$\tau,=\tau$$

Can we find any analogous expression for the effect of the operation tan δ_τ^n, or even tan δ_τ?

$$\tan\delta_\tau = \nabla_\tau^{-1} - (\nabla_\tau^{-2}-1)^{\frac{1}{2}} = \nabla_\tau^{-1}\{1-\sqrt{1-\nabla_\tau^2}\}$$
$$= \frac{1+I_\tau^2-\sqrt{1+2I_\tau^2}}{I_\tau^2} = -\Sigma_{(k)2}^{\infty}[\tfrac{1}{2}]^k[0]^{-k}2^k I_\tau^{2k-2}$$
$$= \Sigma_{(k)0}^{\infty}[-\tfrac{3}{2}]^k[0]^{-(k+2)}2^k I_\tau^{2k+2} = \Sigma_{(k)0}^{\infty}(-1)^k\frac{1.3.5.7\ldots(2k+1)}{2.3.4.5\ldots(k+2)}\left(\int_0^\tau d\tau\right)^{2k+2}.$$

Let this operation be performed on τ^i, and because

$$\left(\int_0^\tau d\tau\right)^{2k+2}\tau^i = \frac{\tau^{2k+2+i}}{[2k+2+i]^{2k+2}},$$

we have

$$\tan\delta_\tau\,.\,\tau^i=\left(\int_0^\tau d\tau\right)^i\Sigma_{(k)\,0}^{\infty}(-1)^k\frac{1\,.\,3\ldots(2k+1)}{2\,.\,3\ldots(k+2)}\,\frac{\tau^{2k+2}\,[i]^i}{[2k+2]^{2k+2}}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\Sigma_{(k)\,0}^{\infty}(-1)^k\,2^{-(k+1)}\,[0]^{-(k+1)}\,[0]^{-(k+2)}\,\tau^{2k+2}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\frac{1}{\tau^2}\int_0^{\tau^2}\left\{1-\Sigma_{(k)\,0}^{\infty}\left(-\frac{t}{2}\right)^k([0]^{-k})^2\right\}dt$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\left\{1-\frac{1}{\tau^2}\int_0^{\tau^2}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos\left(\sqrt{2t}\cos\theta\right)d\theta\right)dt\right\}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\left\{1-\frac{\sqrt{2}}{\tau}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin\left(\tau\sqrt{2}\cos\theta\right)+\frac{1}{\tau^2}\,\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta^2\,\mathrm{vers}\left(\tau\sqrt{2}\cos\theta\right)\right\}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^{i+1}\frac{4}{\pi\tau^3}\int_0^{\frac{\pi}{2}}d\theta\left\{\tau\sqrt{2}\cos\theta\sin\left(\tau\sqrt{2}\cos\theta\right)\right.$$
$$\left.-\tau^2\cos\left(\tau\sqrt{2}\cos\theta\right)\cos\theta^2-\mathrm{vers}\left(\tau\sqrt{2}\cos\theta\right)\right\}\sec\theta^2$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^{i+1}\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^{\frac{\pi}{2}}d\theta\,\tau^2\sin\left(\tau\sqrt{2}\cos\theta\right)\cos\theta.$$

Hence

$$\tan\delta_\tau\,.\,\tau^i=\int_0^\tau d\tau\,(\tau_,-\tau)^i\,\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^{\frac{\pi}{2}}d\theta\,\tau^2\cos\theta\sin\left(\tau\sqrt{2}\cos\theta\right),$$

and generally

$$\tan\delta_\tau\,.\,\phi_\tau=\underset{\tau_,=\tau}{\int_0^\tau}d\tau\,\phi_{\tau_,-\tau}\,\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^2 d\theta\,\tau^2\cos\theta\sin\left(\tau\sqrt{2}\cos\theta\right)$$

$$=\frac{4}{\pi}\underset{\tau=\tau_,}{\int_0^\tau}d\tau\,\phi_{\tau_,-\tau}\int_0^2 d\theta\left\{\tau\sqrt{2}\cos\theta\sin\left(\tau\sqrt{2}\cos\theta\right)\right.$$
$$\left.-\tau^2\cos\theta^2\cos\left(\tau\sqrt{2}\cos\theta\right)-\mathrm{vers}\left(\tau\sqrt{2}\cos\theta\right)\right\}\frac{}{\tau^3}$$

$$=\frac{4\sqrt{2}}{\pi}\underset{\tau_,=\tau}{\int_0^\tau}d\tau\,\phi_{\tau_,-\tau}\,\tau^{-3}\int_0^{\sqrt{2}}dn\left\{n\tau\sin n\tau-1+\left(1-\tfrac{1}{2}n^2\tau^2\right)\cos n\tau\right\}n^{-2}\left(1-\tfrac{1}{2}n^2\right)^{-\frac{1}{2}}.$$

[$\eta_{x,t}$ *expressed by means of operator* D_t.]

(Jan. 22nd.)

It will be interesting to consider the simplest forms to which the repetition of this operation, & its combination with the operation $\tan 2\delta_\tau$, conduct; but at present I prefer to resume the problem in another way, as follows.

We have the system of equations, $\left(\text{in which } D = D_\tau = \dfrac{d}{d\tau}\right)$,

$$\eta_1 = -\beta + (1+D^2)\,\eta_0; \quad \eta_2 = 2\,(1+D^2)\,\eta_1 - \eta_0; \quad \eta_3 = 2\,(1+D^2)\,\eta_2 - \eta_1; \quad \&\text{c.};$$

so that the series

$$\eta_0 + \eta_1 x + \eta_2 x^2 + \ldots + \eta_n x^n + \ldots$$

being multiplied by $1 - 2\,(1+D^2)\,x + x^2$ must reduce itself to the 2 terms

$$\eta_0 + \{\eta_1 - 2\,(1+D^2)\,\eta_0\}\,x = \eta_0 - \{\beta + (1+D^2)\,\eta_0\}\,x;$$

if then we develope the fraction

$$\frac{\eta_0 - \{\beta + (1+D^2)\,\eta_0\}\,x}{(1-x)^2 - 2D^2 x}$$

according to ascending powers of x, the coefficient of x^n in this development will be the function η_n.

The coefficient of x^n in the development of $\dfrac{1}{(1-x)^2 - 2D^2x}$ is

$$\begin{aligned} &\text{coeff}^{\text{t}} \text{ of } x^n \text{ in } (1-x)^{-2} + 2D^2 \times \text{coeff}^{\text{t}} \text{ of } x^{n-1} \text{ in } (1-x)^{-4} \\ &\quad + (2D^2)^2 \times \text{coeff}^{\text{t}} \text{ of } x^{n-2} \text{ in } (1-x)^{-6} + \&\text{c.} \\ &= (n+1) + 2D^2\frac{[n+2]^3}{[3]^3} + (2D^2)^2\frac{[n+3]^5}{[5]^5} + \&\text{c.} \end{aligned}$$

Hence

$$\eta_n - \eta_0 = \frac{n}{2}\left(\frac{[n]^1}{[1]^1}\frac{2D^2}{1} + \frac{[n+1]^3}{[3]^3}\frac{(2D^2)^2}{2} + \frac{[n+2]^5}{[5]^5}\frac{(2D^2)^3}{3} + \ldots\right)\eta_0$$
$$- \left(\frac{[n]^1}{[1]^1} + \frac{[n+1]^3}{[3]^3}\,2D^2 + \frac{[n+2]^5}{[5]^5}\,(2D^2)^2 + \ldots\right)\beta.$$

Now, in so far as η_0 depends on β alone, we have

$$\eta_0 = (\nabla^{-2} - 1)^{-\frac{1}{2}}\,\beta = \frac{I^2\beta}{\sqrt{1+2I^2}} = \tan 2\delta\,.\,\beta = \int_{\substack{0\\ \tau_,=\tau}}^{\tau} d\tau\,\beta_{\tau_,-\tau}\gamma'_{\tau},$$

in which
$$\gamma'_\tau = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sec\theta \sin(\tau\sqrt{2}\cos\theta).$$

Hence
$$D\eta_0 = \beta_0\gamma'_\tau + \int_{\substack{0\\ \tau_,=\tau}}^{\tau} d\tau\,\beta'_{\tau_,-\tau}\gamma'_\tau; \quad D^2\eta_0 = \beta_0\gamma''_\tau + \beta'_0\gamma'_\tau + \int_{\substack{0\\ \tau_,=\tau}}^{\tau} d\tau\,\beta''_{\tau_,-\tau}\gamma'_\tau; \quad \&\text{c.}$$

Let us suppose, which we shall find to be permitted in the present investigation, that β_τ is a linear function of τ, & ∵ that β''_τ, β'''_τ, &c. vanish; ∵ also $D^2\beta$, $D^4\beta$, &c. vanish. [We finally obtain]

$$\eta_{n,\tau} = (\beta_0 D_\tau + \beta'_0)\left\{\gamma^{\backprime}_\tau + \frac{n}{2}\left(\frac{[n]^1}{[1]^1}\,2D_\tau^2 + \frac{[n+1]^3}{[3]^3}\frac{(2D_\tau^2)^2}{2} + \frac{[n+2]^5}{[5]^5}\frac{(2D_\tau^2)^3}{3} + \&\text{c.}\right)\gamma^{\backprime}_\tau - n\tau\right\},$$

in which
$$\gamma^{\backprime}_\tau = \int_0^\tau \gamma_\tau d\tau = \frac{1}{\pi\sqrt{2}}\int_0^{\frac{\pi}{2}} d\theta \sec\theta^3\{\tau\sqrt{2}\cos\theta - \sin(\tau\sqrt{2}\cos\theta)\}.$$

The properties asserted of this set of functions $\eta_{n,\tau}$ are: 1st, that they vanish, and that their 1st differential coefficients relatively to τ vanish, when $\tau=0$; & 2nd, that they satisfy the system of differential equations

$$(1+D_\tau^2)\,\eta_{0,\tau}=\eta_{1,\tau}+\beta_0+\beta_0'\tau, \quad \& \quad (1+D_\tau^2)\,\eta_{n,\tau}=\tfrac{1}{2}(\eta_{n+1,\tau}+\eta_{n-1,\tau}), \quad \text{if } n>0.$$

Reciprocally, we shall have proved the justice of the expression for $\eta_{n,\tau}$ if we prove that it possesses these properties. It is therefore important to make this verification. [The verification follows and incidentally involves the identity

$$\Sigma_{(k)\,0}^{n-1}\frac{(-1)^k}{k+1}([0]^{-k})^2\,\frac{[n+k]^{2k+1}}{2k+1}=1.]^*$$

Suppose now that, when $n>0$, $\eta_{n,\tau}=\eta_{-n,\tau}=$ transversal vibration or displacement of the nth particle before or behind the particle corresponding to $n=0$; but that this last mentioned particle has its displacement represented by $\eta_{0,\tau}-\beta_0-\tau\beta_0'$; then to the expressions lately given we are to add those which would correspond to the introduction of the terms $-\frac{1}{2}\beta_0-\frac{1}{2}\tau\beta_0'$ in the expressions for $D_\tau^2\eta_{1,\tau}$ and $D_\tau^2\eta_{-1,\tau}$; & this again comes to changing, in the integrals, $\eta_{0,\tau}$ to $\eta_{0,\tau}-\eta_{1,\tau}$; $\eta_{1,\tau}$ to $\eta_{1,\tau}-\frac{1}{2}(\eta_{0,\tau}+\eta_{2,\tau})$, &c.; or finally to replacing the displacements at the time τ by the negatives of the 2nd differential coefficients, relatively to τ,† of the expressions lately given: so that these displacements are

$$y_{0,\tau}=y_{0,0}\gamma_\tau''+y_{0,0}'\gamma_\tau'; \ \&\text{c.};$$

or, more generally,

$$y_{x,\tau}=(y_{0,0}D_\tau+y_{0,0}')\left\{1+\frac{x}{2}\left(\frac{[x]^1}{[1]^1}2D_\tau^2+\frac{[x+1]^3}{[3]^3}\frac{(2D_\tau^2)^2}{2}+\frac{[x+2]^5}{[5]^5}\frac{(2D_\tau^2)^3}{3}+\dots\right)\right\}\gamma_\tau',$$

in which, as before,

$$\gamma_\tau'=\frac{\sqrt{2}}{\pi}\int_0^2 d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta),$$

and $y_{0,0}$, $y_{0,0}'$ are the initial displacements & velocities of the particle 0.

---

[*] [We have

$$\frac{(-1)^k([0]^{-k})^2}{k+1}\,\frac{[n+k]^{2k+1}}{2k+1}=\frac{(-1)^k[2k]^{2k}}{[k]^k[k+1]^{k+1}}\,\frac{[n+k]^{2k+1}}{[2k+1]^{2k+1}}=\frac{[\frac{1}{2}]^{k+1}\,2^{2k+1}}{[k+1]^{k+1}}\,\frac{[n+k]^{2k+1}}{[2k+1]^{2k+1}}.$$

Hence

$$\Sigma_{(k)\,0}^{n-1}\frac{(-1)^k([0]^{-k})^2}{k+1}\,\frac{[n+k]^{2k+1}}{2k+1}=\text{coefficient of } x \text{ in } \Sigma_{(k)\,0}^{n-1}\frac{[\frac{1}{2}]^{k+1}\,2^{2k+1}}{[k+1]^{k+1}}\,\frac{(1+x)^{n+k}}{x^{2k}}$$

$$= \quad ,, \quad ,, \ ,, \ ,, \ \tfrac{1}{2}x^2(1+x)^{n-1}\,\Sigma_{(k)\,0}^{\infty}\frac{[\frac{1}{2}]^{k+1}}{[k+1]^{k+1}}\left\{\frac{4(1+x)}{x^2}\right\}$$

$$,, \quad ,, \ ,, \ ,, \ \tfrac{1}{2}x^2(1+x)^{n-1}\left\{1+\frac{4(1+x)}{x^2}\right\}^{\frac{1}{2}}$$

$$,, \ ,, \ \tfrac{1}{2}x(1+x)^{n-1}(x+2)$$

$$=1.]$$

† [$y_{n,\tau}=-\dfrac{d^2\eta_{n,\tau}}{d\tau^2}$. n is also replaced by x. The transformation gives the solution of a new problem as described here.]

For example, the displacement of this very particle at the time τ is, *rigorously*,

$$y_{0,\tau}=y_{0,0}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\cos(\tau\sqrt{2}\cos\theta)+y'_{0,0}\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta).$$

As τ increases without limit this displacement tends to become $=\sqrt{\frac{1}{2}}\,y'_{0,0}$;* it bears therefore a constant ratio $\sqrt{\frac{1}{2}}$ to the initial velocity $y'_{0,0}$. This is a very curious result. The ratio $\sqrt{\frac{1}{2}}$ depends no doubt upon the assumed unit of time, & on the energy of attraction between 2 adjacent particles.

The expression for $y_{x,\tau}$ ought to satisfy *generally* the equation in mixed differences

$$y''_{x,\tau}=\tfrac{1}{2}(y_{x+1,\tau}-2y_{x,\tau}+y_{x-1,\tau}).$$

It ought also to give $y_{x,0}=0$ and $y'_{x,0}=0$, when $x>$ or <0; but $=y_{0,0}$ and $y'_{0,0}$ respectively, when $x=0$. And so it does.

(Jan. 23rd, 1839.)

The last expression for $y_{x,\tau}$ may be thus written:

$$y_{x,\tau}=(y_{0,0}D_\tau+y'_{0,0})\left\{1+\frac{1}{2}\frac{x^2}{1}2D_\tau^2+\frac{1}{2}\frac{x^2(x^2-1)}{1\,.\,2\,.\,3}\frac{(2D_\tau)^2}{2}\right.$$
$$\left.+\frac{1}{2}\frac{x^2(x^2-1)(x^2-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\frac{(2D_\tau^2)^3}{}+\dots\right\}\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta);$$

and from it we may deduce the more general one:

$$y_{x,\tau}=\frac{1}{\pi\sqrt{2}}\Sigma_{(i)-\infty}^{\infty}(y_{i,0}D_\tau+y'_{i,0})\left\{2+\frac{(x-i)^2}{1}\frac{(2D_\tau^2)}{1}+\frac{(x-i)^2\{(x-i)^2-1\}}{1\,.\,2\,.\,3}\frac{(2D_\tau^2)^2}{2}\right.$$
$$\left.+\frac{(x-i)^2\{(x-i)^2-1\}\{(x-i)^2-4\}}{1\,.\,2\,.\,3\,.\,4\,.\,5}(2D_\tau^2)^3+\dots\right\}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta);$$

in which $D_\tau=\frac{d}{d\tau}$, and the function $y_{x,\tau}$ is obliged to satisfy generally the differential equation

$$D_\tau^2 y_{x,\tau}+y_{x,\tau}=\tfrac{1}{2}(y_{x+1,\tau}+y_{x-1,\tau}),$$

and also to become $y_{i,0}$ when $\tau=0$, & to give $D_\tau y_{x,\tau}=y'_{i,0}$ when $\tau=0$.

Restoring ∵ $ta\sqrt{2}$ for τ, and ∵ putting $\frac{1}{a\sqrt{2}}D_t$ for D_τ; changing also $y_{i,0}$ to $\eta_{i,0}$ and $y'_{i,0}$ to $\frac{1}{a\sqrt{2}}\eta'_{i,0}$, we get the expression

$$\eta_{x,t}=\frac{1}{2a\pi}\Sigma_{(i)-\infty}^{\infty}(\eta_{i,0}D_t+\eta'_{i,0})\left\{2+\frac{(x-i)^2}{1}\frac{D_t^2}{1a^2}+\frac{(x-i)^2\{(x-i)^2-1\}}{1\,.\,2\,.\,3}\frac{D_t^4}{2a^4}\right.$$
$$\left.+\frac{(x-i)^2\{(x-i)^2-1\}\{(x-i)^2-4\}}{1\,.\,2\,.\,3\,.\,4\,.\,5}\frac{D_t^6}{3a^6}+\dots\right\}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(2at\cos\theta),$$

[The first integral is $J_0(\tau\sqrt{2})$. The asymptotic form for the second can be obtained by considering the integral $\int e^{izt}(1-t^2)^{-\frac{1}{2}}t^{-1}\,dt$ over a suitable contour.]

as the complete integral of the equation (1) of page 527,

$$\eta''_{x,t}=a^2(\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t}).$$

The expression may be written more concisely thus:*

$$\eta_{x,t}=\frac{1}{2a\pi}\Sigma_{(i)-\infty}^{\infty}(\eta_{i,0}D_t+\eta'_{i,0})\left\{2+(x-i)\Sigma_{(k)\,0}^{\infty}\frac{[x-i+k]^{2k+1}}{[2k+1]^{2k+1}}\frac{D_{\tau}^{2k+2}}{(k+1)\,a^{2k+2}}\right\}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin(2at\cos\theta)}{\cos\theta}.$$

In it $\eta_{i,0}$ and $\eta'_{i,0}$ may be considered as the 2 arbitrary functions, and if we denote them by F_i and f_i we may write

$$\eta_{x,t}=\frac{1}{a\pi}\Sigma_{(i)-\infty}^{\infty}(F_iD_t+f_i)\left\{1+\frac{x-i}{2}\Sigma_{(k)\,0}^{\infty}\frac{[x-i+k]^{2k+1}}{[2k+1]^{2k+1}}\frac{D_{\tau}^{2k+2}}{(k+1)\,a^{2k+2}}\right\}\int_0^{\frac{\pi}{2}}\frac{\sin(2at\cos\theta)\,d\theta}{\cos\theta}$$

[$\eta_{x,t}$ *expressed as an integral.*]

(Jan. 25th, 1839.)

Employing the very simple expression†

$$\eta_{x,t}=F_x+\frac{a^2t^2}{1.2}\Delta_x^2F_{x-1}+\frac{a^4t^4}{1.2.3.4}\Delta_x^4F_{x-2}+\dots$$
$$+tf_x+\frac{a^2t^3}{1.2.3}\Delta_x^2f_{x-1}+\frac{a^4t^5}{1.2.3.4.5}\Delta_x^4f_{x-2}+\dots$$

for the integral of the equation in mixed differences

$$D_t^2\eta_{x,t}=a^2\Delta_x^2\eta_{x-1,t},$$

F_x and f_x being the initial values of $\eta_{x,t}$ and of $D_t\eta_{x,t}$; or still more simply employing the expression

$$\eta_{x,t}=F_x+\frac{t^2}{1.2}\Delta_x^2F_{x-1}+\frac{t^4}{1.2.3.4}\Delta_x^4F_{x-2}+\dots$$
$$+tf_x+\frac{t^3}{1.2.3}\Delta_x^2f_{x-1}+\frac{t^5}{1.2.3.4.5}\Delta_x^4f_{x-2}+\dots$$

as the integral of the equation $D_t^2\eta_{x,t}=\Delta_x^2\eta_{x-1,t}$; let us suppose that f_x is $=0$ for all values of x, & that F_x also vanishes except for negative and odd values, for which we shall suppose it to be constantly $=1$; and let us seek the corresponding form of the function $\eta_{0,t}$, with a view to show that it reduces itself to the form $\frac{1}{2}(\sin t)^2$.

We are ∵ to have, for all values of t,

$$\tfrac{1}{2}(\sin t)^2=F_0+\frac{t^2}{1.2}\Delta_0^2F_{0-1}+\frac{t^4}{1.2.3.4}\Delta_0^4F_{0-2}+\text{\&c.},$$

---

* [Hamilton verifies this expression and in the verification the following results appear

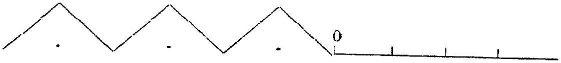
$$\tfrac{1}{2}(\sin at)^2=\Sigma_{(i)\,0}^{\infty}\left\{1+\frac{2i+1}{2}\Sigma_{(k)\,0}^{2i}\frac{[2i+k+1]^{2k+1}}{[2k+1]^{2k+1}}\frac{D^{2k+2}}{(k+1)\,a^{2k+2}}\right\}\Sigma_{(r)\,0}^{\infty}(-a^2t^2)^r([0]^{-r})^2;$$

and

$$0=\Sigma_{(i)\,0}^{\infty}\left\{1-\frac{(2i+1)^2}{1^2}+\frac{(2i+1)^2\{(2i+1)^2-1^2\}}{1^2.2^2}-\dots\right\}.]$$

† [This expression was arrived at in the course of the verification referred to in the previous note.]

in which
$$\Delta_0^2 F_{0-1} = F_{-1} - 2F_0 + F_1,$$
$$\Delta_0^4 F_{0-2} = F_{-2} - 4F_{-1} + 6F_0 - 4F_1 + F_2, \ \&\text{c.},$$
& $F_0 = 0,\ F_1 = 0,\ F_2 = 0,\ \ldots;$ also $F_{-2} = 0,\ F_{-4} = 0,\ \ldots;$ but $F_{-1} = F_{-3} = \&\text{c.} = 1$.
So that we ought to have the equations (because $\frac{1}{2}\sin t^2 = \frac{1}{4}(1 - \cos 2t)$),
$$+1 = -\Delta_0^2 F_{0-1} = F_{-1}, \quad -4 = \Delta_0^4 F_{0-2} = -4F_{-1},$$
$$+4^2 = \Delta_0^6 F_{0-3} = F_{-3} + \frac{6\,.\,5}{1\,.\,2} F_{-1},$$
$$-4^3 = \Delta_0^8 F_{0-4} = -8F_{-3} - \frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3} F_{-1}, \ \&\text{c.};$$
which are true. In general, we ought to have
$$4^{2l} = \Delta_0^{4l+2} F_{0-2l-1} = 1 + \frac{[4l+2]^2}{[2]^2} + \frac{[4l+2]^4}{[4]^4} + \ldots,$$
$$4^{2l+1} = (-1)^l \Delta_0^{4l+4} F_{0-2l-2} = \frac{[4l+4]^1}{[1]^1} + \frac{[4l+4]^3}{[3]^3} + \ldots,$$
& accordingly these series are the quarters of the developments of $(1+1)^{4l+2}$, $(1+1)^{4l+4}$.



(Jan. 26th, 1839.)

Had we supposed that $F_x = 1$ for *all* odd values of x, (positive as well as negative), being still $= 0$ for all even values of x, & f_x being still $= 0$ for *all* values of that variable, we should have just doubled the last found expression for $\eta_{0,t}$, & thereby found $\eta_{0,t} = \sin t^2$.

I think that we ought to have found $\eta_{0,t} = \cos t^2$, if we had supposed $f_x = 0$ for all values of x, & $F_x = 0$ for all odd but $= 1$ for all even values, the value 0 included.

In general, one integral of the equation in mixed differences is
$$\eta_{x,t} = \eta \operatorname{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right),$$
η & μ being arbitrary constants. In fact this expression gives
$$\Delta_x^2 \eta_{x-1,t} = \eta\left\{2\cos\left(\mu x - 2t\sin\frac{\mu}{2}\right) - \cos\left(\mu x + \mu - 2t\sin\frac{\mu}{2}\right) - \cos\left(\mu x - \mu - 2t\sin\frac{\mu}{2}\right)\right\}$$
$$= 2\eta\cos\left(\mu x - 2t\sin\frac{\mu}{2}\right)\operatorname{vers}\mu = D_t^2 \eta_{x,t}.$$

We ought ∴ to have $\eta_{x,t} =$ the above cited expression for *all* values of t, if it be so for two values infinitely near each other; ∴ if, for all values of x, we have
$$F_x = \eta \operatorname{vers}\mu x, \quad f_x = -2\eta \sin\mu x \sin\frac{\mu}{2}.$$
These conditions give
$$F_{x+i} = \eta - \eta\cos\mu x\cos\mu i + \eta\sin\mu x\sin\mu i,$$
$$f_{x+i} = -2\eta\sin\frac{\mu}{2}\sin\mu x\cos\mu i - 2\eta\sin\frac{\mu}{2}\cos\mu x\sin\mu i;$$

$$\because \quad \Delta_x^2 F_{x-1} = -\eta \cos \mu x \{\cos \mu - 2\cos 0 + \cos(-\mu)\} + \eta \sin \mu x \{\sin \mu - 2 \sin 0 + \sin(-\mu)\}$$

$$= 4\eta \cos \mu x \left(\sin \frac{\mu}{2}\right)^2,$$

$$\Delta_x^4 F_{x-2} = -\eta \cos \mu x \{\cos 2\mu - 4\cos \mu + 6 - 4\cos(-\mu) + \cos(-2\mu)\}$$

$$= -\eta \cos \mu x \left(e^{2\mu\sqrt{-1}} - 4e^{\mu\sqrt{-1}} + 6 - 4e^{-\mu\sqrt{-1}} + e^{-2\mu\sqrt{-1}}\right)$$

$$= -\eta \cos \mu x \left(e^{\frac{\mu}{2}\sqrt{-1}} - e^{-\frac{\mu}{2}\sqrt{-1}}\right)^4 = -2^4 \eta \cos \mu x \left(\sin \frac{\mu}{2}\right.$$

$$\Delta_x^{2r} F_{x-r} = (-1)^{r+1}\, 2^{2r}\, \eta \cos \mu x \left(\sin \frac{\mu}{2}\right)^{2r}, \quad (r > 0),$$

$$\Delta_x^{2r} f_{x-r} = (-1)^{r+1}\, 2^{2r+1}\, \eta \sin \mu x \left(\sin \frac{\mu}{2}\right)^{2r+1};$$

$$\frac{t^{2r}}{1\,.\,2\,.\,3 \ldots (2r)} \Delta_x^{2r} F_{x-r} = -\eta \cos \mu x\, (-1)^r \left(2t \sin \frac{\mu}{2}\right)^{2r} [0]^{-2r}, \quad (r > 0),$$

$$\frac{t^{2r+1}}{1\,.\,2\,.\,3 \ldots (2r+1)} \Delta_x^{2r} f_{x-r} = -\eta \sin \mu x\, (-1)^r \left(2t \sin \frac{\mu}{2}\right)^{2r+1} [0]^{-(2r+1)}.$$

$\because$ summing these 2 last expressions from $r = 0$ to $r = \infty$, then adding them to each other & to the additional term η (corresponding to $\Delta_x^0 F_x$), we get

$$\eta_{x,t} = \eta \operatorname{vers}\left(\mu x - 2t \sin \frac{\mu}{2}\right),$$

as I expected.

Resuming now the original problem of this book, let the expressions on the preceding page for F_x and f_x hold only for negative values of x, & for the value $x = 0$; the functions F_x & f_x vanishing for $x = 0$ or > 0. And let me try to calculate the function $\eta_{0,t}$ by calculating the differences

$$\Delta_0^{2r} F_{0-r}, \quad \Delta_0^{2r} f_{0-r}. \quad \text{(We may suppose for simplicity } \eta = 1.)$$

$$\Delta_0^{2r} F_{0-r} = F_{-r} - \frac{[2r]^1}{[1]^1} F_{-r+1} + \frac{[2r]^2}{[2]^2} F_{-r+2} - \ldots$$

$$= 1 - \frac{[2r]^1}{[1]^1} + \frac{[2r]^2}{[2]^2} - \ldots + \frac{(-1)^r [2r]^r}{[r]^r}$$

$$-1 \cos r\mu + \frac{[2r]^1}{[1]^1} \cos(r\mu - \mu) - \frac{[2r]^2}{[2]^2} \cos(r\mu - 2\mu) + \ldots - \frac{(-1)^r}{2} \frac{[2r]^r}{[r]^r}$$

$$= -\tfrac{1}{2}\left(e^{\frac{\mu}{2}\sqrt{-1}} - e^{-\frac{\mu}{2}\sqrt{-1}}\right)^{2r} = (-1)^{r+1}\, 2^{2r-1} \left(\sin \frac{\mu}{2}\right)^{2r}, \quad r > 0;$$

& the part of $\eta_{0,t}$ corresponding to F (that is to the initial displacements) is simply

$$\tfrac{1}{2} \operatorname{vers}\left(2t \sin \frac{\mu}{2}\right).$$

The question then which remains to be solved is this: *How nearly will the initial velocities of the particles behind the origin produce the remaining half of this versed sine in the complete expression of the displacement $\eta_{0,t}$?*

$$\Delta_0^{2r} f_{0-r} = 2 \sin \frac{\mu}{2} \left(\sin r\mu - \frac{[2r]^1}{[1]^1} \sin(r\mu - \mu) + \ldots + \frac{(-1)^{r-1} [2r]^{r-1}}{[r-1]^{r-1}} \sin \mu\right).$$

If we seek only the limit to which $\frac{1}{\mu^2}\Delta_0^{2r} f_{0-r}$ tends, as μ tends to 0, we have, for this limit, the expression

$$r - \frac{[2r]^1}{[1]^1}(r-1) + \frac{[2r]^2}{[2]^2}(r-2) - \ldots + \frac{(-1)^{r-1}[2r]^{r-1}}{[r-1]^{r-1}};$$

of which the values are

$$1;\quad 2-4=-2;\quad 3-6.2+15=+6;$$

$$4-8.3+28.2-56=-20;\quad 5-10.4+45.3-120.2+210=+70;$$

& the general expression seems to be

$$4^{r-1}[-\tfrac{1}{2}]^{r-1}[0]^{-(r-1)} = (-1)^{r-1}[2r-2]^{r-1}[0]^{-(r-1)}.$$

[A proof by induction of this expression is given.]

We have ∵ at present, by neglecting μ^4,

$$\Delta_0^{2r} f_{0-r} = \mu^2(-1)^{r-1}[2r-2]^{r-1}[0]^{-(r-1)};$$

which is to be multiplied by $t^{2r+1}[0]^{-(2r+1)}$ and summed from $r=1$ to $r=\infty$; & thus we get, by changing r to $r+1$ & summing from 0 to ∞,

$$\mu^2\left(\int_0^t dt\right)^3 \Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2;$$

that is,

$$\lim_{\mu=0}\frac{1}{\mu^2}\eta_{0,t} = \frac{t^2}{4} + \left(\int_0^t dt\right)^3 \Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2,$$

a result which agrees with former investigations in this book, & in which the part added to $\frac{t^2}{4}$ tends to become $=\frac{t^2}{4}$ as t increases indefinitely; or rather, perhaps, tends to become $=\frac{t^2}{4}-\frac{1}{16}$.*

Without neglecting any power of μ

$$\Delta_0^{2r+2} f_{0-(r+1)} = 2\sin\frac{\mu}{2}\left\{\sin(r\mu+\mu) - \frac{[2r+2]^1}{[1]^1}\sin r\mu + \ldots\right\}$$

$$= 2\sin\frac{\mu}{2}\{\sin(r\mu+\mu) - ([2r]^1[0]^{-1}+2)\sin r\mu + ([2r]^2[0]^{-2} + 2[2r]^1[0]^{-1}+1)\sin(r\mu-\mu)$$

$$-([2r]^3[0]^{-3} + 2[2r]^2[0]^{-2} + [2r]^1[0]^{-1})\sin(r\mu-2\mu) + \ldots$$

$$+(-1)^{r-1}([2r]^{r-1}[0]^{-(r-1)} + 2[2r]^{r-2}[0]^{-(r-2)} + [2r]^{r-3}[0]^{-(r-3)})\sin 2\mu$$

$$+(-1)^r([2r]^r[0]^{-r} + 2[2r]^{r-1}[0]^{-(r-1)} + [2r]^{r-2}[0]^{-(r-2)})\sin\mu\}$$

$$= 2(-1)^r[2r]^r[0]^{-r}\sin\mu\sin\frac{\mu}{2} - \left(2\sin\frac{\mu}{2}\right)^3\{\sin r\mu - [2r]^1[0]^{-1}\sin(r\mu-\mu)$$

$$+[2r]^2[0]^{-2}\sin(r\mu-2\mu) - \ldots + (-1)^{r-1}[2r]^{r-1}[0]^{-(r-1)}\sin\mu\}$$

$$= -\left(2\sin\frac{\mu}{2}\right)^2\Delta_0^{2r} f_{0-r} + 2(-1)^r[2r]^r[0]^{-r}\sin\mu\sin\frac{\mu}{2}.$$

If then we consider $\Delta_0^{2r} f_{0-r}$ as a function of r, & denote it by $R_r\left(2\sin\frac{\mu}{2}\right)^{2r}$,

$$R_{r+1} + R_r = \cos\frac{\mu}{2}(-1)^r[2r]^r[0]^{-r}\left(2\sin\frac{\mu}{2}\right)^{-2r},$$

---

* [See p. 539.]

which is an ordinary equation in differences of the 1st order and 1st degree, with constant coefficients, but with a variable term. It gives

$$-(R_r+R_{r-1})=\cos\frac{\mu}{2}(-1)^r[2r-2]^{r-1}[0]^{-(r-1)}\left(2\sin\frac{\mu}{2}\right)^{-2(r-1)},$$

& also $R_0=0$,

$$\because\quad \Delta_0^{2r+2}f_{0-(r+1)}=\left(2\sin\frac{\mu}{2}\right)^2\cos\frac{\mu}{2}(-1)^r\left\{[2r]^r[0]^{-r}+[2r-2]^{r-1}[0]^{-(r-1)}\left(2\sin\frac{\mu}{2}\right)^2\right.$$
$$\left.+[2r-4]^{r-2}[0]^{-(r-2)}\left(2\sin\frac{\mu}{2}\right)^4+\ldots+[2]^1[0]^{-1}\left(2\sin\frac{\mu}{2}\right)^{2r-2}+\left(2\sin\frac{\mu}{2}\right)^{2r}\right\}.$$

Multiplying this by $t^{2r+3}[0]^{-(2r+3)}$ & summing relatively to r from 0 to ∞, & then adding the result to

$$\tfrac{1}{2}\operatorname{vers}\left(2t\sin\frac{\mu}{2}\right),$$

we get

$$\eta_{0,t}=\tfrac{1}{2}\operatorname{vers}\left(2t\sin\frac{\mu}{2}\right)$$
$$+\cos\frac{\mu}{2}\left(2\sin\frac{\mu}{2}\right)^2\left(\int_0^t dt\right)^3\left\{1-\left(2\sin\frac{\mu}{2}\right)^2\left(\int_0^t dt\right)^2+\left(2\sin\frac{\mu}{2}\right)^4\left(\int_0^t dt\right)^4-\ldots\right\}\Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2;$$

a result agreeing with those obtained at an earlier stage of these investigations.*

(Jan. 31st, 1839.)

[Hamilton identifies this result with]

$$\eta_{0,v/m\sqrt{2}}-\tfrac{1}{2}\operatorname{vers}v=\frac{1}{\pi}\int_0^{\frac{\sqrt{2}}{m}}\frac{\sin pv-p\sin v}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,\frac{dp}{p},$$

$v=mt\sqrt{2}$, $m=\sqrt{2}\sin\dfrac{\mu}{2}$.

The 2nd member of this equation is rigorously an expression for the part of $\eta_{0,t}$ which depends on the initial velocities; & it may be still more simply written thus:

$$\frac{1}{\pi}\int_0^{\frac{\sqrt{2}}{m}}\frac{\sin pv}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,\frac{dp}{p};$$

because we have, exactly,

$$0=\int_0^{\frac{\sqrt{2}}{m}}\frac{dp}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}.\quad\text{[Cauchy's Principal Value.]}$$

At the same time we see that we may write

$$\eta_{0,t}-\tfrac{1}{2}\operatorname{vers}v=\frac{1}{\pi}\int_0^{\infty}\frac{\sin pv}{p}\,\frac{d\varpi}{1-\varpi^2},$$

if $\varpi^{-2}(1-\frac{1}{2}m^2)=p^{-2}-\frac{1}{2}m^2$, or $p^{-2}=\frac{1}{2}m^2+(1-\frac{1}{2}m^2)\varpi^{-2}$,

that is, $p=\varpi\{1-\frac{1}{2}m^2(1-\varpi^2)\}^{-\frac{1}{2}}$.

---

* [See p. 540.]

Or we may write

$$\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \frac{1}{\pi}\int_0^1 \left(\frac{\sin pv}{p} - \frac{\sin qv}{q}\right)\frac{d\varpi}{1-\varpi^2},$$

in which

$$p = \varpi\{1-\tfrac{1}{2}m^2(1-\varpi^2)\}^{-\frac{1}{2}}, \quad q = \varpi^{-1}\{1-\tfrac{1}{2}m^2(1-\varpi^{-2})\}^{-\frac{1}{2}}.$$

If $v = mt\sqrt{2} = 2t\sin\frac{\mu}{2}$ be large and if m be $<\sqrt{2}$, we need only retain those parts of the integral which have small divisors; but p increases from 0 to 1, & q decreases from $\frac{\sqrt{2}}{m}$ to 1; also when ϖ is nearly 0, we have nearly $\varpi = \sqrt{1-\frac{1}{2}m^2}p$; and when ϖ is nearly 1, we have nearly

$$p = 1-(1-\tfrac{1}{2}m^2)(1-\varpi), \quad q = 1+(1-\tfrac{1}{2}m^2)(1-\varpi);$$

also $$\frac{\sin pv}{p} - \frac{\sin qv}{q} = \sin pv - \sin qv = -2\cos v \sin\{v(1-\tfrac{1}{2}m^2)(1-\varpi)\} \text{ nearly};$$

$\because$, v being large, we have nearly

$$\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \frac{1}{\pi}\left\{\sqrt{1-\tfrac{1}{2}m^2}\int_0^\infty \frac{\sin pv}{p}\,dp - \cos v \int_0^\infty \frac{\sin p'v}{p'}\,dp'\right\},$$

if $p' = (1-\frac{1}{2}m^2)(1-\varpi)$; $\because$ finally, we have the following approximate expression for the part of $\eta_{0,t}$ which depends on the initial velocities:

$$\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \tfrac{1}{2}\{\sqrt{1-\tfrac{1}{2}m^2} - \cos v\};$$

in which $v = m\tau = 2t\sin\frac{\mu}{2} =$ a large positive number.

The whole expression for the displacement of the particle 0 at the time t is therefore

$$\eta_{0,t} = \left(\cos\frac{\mu}{4}\right)^2 - \cos\left(2t\sin\frac{\mu}{2}\right) = \operatorname{vers}\left(2t\sin\frac{\mu}{2}\right) - \left(\sin\frac{\mu}{4}\right)^2,$$

if $t\sin\frac{\mu}{2}$ be large.

[By a lengthy process of summation the following result is arrived at.]

The expression

$$\eta_{x,t} = \tfrac{1}{2}\operatorname{vers}\left(\mu x - 2t\sin\frac{\mu}{2}\right)$$
$$+ \operatorname{vers}\mu \int_0^{\frac{\mu}{2}} \frac{\cos\frac{\mu}{2}\cos 2x\theta \sin(2t\sin\theta) - \cos\theta \sin 2x\theta \cos(2t\sin\theta)}{\sin\theta(\cos 2\theta - \cos\mu)}\cdot d\theta$$

contains, *for all values of x and t*, the *rigorous solution of the problem* proposed at the commencement of this manuscript; namely to find a function $\eta_{x,t}$ such that it shall satisfy generally the equation in mixed differences

$$D_t^2\eta_{x,t} = \Delta_x^2\eta_{x-1,t},$$

and also the initial conditions

$$\eta_{x,0} = \operatorname{vers}\mu x \quad \text{or} \quad = 0,$$

according as $x \not> 0$ or > 0, and

$$D_0 \eta_{x,0} = -2 \sin\frac{\mu}{2} \sin \mu x \quad \text{or} \quad = 0,$$

according as $x \not> 0$ or > 0.

For a complete *à posteriori* proof of the correctness of the solution, it is sufficient to prove that*

1st: $\dfrac{\text{vers}\,\mu}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin 2x\theta\, d\theta}{\sin\theta(\cos 2\theta - \cos\mu)} = \pm \tfrac{1}{2} \text{vers}\,\mu x$, according as x is $\gtrless 0$;

2nd: $\dfrac{\sin\mu}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 2x\theta\, d\theta}{\cos 2\theta - \cos\mu} = \pm \tfrac{1}{2} \sin \mu x$, according as x is $\gtrless 0$;

3rd: the latter integral, like the former, vanishes with x.

And whatever x may be, we have, when t is large and positive,

$$\eta_{x,t} = \text{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right) - \left(\sin\frac{\mu}{4}\right)^2;\dagger$$

but when t is large and negative, $\eta_{x,t} = \left(\sin\dfrac{\mu}{4}\right)^2.$

[*Discussion of asymptotic value of this integral.*]

(Feb. 5th.)

The expression for $\eta_{x,t}$ on the previous page can be written

$$\eta_{x,t} = \tfrac{1}{2} \text{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right) - \frac{\text{vers}\,\mu}{4\pi} \quad {}_0\frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta\left(\cos\theta - \cos\dfrac{\mu}{2}\right)}\, d\theta,$$

or this more elegant one $\left(\text{in which } \nu = \dfrac{\mu}{2}\right)$,

$$\eta_{x,t} = \sin(x\nu - t\sin\nu)^2 - \frac{\sin\nu^2}{2\pi}\int_0^{\pi} \frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta(\cos\theta - \cos\nu)}\, d\theta.$$

This is a *rigorous* form for that particular integral of the equation

$$\left(D_t^2 - \frac{\Delta_x^2}{1 + \Delta_x}\right)\eta_{x,t} = 0,$$

which gives $\eta_{x,0} = 2(\sin x\nu)^2$ or $= 0$, and $D_0\eta_{x,0} = -2\sin\nu \sin 2x\nu$ or $= 0$, according as $x \not> 0$ or > 0.

---

* [The verification here suggested is similar to that on pages 469–472.]

† [The integral may be written

$$\int_0^{\frac{\pi}{2}} \frac{\left(\cos\dfrac{\mu}{2} - \cos\theta\right)\cos 2x\theta \sin(2t\sin\theta)\, d\theta}{\sin\theta(\cos 2\theta - \cos\mu)} - + \int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin(2t\sin\theta - 2x\theta)\, d\theta}{\sin\theta(\cos 2\theta - \cos\mu)}.$$

The value of the first part when $t \to \infty$ is $\dfrac{-\pi\left(\sin\dfrac{\mu}{4}\right)^2}{\text{vers}\,\mu}$, the asymptotic value of the second can be obtained by taking the complex integral $\displaystyle\int \frac{e^{2itu - 2ix\sin^{-1}u}\, du}{u(u^2 - \sin^2\frac{1}{2}\mu)}$ about the contour $(\sin\frac{1}{2}\mu, 0)$, $(-\sin\frac{1}{2}\mu, 0)$, $(\sin\frac{1}{2}\mu, i\infty)$, $(-\sin\frac{1}{2}\mu, i\infty)$ with indentations at $u = \sin\frac{1}{2}\mu$, 0, $-\sin\frac{1}{2}\mu$.]

We shall suppose that ν is > 0 but $< \frac{\pi}{2}$; and that t and x are large positive numbers, x being also integer. And on these suppositions we shall seek the *approximate* value of the definite integral in $\eta_{x,t}$.

Because x and t are large, the arc $2x\theta - 2t \sin \theta$ will in general vary much & rapidly while θ varies from 0 to π; & ∵ the sine of this arc will fluctuate often & rapidly between the finite limits ± 1, while θ varies so. Hence we need only consider those particular values of θ for which this sine varies less rapidly than usual, or is divided by an unusually small quantity. But the rate of variation of the arc is expressed by $2(x - t \cos \theta)$; if then we at first suppose that x is $> t$. (corresponding to the case of *darkness*,) we may omit the consideration of the case of slow variation, & may confine ourselves to the cases of $\theta =$ nearly 0 or ν or π.

When θ is nearly 0, x being $> t$, we get $-\frac{\sin \nu^2}{4 \operatorname{vers} \nu} = -\frac{1}{4}(1 + \cos \nu)$ as the corresponding part of the integral. When θ is nearly π, we get the part $-\frac{\sin \nu^2}{4(1 + \cos \nu)} = -\frac{1}{4}(1 - \cos \nu)$; so that the sum of those two parts, corresponding to the divisor $\sin \theta$, is $-\frac{1}{2}$. When θ is nearly ν, the arc $2x\nu - 2t \sin \nu + 2x(\theta - \nu) - 2t(\sin \theta - \sin \nu) =$ nearly

$$2x\nu - 2t \sin \nu + 2(x - t \cos \nu)(\theta - \nu);$$

also the small divisor $\cos \theta - \cos \nu =$ nearly $-\sin \nu (\theta - \nu)$; & the corresponding part of the integral is $\frac{1}{2} \cos (2x\nu - 2t \sin \nu)$. Thus the whole integral is nearly

$$= -\tfrac{1}{2} \operatorname{vers} (2x\nu - 2t \sin \nu) = -\sin (x\nu - t \sin \nu)^2;$$

and $\eta_{x,t} = 0$, if x be large and positive, & $> t$.

Next let $x < t$, but let t be large and positive, & let x be also > 0. The parts corresponding to $\theta =$ nearly 0 & π are now $\frac{1}{4}(\pm 1 + \cos \nu)$, & their sum is $\frac{1}{2} \cos \nu$. The part corresponding to $\theta =$ nearly ν is, if $x \lessgtr t \cos \nu$, $\mp \frac{1}{2} \cos (2x\nu - 2t \sin \nu)$. What is the part corresponding to $\cos \theta =$ nearly $\frac{x}{t}$? It is insensible.

(Feb. 6th.)

But it is important to observe that the foregoing calculations suppose that x differs from t & from $t \cos \nu$ by *large* quantities, positive or negative. When x is *nearly equal* to t or to $t \cos \nu$, we must employ some new considerations.

If $x = t \cos \nu$, then *rigorously*

$$\eta_{x,t} = \eta_{t \cos \nu, t} = \sin t (\nu \cos \nu - \sin \nu)^2 - \frac{\sin \nu^2}{2\pi} \int_0^\pi \frac{\sin 2t (\theta \cos \nu - \sin \theta)}{\sin \theta (\cos \theta - \cos \nu)} d\theta;$$

& when θ is nearly $= \nu$, we have

$$\theta \cos \nu - \sin \theta = \nu \cos \nu - \sin \nu + (\theta - \nu)^2 \frac{\sin \nu}{2} \text{ nearly;}$$

$$2t\,(\theta\cos\nu-\sin\theta)=2t\,(\nu\cos\nu-\sin\nu)+t\sin\nu\,(\theta-\nu)^2\text{ nearly;}$$

$$\theta-\nu=\sqrt{\frac{\delta\phi}{t\sin\nu}}\text{ nearly, if }\phi=2t\,(\nu\cos\nu-\sin\nu)\text{ and }\phi+\delta\phi=2t\,(\theta\cos\nu-\sin\theta);\text{ also}$$

$$\cos\theta-\cos\nu=-\sqrt{\frac{\sin\nu\,\delta\phi}{t}}\text{ nearly, and }d\theta=\frac{d\,\delta\phi}{2\sqrt{t\sin\nu\,\delta\phi}};$$

$$\because\qquad \frac{d\theta}{\cos\theta-\cos\nu}=-\frac{d\,\delta\phi}{2\sin\nu\,\delta\phi},$$

but to a given positive value of $\delta\phi$ correspond two opposite values of $\theta-\nu$; ∵ the corresponding part of $\eta_{t\cos\nu,t}$ is 0; whereas the corresponding part of $\eta_{x,t}$ is $\mp\frac{1}{2}\cos 2\,(x\nu-t\sin\nu)$, if x be considerably less or considerably more than $t\cos\nu$. And if we take in the connected periodic part outside the integral sign, we have

$$-\cos(2x\nu-2t\sin\nu),\qquad -\tfrac{1}{2}\cos(2t\nu\cos\nu-2t\sin\nu),\qquad\text{or }0,$$

according as x is sensibly less than, or exactly equal to, or sensibly greater than $t\cos\nu$.

It seems then that we may regard the *velocity of propagation* as being $=\cos\nu$, and *not* as being $=\dfrac{\sin\nu}{\nu}$.

This distinction between the velocity of propagation of a wave, and the ratio of the space-period to the time-period of its vibrations, appears to me to be entirely new; and to be one of the most curious results hitherto obtained, by introducing the consideration of finite intervals.

Let $\nu=\dfrac{\pi}{n}$, n being an integer number, expressing the number of molecular intervals contained in the length of the wave. And let only i such lengths behind the origin of x be disturbed at the origin of t. We meet these suppositions by changing x in the recent $\eta_{x,t}$ to $x+in$, & subtracting the new expression from the old. *We therefore obtain*

$$y_{x,t}=\frac{1}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin in\theta}{\sin\theta}\,\frac{\cos(2x\theta+in\theta-2t\sin\theta)}{\cos\theta-\cos\frac{\pi}{n}}\,d\theta;$$

as the rigorous expression for a function which satisfies the equation in mixed differences

$$\left(D_t^2-\frac{\Delta_x^2}{1+\Delta_x}\right)y_{x,t}=0,$$

and also the initial conditions

$$y_{x,0}=\operatorname{vers}\frac{2x\pi}{n},\ \text{ or }\ =0,$$

and

$$D_0y_{x,0}=-2\sin\frac{\pi}{n}\sin\frac{2x\pi}{n},\ \text{ or }\ =0,$$

according as x does or does not satisfy the conditions

$$x\not>0,\ x\not<-in.$$

[Here follows a verification.]

Let us consider the cases which correspond to x and t being both large and positive, & inquire, for these cases, the *approximate* values of $y_{x,t}$.

In this research it is useful to remember that we may write, *rigorously*,

$$y_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin(2x\theta+2in\theta-2t\sin\theta)}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}\,d\theta-\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin(2x\theta-2t\sin\theta)}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}\,d\theta.$$

It is ∵ sufficient to study the latter of these 2 integrals; which was indeed considered in part before, (see pages 554, 555,) but shall now be considered anew. For simplicity we shall write the above equation in the form

$$y_{x,t}=z_{x+in,t}-z_{x,t}.$$

Now, because x and t are large, $\sin(2x\theta-2t\sin\theta)$ fluctuates often and rapidly between the finite limits ± 1, while θ varies from 0 to π; & these rapid & repeated fluctuations destroy sensibly all those parts of the integral $z_{x,t}$ which are not rendered sensible by having small divisors. We need therefore attend only, in the calculation of $z_{x,t}$, to those values of θ which are nearly 0 or π or $\frac{\pi}{n}$. And the chief theorem for all these values is that

$$\int_0^\infty\frac{\sin\psi\,d\psi}{\psi}=\frac{\pi}{2}.$$

When θ is nearly 0, we have

$$\frac{\sin\theta}{2x\theta-2t\sin\theta}=\frac{1}{2(x-t)}\text{ nearly;}$$

this relation becomes more and more exact as x and t increase, while the denominator of the fraction remains constant; because, under these conditions, θ diminishes more and more & tends to 0. At the same time

$$\cos\theta-\cos\frac{\pi}{n}\text{ tends to }\operatorname{vers}\frac{\pi}{n};\text{ and }\frac{d\theta}{d\psi}\text{ tends to }\frac{1}{2(x-t)},\text{ if }\psi=2x\theta-2t\sin\theta.$$

Thus the corresponding part of the element of $z_{x,t}$ tends to this limit:

$$\frac{1}{2\pi}\left(1+\cos\frac{\pi}{n}\right)\frac{\sin\psi\,d\psi}{\psi},$$

at least if $x-t$ be considerably $>$ or <0.

It must also be observed that while θ increases from 0 to $+$, that is to a small positive quantity, ψ is to be considered as increasing or diminishing from 0 to a very large positive or negative quantity, according as $x-t$ is $>$ or <0; $x-t$ being supposed to be always large.

In the first of these 2 last mentioned cases, we get ∵, for the corresponding part of the integral $z_{x,t}$,

$$+\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right);\qquad(x>t);$$

and, in the 2nd case,

$$-\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right);\qquad(x<t);$$

which may be explained by saying that while $\frac{d\psi}{\psi}$ is in both cases positive, the factor $\sin\psi$ is in the first case positive, but in the second case negative.

If x were exactly $=t$, we should have, rigorously,

$$z_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin 2t\,(\theta-\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)};$$

&, nearly, for the part which corresponds to θ nearly $=0$,

$$\psi=2t\,(\theta-\sin\theta)=\frac{t\theta^3}{3},\quad d\psi=t\theta^2 d\theta,\quad \frac{d\theta}{\sin\theta}=\frac{d\theta}{\theta}=\frac{d\psi}{3\psi};$$

and the part would be $\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$. Such are the parts which correspond to values of θ a little greater than 0.

Supposing next that θ is only a little less than π; we have, nearly,

$$\frac{2x\,(\pi-\theta)+2t\sin(\pi-\theta)}{\sin(\pi-\theta)}=2\,(x+t),$$

& this relation becomes more and more exact as $x+t$ increases, the arc

$$\psi=2x\,(\pi-\theta)+2t\sin(\pi-\theta)$$

remaining constant; because under these conditions the arc $\pi-\theta$ tends indefinitely to 0.

Also $-\frac{d\theta}{d\psi}$ tends to be $=\frac{+1}{2\,(x+t)}$, being rigorously $=\frac{+1}{2\,(x-t\cos\theta)}$; therefore, since $x+t$ is large & >0, the part of $z_{x,t}$ which is now under consideration is $+\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)$.

Thus the sum of these 2 first parts of the integral $z_{x,t}$ is

$$+\tfrac{1}{2},\qquad \text{if } x \text{ be much } >t;$$

$$-\tfrac{1}{2}\cos\frac{\pi}{n},\qquad \text{if } x \text{ be much } <t;$$

and

$$\frac{1}{6}\left(2-\cos\frac{\pi}{n}\right),\ \text{if } x \text{ be } =t;$$

t and x being both large & positive.

Supposing finally that θ is nearly $=\frac{\pi}{n}$, & putting

$$\psi=2x\theta-2t\sin\theta-\phi,\quad \phi=\frac{2x\pi}{n}-2t\sin\frac{\pi}{n},$$

we have nearly

$$\psi=2\left(\theta-\frac{\pi}{n}\right)\left(x-t\cos\frac{\pi}{n}\right)+t\sin\frac{\pi}{n}\left(\theta-\frac{\pi}{n}\right)^2;$$

and if $x-t\cos\frac{\pi}{n}$ be large, we may neglect the 2nd term.

In this manner we get, for the 3rd part of $z_{x,t}$,

$$\mp\frac{\cos\phi}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\psi\,d\psi}{\psi}=\mp\frac{\cos\phi}{2}=\mp\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

according as x is much $>$ or much $<$ $t\cos\frac{\pi}{n}$. If $x=t\cos\frac{\pi}{n}$, then

$$\theta-\frac{\pi}{n}=t^{-\frac{1}{2}}\sqrt{\frac{\psi}{\sin\frac{\pi}{n}}};\quad \frac{d\theta}{\theta-\frac{\pi}{n}}=\frac{d\psi}{2\psi};\quad \phi=2t\left(\frac{\pi}{n}\cos\frac{\pi}{n}-\sin\frac{\pi}{n}\right);$$

to each value of $\phi + \psi = 2x\theta - 2t \sin \theta$ correspond now two values of $\theta - \frac{\pi}{n}$ which are nearly equal and opposite; ∵ the corresponding part of $z_{x,t}$ is insensible.*

If then $\cos \frac{\pi}{n}$ be sensibly less than 1, so that $t \cos \frac{\pi}{n}$ may be regarded as considerably less than t, when t is large and positive; if x be also large and positive; we have the following approximate values for the integral $z_{x,t}$:

$$\begin{array}{rll} z_{x,t} = & -\frac{1}{2} \cos \frac{\pi}{n} + \frac{1}{2} \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right), & \text{if } x \text{ be much } < t \cos \frac{\pi}{n}; \\ & -\frac{1}{2} \cos \frac{\pi}{n}, & \text{if } x = t \cos \frac{\pi}{n}; \\ & -\frac{1}{2} \cos \frac{\pi}{n} - \frac{1}{2} \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right), & \text{if } x \text{ be much } > t \cos \frac{\pi}{n} \text{ but } < t; \\ & \frac{1}{2} - \frac{1}{3} \left(\cos \frac{\pi}{2n} \right)^2 - \frac{1}{2} \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right. & \text{if } x = t; \\ & \frac{1}{2} - \frac{1}{2} \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right), & \text{if } x \text{ be much } > t; \end{array}$$

and consequently, under the same 5 different suppositions, the function

$$\eta_{x,t} = \tfrac{1}{2} \operatorname{vers} \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right) - z_{x,t}$$

takes the 5 following values:

$$\left(\cos \frac{\pi}{2n} \right)^2 - \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right) = \operatorname{vers} \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right) - \left(\sin \frac{\pi}{2n} \right)^2;$$

$$\left(\cos \frac{\pi}{2n} \right)^2 - \tfrac{1}{2} \cos \left(\frac{2x\pi}{n} - 2t \sin \frac{\pi}{n} \right); \quad \left(\cos \frac{\pi}{2n} \right)^2; \quad \tfrac{1}{3} \left(\cos \frac{\pi}{2n} \right)^2; \quad 0.$$

What would happen if $x - t \cos \frac{\pi}{n}$ were of the same order as $t^{\frac{1}{2}}$? Let $x - t \cos \frac{\pi}{n} = \xi \sqrt{t}$; then, by the last page,

$$\psi = 2\xi t^{\frac{1}{2}} \left(\theta - \frac{\pi}{n} \right) + \sin \frac{\pi}{n} . t \left(\theta - \frac{\pi}{n} \right)^2;$$

∵ if ξ be positive and a good deal larger than $\sqrt{\frac{1}{4} \sin \frac{\pi}{n}}$, we shall have, nearly,

$$t^{\frac{1}{2}} \left(\theta - \frac{\pi}{n} \right) = \frac{\psi}{2\xi} \left(1 - \frac{\psi}{4\xi^2} \sin \frac{\pi}{n} \right);$$

or, more accurately,

$$\psi \sin \frac{\pi}{n} + \xi^2 = \left\{ \xi + \sin \frac{\pi}{n} . t^{\frac{1}{2}} \left(\theta - \frac{\pi}{n} \right) \right\}^2,$$

$$t^{\frac{1}{2}} \left(\theta - \frac{\pi}{n} \right) \sin \frac{\pi}{n} = \xi \left\{ -1 + \sqrt{1 + \frac{\psi}{\xi^2} \sin \frac{\pi}{n}} \right\},$$

[See p. 555.]

that is,

$$t\left(\theta-\frac{\pi}{n}\right)\sin\frac{\pi}{n}=\left(x-t\cos\frac{\pi}{n}\right)\left\{-1+\sqrt{1+\frac{t\psi\sin\frac{\pi}{n}}{\left(x-t\cos\frac{\pi}{n}\right)^2}}\right\},$$

$$\theta-\frac{\pi}{n}=\frac{1}{t\sin\frac{\pi}{n}}\left\{-\left(x-t\cos\frac{\pi}{n}\right)+\sqrt{\left(x-t\cos\frac{\pi}{n}\right)^2+t\sin\frac{\pi}{n}\cdot\psi}\right.$$

$$=\text{(by hypothesis) a quantity of the same order as }\frac{1}{\sqrt{t}};$$

we may ∵ neglect its square in the denominator of the element of $z_{x,t}$; & thus we have to calculate

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\phi+\psi)\,d\,.\,t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)}{t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\left(\phi+2\xi\theta'+\sin\frac{\pi}{n}\theta'^2\right)d\theta'}{\theta'}=-\frac{1}{\pi}\int^{\infty}\frac{\cos\left(\phi+\theta'^2\sin\frac{\pi}{n}\right)\sin 2\xi\theta'\,d\theta'}{\theta'}$$

which tends to $-\frac{\cos\phi}{2}$, when ξ, being positive, tends to become infinitely greater than $\sin\frac{\pi}{n}$.

Thus by taking $\frac{x}{t}=\cos\frac{\pi}{n}+\frac{\xi}{\sqrt{t}}$, in which ξ is a large positive constant number, we shall have with a given, and great, degree of accuracy:

$$z_{x,t}=-\tfrac{1}{2}\cos\frac{\pi}{n}-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

$$\eta_{x,t}=\left(\cos\frac{\pi}{n}\right)^2;$$

so that we are justified in considering $\cos\frac{\pi}{n}$ as the *velocity of propagation.*

If ξ were much less than $\frac{1}{2}\sqrt{\sin\frac{\pi}{n}}$, or more precisely if $\theta'^2\sin\frac{\pi}{n}$ increased much more rapidly than $(2\xi\theta')^2$, so that the ratio $\frac{1}{4}\xi^{-2}\sin\frac{\pi}{n}=s's^{-2}$ is large, s being a number such that $\int_0^s\theta^{-1}\sin\theta\,d\theta$ is nearly $=\frac{\pi}{2}$; then, for the integral

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\cos(\phi+s^{-2}s'\psi'^2)\sin\psi'\,d\psi'}{\psi'},$$

in which $\psi'=2\xi\theta'$, we cannot correctly substitute

$$-\frac{1}{\pi}\int_0^{s}\frac{\cos\phi\sin\psi'}{\psi'}\,d\psi'=-\frac{\cos\phi}{2},$$

but are rather to substitute, as the opposite limit,

$$-\frac{1}{\pi}\int_0^\infty \cos\left(\phi+\frac{s'}{s^2}\psi'^2\right)d\psi';$$

in which we are to remember that $s's^{-2}$ is now large. It is remarkable that this last limit, if it do not vanish, must be proportional to $s\,(s')^{-\frac{1}{2}}$; for it may be written thus:

$$-\frac{1}{\pi}\frac{s}{\sqrt{s'}}\int_0^\infty \cos(\phi+\psi''^2)\,d\psi'';$$

in which $\psi''=\frac{\sqrt{s'}}{s}\psi'$. Also $\frac{s}{\sqrt{s'}}=\frac{2\xi}{\sin\frac{\pi}{n}}$, which we have supposed to be small; it is

$$-\frac{2\left(x-t\cos\frac{\pi}{n}\right)}{\sqrt{t\sin\frac{\pi}{n}}}$$

Cauchy gives in his Memoir on Definite and Singular Integrals (*Mém. Sav. Etr.*, Tome

$$\int_0^\infty \cos(\psi^2)\,d\psi=\int_0^\infty \sin(\psi^2)\,d\psi=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}.$$

We have then $$-\frac{1}{\pi}\frac{s}{\sqrt{s'}}\int_0^\infty \cos(\phi+\psi''^2)\,d\psi''=\frac{x-t\cos\frac{\pi}{n}}{\sqrt{\pi t\sin\frac{\pi}{n}}}\sin\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}-\frac{\pi}{4}\right).$$

There is something here analogous to a *change of phase.**

(Feb. 7th.)

Admitting that $$\int_0^\infty \cos(\psi^2)\,d\psi=\int_0^\infty \sin(\psi^2)\,d\psi=\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

we have, as Cauchy remarks,

$$\begin{aligned}\frac{1}{2}\sqrt{\frac{\pi}{2}}=\frac{1}{2}\int_{-\infty}^\infty \cos(\psi^2)\,d\psi&=\frac{1}{4}\int_{-\infty}^\infty \cos(\overline{\psi+\varpi}^2)\,d\psi+\frac{1}{4}\int_{-\infty}^\infty \cos(\overline{\psi-\varpi}^2)\,d\psi\\&=\frac{1}{4}\int_{-\infty}^\infty\{\cos(\psi^2+\varpi^2+2\varpi\psi)+\cos(\psi^2+\varpi^2-2\varpi\psi)\}\,d\psi\\&=\frac{1}{2}\int_{-\infty}^\infty \cos(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi=\int_0^\infty \cos(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi;\end{aligned}$$

& for the same reason $$\frac{1}{2}\sqrt{\frac{\pi}{2}}=\int_0^\infty \sin(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi;$$

$$\because\quad \int_0^\infty \cos(\psi^2)\cos(2\varpi\psi)\,d\psi=\frac{\sqrt{\pi}}{2}\cos\left(\varpi^2-\frac{\pi}{4}\right);$$

$$\int_0^\infty \sin(\psi^2)\cos(2\varpi\psi)\,d\psi=\frac{\sqrt{\pi}}{2}\cos\left(\varpi^2+\frac{\pi}{4}\right).$$

---

* [Cf. Kelvin, *Proc. Roy. Soc.* Vol. XLII (1887), p. 80.]

Hence I deduce, by integration relatively to ϖ,

$$\int_0^\infty \frac{\cos(\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \sqrt{\pi}\int_0^\varpi \cos\left(\varpi^2 - \frac{\pi}{4}\right)d\varpi;$$

$$\int_0^\infty \frac{\sin(\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \sqrt{\pi}\int_0^\varpi \cos\left(\varpi^2 + \frac{\pi}{4}\right)d\varpi.$$

Hence
$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\cos(\phi+\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \int_0^\varpi \cos\left(\phi + \frac{\pi}{4} - \varpi^2\right)d\varpi;$$

which, when ϖ is very small, is nearly $= \varpi\cos\left(\phi+\frac{\pi}{4}\right)$; but, when ϖ is very large & positive, is nearly

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\cos\left(\phi+\frac{\pi}{4}\right) + \sin\left(\phi+\frac{\pi}{4}\right)\right\} = \tfrac{1}{2}\sqrt{\pi}\cos\phi;$$

& these two extreme values agree with those lately obtained.

To calculate ∵

$$-\frac{1}{\pi}\int_0^\infty \frac{\cos(\phi + s's^{-2}\psi'^2)\sin\psi'\,d\psi'}{\psi'}$$

we may put, as on the last page,

$$\psi' = \frac{s}{\sqrt{s'}}\psi'' = 2\varpi\psi'',$$

& we get

$$-\frac{1}{\pi}\int_0^\infty \frac{\cos(\phi + s's^{-2}\psi'^2)}{\psi'}\sin\psi'\,d\psi' = -\frac{1}{\sqrt{\pi}}\int_0^{\frac{s}{2\sqrt{s'}}} \cos\left(\phi+\frac{\pi}{4}-\varpi^2\right)d\varpi$$

$$= -\frac{1}{\sqrt{\pi}}\int_0^{\frac{x-t\cos\frac{\pi}{n}}{\sqrt{t\sin\frac{\pi}{n}}}} \cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n} + \frac{\pi}{4} - \varpi^2\right)d\varpi.$$

We may even extend this expression to *all values of* x, t being large and positive; for as soon as $x - t\cos\frac{\pi}{n}$ is of the same order as t, whether positive or negative, it becomes sensibly

$$= \mp\frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n} + \frac{\pi}{4} - \varpi^2\right)d\varpi = \mp\tfrac{1}{2}\cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n}\right).*$$

---

* [It is to be observed that the fundamental integral, $z_{x,t}$, written as a usual integral, is $\left(\text{putting } \alpha = \frac{\pi}{n}\right)$

$$\frac{1}{2\pi}(\sin\alpha)^2\int_0^\pi \left\{\frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta} - \frac{\sin(2x\alpha - 2t\sin\alpha)}{\sin\alpha}\right\}\frac{d\theta}{\cos\theta - \cos\alpha}.$$

For values of θ nearly equal to α this becomes

$$\frac{\cos(2x\alpha - 2t\sin\alpha)}{2\pi}\int_0^\pi \frac{\sin\{x(\theta-\alpha) - t(\sin\theta - \sin\alpha)\}\,d\theta}{\sin\frac{1}{2}(\theta-\alpha)}.]$$

[*Airy's integral.*]

Can we find any analogous general expression for the part of the same integral [$z_{x,t}$], which corresponds to values of θ nearly $=0$?

For such values, $\psi = 2x\theta - 2t\sin\theta = 2(x-t)\,\theta + \frac{t}{3}\theta^3 \pm$ terms which may be neglected in comparison with these, even if $\frac{x-t}{t}$ be small; because θ is to be at largest of the same order as $\sqrt{\frac{s}{t}}$, in which s is a large but constant number, independent of t, & such that $\int_0^s \frac{\sin\psi}{\psi}\,d\psi$ is sufficiently near to $\frac{\pi}{2}$ for the purposes of our approximation. For the same reason, we may treat the denominator $\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)$ as being $=\theta\operatorname{vers}\frac{\pi}{n}$; & $\because$ shall have, for the corresponding part of $z_{x,t}$, the expression

$$\frac{1}{2\pi}\left(1+\cos\frac{\pi}{n}\right)\int_0^\infty \frac{\sin\left(2\xi_{,}\theta_{,} + \frac{1}{3}\theta_{,}^3\right)d\theta_{,}}{\theta_{,}},$$

in which $$\theta_{,} = \theta\sqrt[3]{t}, \quad \text{and} \quad \xi_{,} = \frac{x-t}{\sqrt[3]{t}}.$$

If $\xi_{,}$ be very large, positive or negative, this expression is evidently $=\pm\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)$. If on the other hand $\xi_{,}$ be very small, the expression becomes $\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$. But according to what law does its value vary for moderate values of $\xi_{,}$?

We have, accurately,

$$\int_0^\infty \frac{\sin\left(2\xi_{,}\theta_{,} + \frac{1}{3}\theta_{,}^3\right)d\theta_{,}}{2\theta_{,}} = \frac{\pi}{12} + \int_0^{\xi_{,}} d\xi_{,} \int_0^\infty d\theta_{,} \cos\left(2\xi_{,}\theta_{,} + \tfrac{1}{3}\theta_{,}^3\right);$$

so that it is desirable to try whether we can obtain a finite expression for this last integral, $\int_0^\infty \cos\left(2\xi\theta + \frac{1}{3}\theta^3\right)d\theta$.*

Can we calculate $\int_0^\infty e^{-\alpha^2(\theta^3+6\xi\theta)}\,d\theta$, when α is real, or $\int_0^\infty e^{-\alpha^2x^3}\cos rx\,dx$, which bears a strong affinity to the integral discussed by Laplace, *Calc. Prob.*, page 97, $\int e^{-\alpha^2x^2}\cos rx\,dx$?

[The calculation is reduced to the solution of a Riccati equation but is not carried further.]

[*Detailed description of wave motion.*]

(Feb. 8th.)

Waiving for the present the discussion begun on this page, let me resume the former discussion of $z_{x,t}$.

---

* [Airy's integral. Cf. Watson, *Theory of Bessel Functions*, pp. 188–190.]

It follows from that discussion that the function

$$z_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \quad \frac{\sin(2x\theta-2t\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}$$

involves a first part corresponding to θ nearly 0, which is $\pm\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)$, according as $x-t$, being large, is positive or negative; but is $=\pm\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$, according as t, being large, is positive or negative, if $x=t$.

The same function or integral $z_{x,t}$ involves a 2nd part, corresponding to θ nearly $=\pi$; which 2nd part is $\pm\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)$, according as $x+t$, being large, is positive or negative; but is $=\mp\frac{1}{12}\left(1-\cos\frac{\pi}{n}\right)$, according as t, being large, is positive or negative, if $x=-t$.

The function $z_{x.t}$ involves finally a 3rd part, corresponding to θ nearly $=\frac{\pi}{n}$; which 3rd part vanishes when $x=t\cos\frac{\pi}{n}$, if t be large, positive or negative; but is $=\mp\frac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)$, according as $x-t\cos\frac{\pi}{n}$, being large, is positive or negative.

More precisely, by saying that $x-t\cos\frac{\pi}{n}$ is *large*, we mean that it is numerically equal to or numerically greater than $\frac{1}{2}s\sqrt{\pm t\sin\frac{\pi}{n}}$, s being a positive number so great that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\left(\phi\pm\frac{\psi^2}{s^2}\right)\frac{\sin\psi}{\psi}\,d\psi$$

is sensibly equal to $\cos\phi$. Perhaps the number $s=1000$ may sufficiently satisfy this condition. At all events, whatever degree of accuracy may be required, this number s is independent of t & of x.

By saying that $x-t$ is *large*, we mean that it is numerically equal to or greater than $\frac{1}{2}s't^{\frac{1}{3}}$, s' being a positive number so great that $\frac{2}{\pi}\int_0^{\infty}\sin\left(\psi\pm\frac{\psi^3}{3s'^3}\right)\frac{d\psi}{\psi}$ is sensibly $=1$. Perhaps s' may be taken $=200$.

Finally, by saying that $x+t$ is large, we mean that it is numerically equal to, or numerically greater than, the same quantity $\frac{1}{2}s't^{\frac{1}{3}}$, s' being the same positive number.

Now let the number in be considerably greater than $\frac{s}{2}\sqrt{\pm t\sin\frac{\pi}{n}}$, & also than $\frac{s}{2}\sqrt[3]{\pm t}$. (What are the relations between these 2 limits?)

And first, let i be infinite; that is, let us return to the *original problem of this manuscript*; in which it is supposed that, at the origin of time, all particles behind the origin are disturbed according to the law of the versed sine, & all beyond it are at rest. The number n of molecular intervals in *one* wave-length is not obliged to be large. We shall suppose that $1-\cos\frac{\pi}{n}$ is sensibly different from 0.

We have now, (compare p. 554,)

$$\eta_{x,t}=\tfrac{1}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-z_{x,t}=\tfrac{1}{2}\operatorname{vers}\phi-z_{x,t}.$$

The law of $z_{x,t}$ is sufficiently stated on the preceding page; and we are to deduce the corresponding law of $\eta_{x,t}$.

I. *Let t be large and positive.**

(1.) If x be $=$ or $>t+\frac{1}{2}s't^{\frac{1}{3}}$, then

$$z_{x,t}=\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)+\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)-\frac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)=\frac{1}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

and $\eta_{x,t}=0$; the disturbance is insensible at a distance beyond the origin so great as $t+\frac{1}{2}s't^{\frac{1}{3}}$.

(2.) If x be $=t$, then $\eta_{x,t}=\frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$; there is then a certain constant and positive displacement, which travels with a velocity $=1=$ the square root of the attractive (accelerative) force exerted by any one particle on its next neighbour, and therefore with a velocity independent of the wave-length.

(3.) If x be $=$ or $<t-\frac{1}{2}s't^{\frac{1}{3}}$, but $>$ or $=t\cos\frac{\pi}{n}+\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$, then $\eta_{x,t}=\left(\cos\frac{\pi}{2n}\right)^2=$ a certain other constant & positive displacement, three times as great as that last mentioned. This new displacement is sensibly constant, within an extent $=t\operatorname{vers}\frac{\pi}{n}-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\frac{1}{2}s't^{\frac{1}{3}}$; whereas for a range $=s't^{\frac{1}{3}}$ next following, there is a variable displacement, which vanishes (sensibly) at the end of that range, & is, at the middle, reduced to one third of what it was at the beginning.

(4.) If x be $=t\cos\frac{\pi}{n}$, then $\eta_{x,t}=\frac{1}{2}\operatorname{vers}\phi+\frac{1}{2}\cos\frac{\pi}{n}=$ a periodical displacement, which travels with a constant velocity $\left(=\cos\frac{\pi}{n}=\text{a function of the wave-length}\right)$, & of which the amplitude is $=1=$ half the amplitude that corresponds to the most complete effect of the initial disturbance.

(5.) If x be $=$ or $<t\cos\frac{\pi}{n}-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$, but $>$ or $=-t+\frac{1}{2}s't^{\frac{1}{3}}$, then $\eta_{x,t}=\operatorname{vers}\phi-\left(\sin\frac{\pi}{2n}\right)^2=$ a periodical displacement, which has a constant amplitude $=2$, & of which the period, for any given value of x, is $\pi\operatorname{cosec}\frac{\pi}{n}$ [wave-length $=n$]. This part of space corresponds very well to the phenomena of light. Its extent is $t\left(1+\cos\frac{\pi}{n}\right)-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\frac{1}{2}s't^{\frac{1}{3}}$.

(6.) If x be $=-t$, then $\eta_{x,t}=\operatorname{vers}\phi-\frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$.

---

* [See Appendix, Note 12, p. 640.]

(7.) If x be $=$ or $< -t - \frac{1}{2}s't^{\frac{1}{3}}$, then $\eta_{x,t} = \text{vers}\,\phi$.

Thus the law of the versed sine may be considered as holding good, without sensible error, for all those particles which are at least as far behind the origin as the position $x = -t - \frac{1}{2}s't^{\frac{1}{3}}$; but during the range $= s't^{\frac{1}{3}}$ next following in the positive direction, that is, between the bounds $x = -t \mp \frac{1}{2}s't^{\frac{1}{3}}$, a negative term is added to vers ϕ, which negative term is sensibly $= -\frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$ at the middle of this interval, & $= -\left(\sin\frac{\pi}{2n}\right)^2$ at the end thereof. It then remains sensibly constant, as also does the law vers ϕ remain unchanged, during a large range

$$= 2t\left(\cos\frac{\pi}{2n}\right)^2 - \tfrac{1}{2}s\sqrt{t\sin\frac{\pi}{n}} - \tfrac{1}{2}s't^{\frac{1}{3}},$$

that is, till $x = t\cos\frac{\pi}{n} - \frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$; but in the interval $= s\sqrt{t\sin\frac{\pi}{n}}$ next following, the displacement changes from $\left(\cos\frac{\pi}{2n}\right)^2 - \cos\phi$ to $\left(\cos\frac{\pi}{2n}\right)^2$, the coefficient of $-\cos\phi$ decreasing from 1 to 0 & being $= \frac{1}{2}$ at the middle of this interval, that is, when $x = t\cos\frac{\pi}{n}$. In another range

$$= 2t\left(\sin\frac{\pi}{2n}\right)^2 - \tfrac{1}{2}s\sqrt{t\sin\frac{\pi}{n}} - \tfrac{1}{2}s't^{\frac{1}{3}},$$

that is, till $x = t - \frac{1}{2}s't^{\frac{1}{3}}$, the constant term $\left(\cos\frac{\pi}{2n}\right)^2$ remains sensibly the same; but in the interval $= s't^{\frac{1}{3}}$ next following, this term is reduced to an insensible quantity, being $= \frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$ at the middle of the interval, that is, for $x = t$. Finally, for $x > t + \frac{1}{2}s't^{\frac{1}{3}}$, the disturbance or displacement is insensible.

II. *Let t be large and negative.*

That is, let us trace back what *must have been* the state of the disturbances before the origin of t.

(1.)′ If $x =$ or $> -t - \frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t} = 0$.

(2.)′ If $x = -t$; $\eta_{x,t} = \frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$.

(3.)′ If $x =$ or $< -t + \frac{1}{2}s't^{\frac{1}{3}}$, but $>$ or $= t\cos\frac{\pi}{n} + \frac{1}{2}s\sqrt{-t\sin\frac{\pi}{n}}$; $\eta_{x,t} = \left(\sin\frac{\pi}{2n}\right)^2$.

(4.)′ If $x = t\cos\frac{\pi}{n}$; $\eta_{x,t} = \left(\sin\frac{\pi}{2n}\right)^2 - \frac{1}{2}\cos\phi$.

(5.)′ If $x =$ or $< t\cos\frac{\pi}{n} - \frac{1}{2}s\sqrt{-t\sin\frac{\pi}{n}}$, but $>$ or $= t - \frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t} = \text{vers}\,\phi - \left(\cos\frac{\pi}{2n}\right)^2$.

(6.)′ If $x = t$; $\eta_{x,t} = \text{vers}\,\phi - \frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$.

(7.)′ If $x =$ or $< t + \frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t} = \text{vers}\,\phi$.

[*Extension of original problem.*]

(June 14th, 1839.)

If we have the equation

$$\eta''_{x,t} = a_1^2(\eta_{x+1,t} - 2\eta_{x,t} + \eta_{x-1,t}) + a_2^2(\eta_{x+2,t} - 2\eta_{x,t} + \eta_{x-2,t}),$$

we may satisfy it, generally & rigorously, by the expression

$$\eta_{x,t} = \frac{2}{\pi}\Sigma_{(l)-\infty}^{\infty}\eta_{l,0}\int_0^{\frac{\pi}{2}} d\theta\cos(2l\theta - 2x\theta)\cos(2t\Theta_\theta)$$

$$+\frac{1}{\pi}\Sigma_{(l)-\infty}^{\infty}\eta'_{l,0}\int_0^{\frac{\pi}{2}} d\theta\cos(2l\theta - 2x\theta)\sin(2t\Theta_\theta)\,\Theta_\theta^{-1},$$

in which $\Theta_\theta = \sqrt{a_1^2\sin\theta^2 + a_2^2\sin 2\theta^2}$; x and l being integers.

Let the initial conditions be

$$\eta_{-l,0} = \eta\text{ vers}\left(\frac{2l\pi}{n}\right) \quad \text{or } = 0,$$

and

$$\eta'_{-l,0} = 2\eta\sin\left(\frac{2l\pi}{n}\right)\Theta_{\frac{\pi}{n}} \quad \text{or } = 0,$$

according as $-l$ is or is not included between 0 and in; then

$$\eta_{x,t} = \frac{2\eta}{\pi}\Sigma_{(l)0}^{in}\left\{\text{vers}\left(\frac{2l\pi}{n}\right)\int_0^{\frac{\pi}{2}} d\theta\cos(2l\theta + 2x\theta)\cos(2t\Theta_\theta)\right.$$

$$\left.+\Theta_{\frac{\pi}{n}}\sin\frac{2l\pi}{n}\int_0^{\frac{\pi}{2}} d\theta\cos(2l\theta + 2x\theta)\sin(2t\Theta_\theta)\,\Theta_\theta^{-1}\right\}.$$

We have

$$\Sigma_{(l)0}^{in}\text{ vers}\frac{2l\pi}{n}\cos(2l\theta + 2x\theta) = \frac{\sin in\theta}{2\sin\theta}\cos(2x\theta + in\theta)\left(\sin\frac{\pi}{n}\right)^2\left\{\frac{1}{\cos\theta - \cos\frac{\pi}{n}} + \frac{1}{\cos\theta + \cos\frac{\pi}{n}}\right\},$$

$$\Sigma_{(l)0}^{in}\sin\frac{2l\pi}{n}\cos(2l\theta + 2x\theta) = \frac{\sin in\theta}{2}\sin(2x\theta + in\theta)\sin\frac{\pi}{n}\left\{\frac{1}{\cos\theta - \cos\frac{\pi}{n}} - \frac{1}{\cos\theta + \cos\frac{\pi}{n}}\right\};$$

$$\because\quad \eta_{x,t} = \frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi} d\theta\,\frac{\sin in\theta}{\sin\theta}\,\frac{\cos(2x\theta + in\theta)}{\cos\theta - \cos\frac{\pi}{n}}\cos(2t\Theta_\theta)$$

$$+\frac{\eta}{\pi}\Theta_{\frac{\pi}{n}}\sin\frac{\pi}{n}\int_0^{\pi} d\theta\,\frac{\sin in\theta}{\Theta_\theta}\,\frac{\sin(2x\theta + in\theta)}{\cos\theta - \cos\frac{\pi}{n}}\sin(2t\Theta_\theta).$$

In general

$$4\sin a\cos b\cos c = \sin(b+c+a) - \sin(b+c-a) + \sin(b-c+a) - \sin(b-c-a),$$

$$4\sin a\sin b\sin c = -\sin(b+c+a) + \sin(b+c-a) + \sin(b-c+a) - \sin(b-c-a);$$

$$\eta_{x,t} = A - B + C - D,$$

in which

$$A=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}-\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta+2in\theta+2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}}.$$

$$B=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}-\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta+2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}},$$

$$C=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}+\Theta_{\frac{\pi}{n}}\right)\frac{\sin(2x\theta+2in\theta-2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}},$$

$$D=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}+\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta-2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}}.$$

[*Two-dimensional waves.*]

(June 14th, 1839.)

If we consider the vibrations of an indefinite *square system* of points & suppose that each attracts only the 4 nearest to itself, the distance between any 2 nearest points being $=1$ in the state of equilibrium, & the attraction being sensibly $=a^2+b^2(r-1)$ for any distance r differing little from 1, but being sensibly $=0$ for any much greater distance such as even $\sqrt{2}$; we shall then have the 2 indefinite equations in mixed differences

$$0=\left\{-\left(\frac{d}{dt}\right)^2+\frac{a^2\Delta_y^2}{1+\Delta_y}+\frac{b^2\Delta_x^2}{1+\Delta_x}\right\}\xi_{x,y,t},$$

$$0=\left\{-\left(\frac{d}{dt}\right)^2+\frac{a^2\Delta_x^2}{1+\Delta_x}+\frac{b^2\Delta_y^2}{1+\Delta_y}\right\}\eta_{x,y,t};$$

of which the complete and general integrals may be thus written:

$$\xi_{x,y,t}=\frac{4}{\pi^2}\Sigma_{(l,m)-\infty;\,-\infty}^{\infty;\ \infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2x\theta-2l\theta)\cos(2y\phi-2m\phi)\left\{\xi_{l,m,0}\cos(2t\Theta)+\xi'_{l,m,0}\frac{\sin(2t\Theta)}{2\Theta}\right\},$$

$$\eta_{x,y,t}=\frac{4}{\pi^2}\Sigma_{(l,m)-\infty;\,-\infty}^{\infty;\ \infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2y\theta-2m\theta)\cos(2x\phi-2l\phi)\left\{\eta_{l,m,0}\cos(2t\Theta)+\eta'_{l,m,0}\frac{\sin(2t\Theta)}{2\Theta}\right\},$$

if

$$\Theta=\sqrt{a^2\sin\phi^2+b^2\sin\theta^2}.$$

Suppose now that for every integer value of l between 0 & $-in$, and for every integer value of m between $-m_,$ & $+m_,$, (the extremes being included,) we have

$$\eta_{l,m,0}=\eta\operatorname{vers}\frac{2l\pi}{n},\quad \eta'_{l,m,0}=-2a\eta\sin\frac{\pi}{n}\sin\frac{2l\pi}{n};$$

but for all other values of l or m we have $\eta_{l,m,0}=0$, $\eta'_{l,m,0}=0$, & for all values of l & m, $\xi_{l,m,0}=0$, $\xi'_{l,m,0}=0$. Then $\xi_{x,y,t}=0$ and

$$\eta_{x,y,t}=\frac{4\eta}{\pi^2}\Sigma^{m,}_{(m)-m,}\Sigma^{in}_{(l)0}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2y\theta-2m\theta)\cos(2x\phi+2l\phi)$$

$$\text{vers}\,\frac{2l\pi}{n}\cos(2t\Theta)+\frac{a\sin\frac{\pi}{n}}{\Theta}\sin\left(\frac{2l\pi}{n}\right)\sin(2t\Theta)$$

But

$${}_{(m)-m,}\cos(2y\theta-2m\theta)=\frac{\cos 2y\theta\sin(2m,\theta+\theta)}{\sin\theta};$$

$$\therefore\quad \eta_{x,y,t}=\frac{2\eta}{\pi^2}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\frac{\pi}{2}}d\theta\int_0^{\pi}d\phi\,\frac{\cos 2y\theta\sin(2m,\theta+\theta)}{\sin\theta}\,\frac{\sin in\phi}{\sin\phi}$$

$$\left\{\frac{\cos(2x\phi+in\phi)}{\cos\phi-\cos\frac{\pi}{n}}\cos(2t\Theta)+\frac{a\sin\phi}{\Theta}\,\frac{\sin(2x\phi+in\phi)}{\cos\phi-\cos\frac{\pi}{n}}\sin(2t\Theta)\right\}.$$

As $m,$ increases without limit, this expression tends to become independent of y,* namely

$$\eta_{x,y,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi}d\phi\,\frac{\sin in\phi}{\sin\phi}\,\frac{\cos(2x\phi+in\phi-2at\sin\phi)}{\cos\phi-\cos\frac{\pi}{n}},$$

agreeing with the value found before. The same result is obtained if y be large, but $m,$ much larger.

If on the contrary y be much larger than $m,$, then the relation

$$2\cos 2y\theta\sin(2m,\theta+\theta)=\sin(2y\theta+2m,\theta+\theta)-\sin(2y\theta-2m,\theta-\theta)$$

causes $\eta_{x,y,t}$ to be sensibly $=0$; & we have thus an example of the possibility of representing in dynamical calculation the sensible *rectilinearity of the propagation of light.*

Diffraction corresponds to the case of y nearly $=m,$.

[*Three-dimensional waves in dispersive medium.*]

(June 18th, 1839.)

In general, the equations of infinitely small vibrations of a system of attracting or repelling particles,

$$\frac{d^2\delta x}{dt^2}=\Sigma\{\Delta\delta x\,\phi(r)+\Delta x\,\phi'(r)\,\delta r\},\ \&\text{c.},$$

---

* [This means, of course, that in the integrand, with the exception of the factor $\sin(2m,\theta+\theta)/\sin\theta$, θ is made zero so that $\Theta=a\sin\phi$.]

may have their integrals expressed, as Cauchy* has shown, by equations of the form

$$\begin{array}{ll}(\delta x)_t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{d_t \cos(ux+vy+wz) + g_t \sin(ux+vy+wz)\}\, du\, dv\, dw, \\ (\delta y)_t = \qquad\qquad\qquad \{e_t \qquad\qquad\qquad\qquad h_t \qquad\qquad\qquad\qquad \} \qquad , \\ (\delta z)_t = \qquad\qquad\qquad \{f_t \end{array}$$

$d_t, \ldots i_t$ being 6 real functions of t and of u, v, w, depending on the nature of the system.

The initial values & rates of increase of these 6 functions may be deduced from those of $\delta x, \delta y, \delta z$ by the 12 formulae

$$d_0 = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\delta x)_0 \cos(ux+vy+wz)\, dx\, dy\, dz,$$

$$\ldots\ldots$$

$$i'_0 = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\delta z)'_0 \sin(ux+vy+wz)\, dx\, dy\, dz,$$

in which $(\delta x)_0, (\delta y)_0, (\delta z)_0, (\delta x)'_0, (\delta y)'_0, (\delta z)'_0$ are 6 known real functions of x, y, z.

The differential relations between d_t, e_t, f_t are of the forms

$$d''_t = -(Ld_t + Re_t + Qf_t), \quad e''_t = -(Me_t + Pf_t + Rd_t), \quad f''_t = -(Nf_t + Qd_t + Pe_t),$$

in which L, M, N, P, Q, R are 6 real and known functions of u, v, w, depending on the nature of the system; this system being supposed to have a certain symmetry of arrangement & to have the same arrangement throughout; & the same equations hold good when d, e, f are changed to g, h, i.

The expressions for these 6 functions $L, \ldots R$ are

$$L = \Sigma \,.\left(\frac{\Delta x^2}{r}\phi'(r) + \phi(r)\right) \operatorname{vers}(u\Delta x + v\Delta y + w\Delta z), \qquad M = \qquad\qquad , \quad N = \qquad\qquad ,$$

$$P = \Sigma \,.\left(\frac{\Delta x \Delta y}{r}\phi'(r)\right) \operatorname{vers}(u\Delta x + v\Delta y + w\Delta z), \qquad Q = \qquad\qquad , \quad R = \qquad\qquad .$$

The integrals of the differential relations between d_t, e_t, f_t are of the forms

$$A_1 d_t + B_1 e_t + C_1 f_t = (A_1 d_0 + B_1 e_0 + C_1 f_0)\cos s_1 t + (A_1 d'_0 + B_1 e'_0 + C_1 f'_0)\int_0^t \cos s_1 t\, dt,$$

$$\ldots\ldots\ldots$$

$$A_3 d_t + B_3 e_t + C_3 f_t = (A_3 d_0 + B_3 e_0 + C_3 f_0)\cos s_3 t + (A_3 d'_0 + B_3 e'_0 + C_3 f'_0)\int_0^t \cos s_3 t\, dt;$$

$A_1, \ldots C_3$ being 9 real cosines, namely of the inclinations of 3 rectangular lines to the 3 rectangular axes of coordinates, these 3 lines being the axes of a certain surface of the 2nd order, & depending only on u, v, w; but s_1, s_2, s_3 being (perhaps) not necessarily real.

The equations which determine them all are

$$s^2 = L + \frac{BR + CQ}{A} = M + \frac{CP + AR}{B} = N + \frac{AQ + BP}{C}$$

---

* [Cauchy, *Œuvres*, 1e Série, Tome IV, pp. 237–298. It is to be noted that $r\phi(r)$ is the law of force between two points.]

(June 19th.)

Let *only half the medium be originally agitated*; & in particular let the initial conditions b

$$(\delta x)_0=0, \quad (\delta y)_0=\operatorname{vers} u^{\backprime}x, \quad (\delta z)_0=0, \quad (\delta x)_0'=0, \quad (\delta y)_0'=s^{\backprime}\sin(u^{\backprime}x), \quad (\delta z)_0'=0,$$

for all negative values of x, these 6 functions all vanishing for positive values of x.

Then

$$d_0=0, \quad f_0=0, \quad g_0=0, \quad i_0=0, \quad d_0'=0, \quad f_0'=0, \quad g_0'=0, \quad i_0'=0,$$

and

$$e_0=\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\operatorname{vers} u^{\backprime}x\cos(ux+vy+wz),$$

$$h_0=\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\operatorname{vers} u^{\backprime}x\sin(ux+vy+wz),$$

$$e_0'=s^{\backprime}\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\sin u^{\backprime}x\cos(ux+vy+wz),$$

$$h_0'=s^{\backprime}\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\sin u^{\backprime}x\sin(ux+vy+wz).$$

[Hamilton then calculates from integrals in which the integrands above have a factor $e^{-h^2(x^2+y^2+z^2)}$. Inserting these values in the equations

$$e_t=(B_1^2\cos s_1t+B_2^2\cos s_2t+B_3^2\cos s_3t)\,e_0+\left(B_1^2\frac{\sin s_1t}{s_1}+B_2^2\frac{\sin s_2t}{s_2}+B_3^2\frac{\sin s_3t}{s_3}\right)e_0', \ \&\text{c.}$$

and making $h=0$ in the triple integral for $(\delta y)_t$, we get finally the following equation]*

$$(\delta y)_t=\tfrac{1}{2}\operatorname{vers}(u^{\backprime}x+s^{\backprime}t)+\frac{u^{\backprime}}{\pi}\int_0^{\infty}\frac{du}{u^2-u^{\backprime 2}}\left(\frac{u^{\backprime}}{u}\sin ux\cos s_1t+\frac{s^{\backprime}}{s_1}\cos ux\sin s_1t\right).$$

This expression for $(\delta y)_t$ is rigorous in the present question; & accordingly it gives

$$(\delta y)_0=\operatorname{vers} u^{\backprime}x \text{ or } =0, \quad \text{and} \quad (\delta y)_0'=s^{\backprime}\sin u^{\backprime}x \text{ or } =0,$$

according as x is <0 or >0; because, in these 2 respective cases, we have

$$2\int_0^{\infty}\frac{\sin ux\,.\,u\,du}{u^2-u^{\backprime 2}}=\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin ux}{u+u^{\backprime}}+\frac{\sin ux}{u-u^{\backprime}}\right)du=\mp\pi\cos u^{\backprime}x;$$

$$2u^{\backprime}\int_0^{\infty}\frac{\cos ux\,.\,du}{u^2-u^{\backprime 2}}=\pm\pi\sin u^{\backprime}x; \quad 2u^{\backprime 2}\int_0^{\infty}\frac{\sin ux\,.\,du}{u(u^2-u^{\backprime 2})}=\pm\pi\operatorname{vers} u^{\backprime}x.$$

Thus, rigorously,

$$\boxed{(\delta y)_t=\tfrac{1}{2}\operatorname{vers}(u^{\backprime}x+s^{\backprime}t)+\frac{u^{\backprime}}{2\pi}\int_0^{\infty}\frac{du}{u^2-u^{\backprime 2}}\left\{\left(\frac{u^{\backprime}}{u}+\frac{s^{\backprime}}{s_1}\right)\sin(ux+s_1t)+\left(\frac{u^{\backprime}}{u}-\frac{s^{\backprime}}{s_1}\right)\sin(ux-s_1t)\right\};}$$

& its periodic part is 0 or $-\cos(u^{\backprime}x+s^{\backprime}t)$, according as $x+\frac{ds}{du^{\backprime}}t$, being large, is $>$ or <0. *The velocity of propagation of vibration, or rather in this case the velocity of extinction, is expressed*

---

* [See Sommerfeld, Diss., Königsberg, 1891 and Carslaw, *Fourier's Series and Integrals*, p. 293.]

not by the ratio of the coefficients of time and space, but by the ratio of their differentials; & I feel almost sure that this is the general law of the propagation of vibratory motion.[*]

$$\left[\textit{Velocity of propagation } \frac{ds^\backprime}{dk^\backprime}.\right]$$

(June 20th.,

Suppose that only half the medium is originally agitated, namely that half which is on the negative side (relatively to x) of the plane $u^\backprime x+v^\backprime y+w^\backprime z=0$, in such a way that, if

$$x<-\frac{v^\backprime y+w^\backprime z}{u^\backprime}$$

we have

$$(\delta x)_0=A^\backprime\cos(u^\backprime x+v^\backprime y+w^\backprime z),\quad (\delta y)_0=B^\backprime\cos(\qquad),\quad (\delta z)_0=C^\backprime\cos(\qquad$$

$$(\delta x)_0'=A^\backprime s^\backprime\sin(\qquad),\quad (\delta y)_0'=B^\backprime s^\backprime\sin(\qquad),\quad (\delta z)_0'=C^\backprime s^\backprime\sin(\qquad);$$

but

$$(\delta x)_0=(\delta y)_0=(\delta z)_0=(\delta x)_0'=(\delta y)_0'=(\delta z)_0'=0,\ \text{ if }\ x>-\frac{v^\backprime y+w^\backprime z}{u^\backprime}.$$

How are we now to effect the triple integrations for $d_0, \ldots i_0'$?

We may conceive 3 new rectangular coordinates $x^\backprime, y^\backprime, z^\backprime$, of which $x^\backprime$ is the perpendicular distance $\dfrac{u^\backprime x+v^\backprime y+w^\backprime z}{\sqrt{u^{\backprime 2}+v^{\backprime 2}+w^{\backprime 2}}}$ from the plane $u^\backprime x+v^\backprime y+w^\backprime z=0$; ($u^\backprime$ may for simplicity be supposed >0;) $y^\backprime=\dfrac{w^\backprime y-v^\backprime z}{\sqrt{v^{\backprime 2}+w^{\backprime 2}}}=$ perpendicular distance from that plane which passes through [the axes of] x and $x^\backprime$; & the plane of $x^\backprime y^\backprime$ is of the form

$$x+\lambda^\backprime(v^\backprime y+w^\backprime z)=0,\quad \lambda^\backprime=-\frac{u^\backprime}{v^{\backprime 2}+w^{\backprime 2}},$$

$$\therefore\qquad z^\backprime=\frac{(v^{\backprime 2}+w^{\backprime 2})x-u^\backprime(v^\backprime y+w^\backprime z)}{\sqrt{v^{\backprime 2}+w^{\backprime 2}}\sqrt{u^{\backprime 2}+v^{\backprime 2}+w^{\backprime 2}}}.$$

Accordingly these expressions give $x^{\backprime 2}+y^{\backprime 2}+z^{\backprime 2}=x^2+y^2+z^2$. They may be written thus: Making $u^\backprime=k^\backprime\cos\theta^\backprime$, $v^\backprime=k^\backprime\sin\theta^\backprime\cos\phi^\backprime$, $w^\backprime=k^\backprime\sin\theta^\backprime\sin\phi^\backprime$, we have

$$x^\backprime=x\cos\theta^\backprime+(y\cos\phi^\backprime+z\sin\phi^\backprime)\sin\theta^\backprime;\ \ y'=y\sin\phi^\backprime-z\cos\phi^\backprime;$$

$$z^\backprime=x\sin\theta^\backprime-(y\cos\phi^\backprime+z\sin\phi^\backprime)\cos\theta^\backprime;$$

and they give reciprocally

$$x=x^\backprime\cos\theta^\backprime+z^\backprime\sin\theta^\backprime;\ \ y=(x^\backprime\sin\theta^\backprime-z^\backprime\cos\theta^\backprime)\cos\phi^\backprime+y^\backprime\sin\phi^\backprime;$$

$$z=(x^\backprime\sin\theta^\backprime-z^\backprime\cos\theta^\backprime)\sin\phi^\backprime-y^\backprime\cos\phi^\backprime.$$

Hence

$$\begin{aligned}ux+vy+wz=x^\backprime\{u\cos\theta^\backprime+(v\cos\phi^\backprime+w\sin\phi^\backprime)\sin\theta^\backprime\}&\\ +y^\backprime(v\sin\phi^\backprime-w\cos\phi^\backprime)+z^\backprime\{u\sin\theta^\backprime-(v\cos\phi^\backprime+w\sin\phi^\backprime)\cos\theta^\backprime\};&\end{aligned}$$

---

[*] [This is obtained by considering the contribution of the values of u in the neighbourhood of $u=u^\backprime$ to the value of the integral. It is to be observed that $s^\backprime$ is the same function of $u^\backprime$ as s_1 is of u.]

and we are to multiply both the cosine and sine of this expression by the cosine & sine of $k^{\backprime}x^{\backprime}$, & integrate relatively to $x^{\backprime}$ from $-\infty$ to 0 but relatively to $y^{\backprime}$ and $z^{\backprime}$ from $-\infty$ to ∞. The results, multiplied by $\left(\frac{1}{2\pi}\right)^3$ & by $A^{\backprime}$, $B^{\backprime}$, $C^{\backprime}$, $A^{\backprime}s^{\backprime}$, $B^{\backprime}s^{\backprime}$, $C^{\backprime}s^{\backprime}$, will give $d_0, \ldots i_0'$.

We find, in this way,

$$\begin{aligned}\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \cos k^{\backprime}x^{\backprime} \cos(ux+vy+wz) \\ = \int_0^{\infty} dx^{\backprime} \cos k^{\backprime}x^{\backprime} \cos\left[x^{\backprime}\{u\cos\theta^{\backprime} + (v\cos\phi^{\backprime} + w\sin\phi^{\backprime})\sin\theta^{\backprime}\}\right] \\ \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy^{\backprime}dz^{\backprime} \cos\left[y^{\backprime}(v\sin\phi^{\backprime} - w\cos\phi^{\backprime}) + z^{\backprime}\{u\sin\theta^{\backprime} - (v\cos\phi^{\backprime} + w\sin\phi^{\backprime})\cos\theta^{\backprime}\}\right] \\ = \frac{\pi^{\frac{3}{2}}}{4h^3}\left\{e^{-\left(\frac{k_{,}-k^{\backprime}}{2h}\right)^2} + e^{-\left(\frac{k_{,}+k^{\backprime}}{2h}\right)^2}\right\} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)}\end{aligned}$$

if $k_{,} = u\cos\theta^{\backprime} + (v\cos\phi^{\backprime} + w\sin\phi^{\backprime})\sin\theta^{\backprime}$, and $k = \sqrt{u^2+v^2+w^2}$, as $k^{\backprime} = \sqrt{u^{\backprime 2} + v^{\backprime 2} + w^{\backprime 2}}$. In like manner

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \sin k^{\backprime}x^{\backprime} \sin(ux+vy+wz) = \frac{\pi^{\frac{3}{2}}}{4h^3}\left\{e^{-\left(\frac{k_{,}-k^{\backprime}}{2h}\right)^2} - e^{-\left(\frac{k_{,}+k^{\backprime}}{2h}\right)^2}\right\} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \sin k^{\backprime}x^{\backprime} \cos(ux+vy+wz) = \frac{\pi}{h^2}\frac{k^{\backprime}}{k_{,}^2 - k^{\backprime 2}} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \cos k^{\backprime}x^{\backprime} \sin(ux+vy+wz) = -\frac{\pi}{h^2}\frac{k_{,}}{k_{,}^2 - k^{\backprime 2}} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

in which it is to be remembered that

$$k^2 - k_{,}^2 = (v\sin\phi^{\backprime} - w\cos\phi^{\backprime})^2 + \{u\sin\theta^{\backprime} - (v\cos\phi^{\backprime} + w\sin\phi^{\backprime})\cos\theta^{\backprime}\}^2.$$

Let $\quad u_{,} = k_{,}, \quad v_{,} = v\sin\phi^{\backprime} - w\cos\phi^{\backprime}, \quad w_{,} = u\sin\theta^{\backprime} - (v\cos\phi^{\backprime} + w\sin\phi^{\backprime})\cos\theta^{\backprime};$

then

$$k^2 - k_{,}^2 = v_{,}^2 + w_{,}^2, \quad u = u_{,}\cos\theta^{\backprime} + w_{,}\sin\theta^{\backprime},$$

$$v = (u_{,}\sin\theta^{\backprime} - w_{,}\cos\theta^{\backprime})\cos\phi^{\backprime} + v_{,}\sin\phi^{\backprime}, \quad w = (u_{,}\sin\theta^{\backprime} - w_{,}\cos\theta^{\backprime})\sin\phi^{\backprime} - v_{,}\cos\phi^{\backprime};$$

$$du\,dv\,dw = du_{,}\,dv_{,}\,dw_{,};$$

$$ux+vy+wz = u_{,}x^{\backprime} + v_{,}y^{\backprime} + w_{,}z^{\backprime}; \quad u^2+v^2+w^2 = u_{,}^2 + v_{,}^2 + w_{,}^2 = k^2.$$

Hence

$$\frac{d_0}{A^{\backprime}} = \frac{e_0}{B^{\backprime}} = \frac{f_0}{C^{\backprime}} = \frac{1}{4}\left(\frac{1}{2h\sqrt{\pi}}\right)^3 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \left\{e^{-\left(\frac{u_{,}-k^{\backprime}}{2h}\right)^2} + e^{-\left(\frac{u_{,}+k^{\backprime}}{2h}\right)^2}\right\}:$$

$$\frac{g_0'}{A^{\backprime}s^{\backprime}} = \frac{h_0'}{B^{\backprime}s^{\backprime}} = \frac{i_0'}{C^{\backprime}s^{\backprime}} = \frac{1}{4}\left(\frac{1}{2h\sqrt{\pi}}\right)^3 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}}$$

$$\frac{d_0'}{A^{\backprime}s^{\backprime}} = \frac{e_0'}{B^{\backprime}s^{\backprime}} = \frac{f_0'}{C^{\backprime}s^{\backprime}} = \frac{1}{2\pi}\left(\frac{1}{2h\sqrt{\pi}}\right)^2 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \frac{k^{\backprime}}{u_{,}^2 - k^{\backprime 2}},$$

$$\frac{g_0}{A^{\backprime}} = \frac{h_0}{B^{\backprime}} = \frac{i_0}{C^{\backprime}} = -\frac{1}{2\pi}\left(\frac{1}{2h\sqrt{\pi}}\right)^2 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \frac{u_{,}}{u_{,}^2 - k^{\backprime 2}};$$

these are to be multiplied by $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(u_,x^{\backprime}+v_,y^{\backprime}+w_,z^{\backprime})\,du_,dv_,dw_,$, & also by some known functions of $u_,$, $v_,$, $w_,$, & integrated relatively to $u_,$, $v_,$, $w_,$ from $-\infty$ to ∞.

This integration may be simplified, by 1st suppressing

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_,dw_,\left(\frac{1}{2h\sqrt{\pi}}\right)^2 e^{-\frac{v_,^2+w_,^2}{4h^2}},$$

& then changing $u_,$, $v_,$, $w_,$ to k, 0, 0; which latter change is equivalent to making

$$u=k\cos\theta^{\backprime},\quad v=k\sin\theta^{\backprime}\cos\phi^{\backprime},\quad w=k\sin\theta^{\backprime}\sin\phi^{\backprime}.$$

We are ∵ to multiply

$$\frac{1}{8h\sqrt{\pi}}\left(e^{-\left(\frac{k-k^{\backprime}}{2h}\right)^2}\pm e^{-\left(\frac{k+k^{\backprime}}{2h}\right)^2}\right),\quad -\frac{1}{4\pi}\left(\frac{1}{k-k^{\backprime}}\mp\frac{1}{k+k^{\backprime}}\right),$$

by some known functions of k & t, & to perform the operations $\int_{-\propto}^{\infty}\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}kx^{\backprime}\,dk$.

But
$$\lim_{h=0}\int_{-\infty}^{\infty}dk\,(2h\sqrt{\pi})^{-1}\,e^{-\left(\frac{k\mp k^{\backprime}}{2h}\right)^2}f(k)=f(\pm k^{\backprime});$$

the sign of integration is ∵ affixed only (after all reductions) to terms of the form $\dfrac{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}kx^{\backprime}}{k\mp k^{\backprime}}F(k,t)$. This latter function $F(k,t)$ involves $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}st$; & here also the velocity of propagation is $\dfrac{ds^{\backprime}}{dk^{\backprime}}$.

Thus the parts of the 2 respective sets of integrals

$$\begin{array}{cc}
\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}dw\,d_t\cos(ux+vy+wz), & \int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}dw\,g_t\sin(ux+vy+wz),\\
e_t\qquad\qquad , & h_t\qquad\qquad ,\\
f_t\qquad\qquad , & i_t\qquad\qquad ,
\end{array}$$

which are respectively even and odd functions of t, are

$$\begin{array}{ll}
\tfrac{1}{2}A^{\backprime}\cos k^{\backprime}x^{\backprime}\cos s^{\backprime}t, & \tfrac{1}{2}A^{\backprime}\sin k^{\backprime}x^{\backprime}\sin s^{\backprime}t,\\
B^{\backprime}\qquad\qquad , & B^{\backprime}\qquad\qquad ,\\
C^{\backprime}\qquad\qquad , & C^{\backprime}\qquad\qquad ;
\end{array}$$

& ∵ by adding these we obtain

$$\tfrac{1}{2}A^{\backprime}\cos(k^{\backprime}x^{\backprime}-s^{\backprime}t),\quad \tfrac{1}{2}B^{\backprime}\cos(k^{\backprime}x^{\backprime}-s^{\backprime}t),\quad \tfrac{1}{2}C^{\backprime}\cos(k^{\backprime}x^{\backprime}-s^{\backprime}t),$$

as the parts of the final expressions for $(\delta x)_t$, $(\delta y)_t$, $(\delta z)_t$ which do not (after all reductions) involve the sign of integration.

The other parts will be less simple. The parts of $\iiint\ldots d_t\cos(ux+vy+wz)$, &c., which are odd functions of t, are the sums of the three values of

$$\frac{s^{\backprime}k^{\backprime}}{2\pi}\int_{-\infty}^{\infty}\frac{dk\cos kx^{\backprime}\sin st}{s(k^2-k^{\backprime 2})}A\,(AA^{\backprime}+BB^{\backprime}+CC^{\backprime}),\ \text{\&c.},$$

corresponding to the three systems of values of A, B, C and s; and the parts of

$$\iiint \dots g_t \sin(ux+vy+wz), \text{ \&c.},$$

which are even functions of t, are the sums of the 3 values of

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{k \sin kx' \cos st}{k^2 - k'^2} A\,(AA' + BB' + CC'), \text{ \&c.}$$

Thus the whole remaining parts of $(\delta x)_t$, $(\delta y)_t$, $(\delta z)_t$ are the sums of the 3 values of

$$\int_0^{\infty} dk \frac{s'k' \cos kx' \sin st - sk \sin kx' \cos st}{\pi s\,(k^2 - k'^2)} A\,(AA' + BB' + CC'), \text{ \&c.},$$

which sums consequently are

$$= -\tfrac{1}{2}A' \cos(k'x' - s't) + (\delta x)_t, \text{ \&c.}$$

And thus we express rigorously the effect at the time t, corresponding to an initial system of velocities and displacements extending only to half the medium. And we confirm the theorem that the velocity of propagation of vibration is $\frac{ds'}{dk'}$; for we find that the vibration does not sensibly attain a place x' till $x' < t\,\frac{ds'}{}$.

XXI.

RESEARCHES ON THE DYNAMICS OF LIGHT

Read Feb. 11th, 1839.

[*Proceedings Royal Irish Academy,* Vol. I (1836–1840), pp. 267–270.]

An indefinite series of equal and equally distant particles, $\ldots m_{-1}, m_0, m_1, \ldots$, situated in the axis of x, at the points $\ldots -1, 0, +1, \ldots$, being supposed to receive, at the time 0, any very small transversal displacements $\ldots y_{-1,0}, y_{0,0}, y_{1,0}, \ldots$, and any very small transversal velocities $\ldots y'_{-1,0}, y'_{0,0}, y'_{1,0}, \ldots$, it is required to determine their displacements $\ldots y_{-1,t}, y_{0,t}, y_{1,t}, \ldots$ for any other time t; each particle being supposed to attract the one which immediately precedes or follows it in the series, with an energy* $= a^2$, and to have no sensible influence on any of the more distant particles. This problem may be considered as equivalent to that of integrating generally the equation in mixed differences,

$$y''_{x,t} = a^2 (y_{x+1,t} - 2y_{x,t} + y_{x-1,t}); \tag{1}$$

which may also be thus written:

$$\left(\frac{d}{dt}\right)^2 y_{x,t} = \frac{(a\Delta_x)^2}{1+\Delta_x} y_{x,t}. \tag*{(1)$'$}$$

The general integral required may be thus written:

$$y_{x,t} = \left\{ 1 - \frac{a^2\Delta_x^2}{1+\Delta_x} \left(\int_0^t dt \right) \right\} \quad (y_{x,0} + ty'_{x,0}); \tag{2}$$

an expression which may be developed into the sum of two series, as follows,

$$\begin{aligned} y_{x,t} = y_{x,0} &+ \frac{a^2t^2}{1\,.\,2} \Delta_x^2 y_{x-1,0} + \frac{a^4t^4}{1\,.\,2\,.\,3\,.\,4} \Delta_x^4 y_{x-2,0} + \&\text{c}. \\ &+ ty'_{x,0} + \frac{a^2t^3}{1\,.\,2\,.\,3} \Delta_x^2 y'_{x-1,0} + \frac{a^4t^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} \Delta_x^4 y'_{x-2,0} + \&\text{c}.; \end{aligned} \tag*{(2)$'$}$$

and may be put under this other form,

$$\begin{aligned} y_{x,t} = \frac{2}{\pi} \Sigma_{(l)-\infty}^{\infty}\, y_{x+l,0} \int_0^{2} d\theta \cos(2l\theta) \cos(2at \sin\theta) \\ + \frac{1}{a\pi} \Sigma_{(l)-\infty}^{\infty}\, y'_{x+l,0} \int_0^{\overline{2}} d\theta \cos(2l\theta) \operatorname{cosec}\theta \sin(2at\sin\theta); \end{aligned} \tag*{(2)$''$}$$

the first line of (2)$'$ or (2)$''$ expressing the effect of the initial displacements, and the second line expressing the effect of the initial velocities, for all possible suppositions respecting these initial data, or for all possible forms of the two arbitrary functions $y_{x,0}$ and $y'_{x,0}$.

Supposing now that these arbitrary forms or initial conditions are such, that

$$y_{x,0} = \eta \operatorname{vers} \frac{2x\pi}{n}, \quad \text{and} \quad y'_{x,0} = -2a\eta \sin\frac{\pi}{n} \sin\frac{2x\pi}{n} \tag{3}$$

---

* [I.e. force.]

for all values of the integer x between the limits 0 and $-in$, n and i being positive and large, but finite integer numbers, and that for all other values of x the functions $y_{x,0}$ and $y'_{x,0}$ vanish: which is equivalent to supposing that at the origin of t, and for a large number i of wave-lengths (each $=n$) behind the origin of x, the displacements and velocities of the particles are such as to agree with the following law of undulatory vibration,

$$y_x = \eta \operatorname{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right), \tag{3$'$}$$

but that all the other particles are, at that moment, at rest: it is required to determine the motion which will ensue, as a consequence of these initial conditions. The solution is expressed by the following formula, which is a rigorous deduction from the equation in mixed differences (1):

$$y_{x,t} = \frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\sin in\theta}{\sin\theta}\,\frac{\cos(2x\theta + in\theta - 2at\sin\theta)}{\cos\theta - \cos\frac{\pi}{n}}\,d\theta; \tag{4}$$

an expression which tends indefinitely to become*

$$y_{x,t} = \frac{\eta}{2}\operatorname{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\sin(2x\theta - 2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)}, \tag{4$'$}$$

as the number i increases without limit. The approximate values are discussed, which these rigorous integrals acquire, when the value of t is large. It is found that a vibration, of which the phase and the amplitude agree with the law (3)′, is propagated forward, but not backward, so as to agitate successively new and more distant particles, (and to leave successively others at rest, if i be finite,) with a velocity of progress which is expressed by $a\cos\frac{\pi}{n}$, and which is therefore less, by a finite though small amount, than the velocity of passage $a\frac{n}{\pi}\sin\frac{\pi}{n}$ of any given phase, from one vibrating particle to another within that extent of the series which is already fully agitated. In other words, the communicated vibration does not attain a sensible amplitude, until a finite interval of time has elapsed from the moment when one should expect it to begin, judging only by the law of the propagation of phase through an indefinite series of particles, which are all in vibration already. A small disturbance, distinct from the vibration (3)′, is also propagated, backward as well as forward, with a velocity $=a$, independent of the length of the wave. And all these propagations are accompanied with a small degree of terminal diffusion, which, after a very long time, renders all the displacements insensible, if the number i, however large, be finite, that is, if the vibration be originally limited to any finite number of particles.

---

* [See p. 554.]

XXII.

RESEARCHES RESPECTING VIBRATION CONNECTED WITH THE THEORY OF LIGHT*

Read June 24th, 1839.

[*Proceedings Royal Irish Academy*, Vol. I (1836–1840), pp. 341–349.]

It is proposed to integrate the system of equations in mixed differences,

$$D_t^2\,\delta x_{g,h}=\Sigma_{\Delta g}\,\delta\,(R\,.\,\Delta_g x_{g,h}); \tag{1}$$

in which h is any integer number from 1 to n inclusive; $x_{g,h}$ is independent of t, but $\delta x_{g,h}$ is a function of t and of $x_{g,1},\ \dots\ x_{g,n}$, the form of which function it is the object of the problem to discover;

$$R=m_{g+\Delta g}\,\phi\,\{\tfrac{1}{2}\Sigma_{(h)1}^{n}\,(\Delta_g x_{g,h})^2\}, \tag{2}$$

ϕ being any real function of the semi-sum which follows it, and m being any other real function of the index $g+\Delta g$; while g and $g+\Delta g$ represent any integer numbers from negative to positive infinity. The equations to be integrated may also be thus written:

$$\xi''_{g,h,t}=\Sigma_{\Delta g}\,(R\Delta_g\xi_{g,h,t}+R'\Delta_g x_{g,h}\,\Sigma_{(h)1}^{n}\,\Delta_g x_{g,h}\,\Delta_g\xi_{g,h,t}), \tag*{(1)$'$}$$

in which

$$R'=m_{g+\Delta g}\,\phi'\,\{\tfrac{1}{2}\Sigma_{(h)1}^{n}\,(\Delta_g x_{g,h})^2\}; \tag*{(2)$'$}$$

the functions to be found by integration are now those of the form $\xi_{g,h,t}$, considered as depending on t and on $x_{g,1},\ \dots\ x_{g,n}$; their initial values, and initial rates of increase (relatively to t), namely $\xi_{g,h,0}$ and $\xi'_{g,h,0}$, are regarded as arbitrary but given and real functions of $x_{g,1},\ \dots\ x_{g,n}$; it is also supposed, in order to simplify the question, that all the sums of the forms

$$\Sigma_{\Delta g}\,R\,(\Delta_g x_{g,1})^{\alpha_1}\dots(\Delta_g x_{g,n})^{\alpha_n},\quad \Sigma_{\Delta g}\,R'\,(\Delta_g x_{g,1})^{\alpha_1}\dots(\Delta_g x_{g,n})^{\alpha_n} \tag{3}$$

are independent of g, and are $=0$ when any one of the exponents $\alpha_1,\ \dots\ \alpha_n$ is an odd number. These equations are analogous to, and include, those which M. Cauchy has considered in his memoir on the Dispersion of Light, and may be integrated by a similar analysis.

A particular integral system may in the first place be found by assuming

$$\xi_{g,h,t}=X_r A_{h,r}\cos\,(\epsilon_r+s_r t-\Sigma_{(i)1}^{n}\,u_i x_{g,i}); \tag{4}$$

$$\Sigma_{(h)1}^{n}\,A_{h,r}^2=1; \tag{5}$$

$$s_r^2 A_{h,r}=\Sigma_{(i)1}^{n}\,H_{h,i}A_{i,r}; \tag{6}$$

$$H_{h,h}=\Sigma_{\Delta g}\,\{R+R'\,(\Delta_g x_{g,h})^2\}\,\mathrm{vers}\,(\Sigma_{(i)1}^{n}\,u_i\Delta_g x_{g,i}), \tag*{(7)†}$$

$$H_{h,i}=\Sigma_{\Delta g}\,R'\,\Delta_g x_{g,h}\Delta_g x_{g,i}\,\mathrm{vers}\,(\Sigma_{(i)1}^{n}\,u_i\Delta_g x_{g,i}); \tag*{(7)$'$}$$

the index r being any integer from 1 to n, and being introduced in order to distinguish among themselves the n different (and in general real) systems of values of s^2, and of the $n-1$ ratios of $A_1, \dots A_h, \dots A_n$, which are obtained by resolving the system of the n equations of the form

---

* [A rough draft of this paper appears in Note Book 53 (1839).]

† [We easily get $\Delta_g\xi_{g,h,t}=-\xi_{g,h,t}\,\mathrm{vers}\,\Sigma_{(i)1}^{n}u_i\Delta_g x_{g,i}+$ terms which vanish on summation with respect to Δg by (3). Equations (3) in fact define the symmetry of the medium.]

$$s^2 A_h = \Sigma_{(i)1}^{n} H_{h,i} A_i, \tag{6$'$}$$

in which, by (7)$'$,

$$H_{i,h} = H_{h,i}. \tag{7$''$}$$

It is important to observe, that by the form of these equations (6)$'$, (which occur in many researches,) we have the relation

$$\Sigma_{(h)1}^{n} A_{h,q} A_{h,r} = 0, \tag{5$'$}$$

if q be different from r; and that, by (5) and (5)$'$, we have also the relations

$$\Sigma_{(r)1}^{n} A_{h,r}^2 = 1, \tag{8}$$

$$\Sigma_{(r)1}^{n} A_{h,r} A_{i,r} = 0. \tag{8$'$}$$

In the particular integral (4), we may consider $u_1, \dots u_n$ as arbitrary parameters, of which X_r and ϵ_r are real and arbitrary, while s_r^2 and $A_{h,r}$ are real and determined functions; and hence, by summations relatively to the index r, and integrations relatively to the parameters u_i, employing also the relations (5) (5)$'$ (8) (8)$'$, and Fourier's theorem extended to several variables, we deduce this general integral, applying to all arbitrary real values of the initial data:

$$\xi_{g,h,t} = \left(\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} du_i \right) (E_{h,t} \cos + F_{h,t} \sin)\, \Sigma_{(i)1}^{n} u_i x_{g,i}; \tag{9}$$

in which

$$\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} du_i = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \dots \int_{-\infty}^{\infty} du_n; \tag{10}$$

$$\left. \begin{aligned} E_{h,t} &= \Sigma_{(r)1}^{n} A_{h,r} \,(Y_r \cos ts_r + Y'_r s_r^{-1} \sin ts_r), \\ F_{h,t} &= \Sigma_{(r)1}^{n} A_{h,r} \,(Z_r \cos ts_r + Z'_r s_r^{-1} \sin ts_r); \end{aligned} \right\} \tag{11}$$

$$\left. \begin{aligned} Y_r &= \Sigma_{(h)1}^{n} A_{h,r} E_{h,0}, \quad & Y'_r &= \Sigma_{(h)1}^{n} A_{h,r} E'_{h,0}, \\ Z_r &= \Sigma_{(h)1}^{n} A_{h,r} F_{h,0}, \quad & Z'_r &= \Sigma_{(h)1}^{n} A_{h,r} F'_{h,0}; \end{aligned} \right\} \tag{12}$$

$$\left. \begin{aligned} E_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} dx_{g,i} \right) \xi_{g,h,0} \cos\, (\Sigma_{(i)1}^{n} u_i x_{g,i}), \\ E'_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} dx_{g,i} \right) \xi'_{g,h,0} \cos\, (\Sigma_{(i)1}^{n} u_i x_{g,i}), \\ F_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} dx_{g,i} \right) \xi_{g,h,0} \sin\, (\Sigma_{(i)1}^{n} u_i x_{g,i}), \\ F'_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)1}^{n} \int_{-\infty}^{\infty} dx_{g,i} \right) \xi'_{g,h,0} \sin\, (\Sigma_{(i)1}^{n} u_i x_{g,i}). \end{aligned} \right\} \tag{13}$$

This general solution involves multiple integrals, of the order $2n$; but many particular suppositions, respecting the initial data, conduct to simpler expressions, among which the following appear worthy of remark.

Suppose that having assumed some particular set $u_1^{\backprime}, \dots u_n^{\backprime}$, of values of the n arbitrary quantities $u_1, \dots u_n$, we deduce a corresponding set of coefficients $H_{h,h}^{\backprime}, H_{h,i}^{\backprime}$, by the formulae

(7) and (7)′, and represent by $s_1^{\backprime 2}$ and by $A^{\backprime}_{1,1}, \dots A^{\backprime}_{h,1}, \dots A^{\backprime}_{n,1}$, some one corresponding system of quantities which satisfy the equations

$$\Sigma^{n}_{(h)1} A^{\backprime 2}_{h,1} = 1, \tag*{(5)$^{\backprime}$}$$

$$s_1^{\backprime 2} A^{\backprime}_{h,1} = \Sigma^{n}_{(i)1} H^{\backprime}_{h,i} A^{\backprime}_{i,1}; \tag*{(6)$^{\backprime}$}$$

we shall then have, as a particular integral system, that which is thus denoted:

$$\xi_{g,h,t} = X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + s^{\backprime}_1 t - \Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i}); \tag*{(4)$^{\backprime}$}$$

$X^{\backprime}_1$ and $\epsilon^{\backprime}_1$ denoting here any arbitrary real quantities. If therefore we suppose that the initial data $\xi_{g,h,0}$ and $\xi'_{g,h,0}$ are all such as to agree with this particular solution, that is, if we have, for all values of g and h,

$$\xi_{g,h,0} = \quad X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 - \Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i}), \tag*{(14)}$$

$$\xi'_{g,h,0} = -s^{\backprime}_1 X^{\backprime}_1 A^{\backprime}_{h,1} \sin(\epsilon^{\backprime}_1 - \Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i}), \tag*{(14)′}$$

we see, *à priori*, that the multiple integrations ought to admit of being all effected in finite terms, so as to reduce the general expression (9) to the particular form (4)$^{\backprime}$; an expectation which the calculation, accordingly, *à posteriori*, proves to be correct. An analogous but less simple reduction takes place, when we suppose that the initial equations (14) and (14)′ hold good, after their second members have been multiplied by a discontinuous factor such as

$$\frac{1}{2}\left(1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(k\,\Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i})\,dk}{k}\right), \tag*{(15)}$$

which is $=1$, or $=\frac{1}{2}$, or $=0$, according as the sum $\Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i}$ is <0, or $=0$, or >0. It is found that, in this case, the $2n$ successive integrations (required for the general solution) can in part be completely effected, and in the remaining part be reduced to the calculation of a simple definite integral; in such a manner that the expression (9) now reduces itself rigorously to the following:*

$$\xi_{g,h,t} = \tfrac{1}{2} X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + t s^{\backprime}_1 - \Sigma^{n}_{(i)1} u^{\backprime}_i x_{g,i}) + \frac{1}{\pi} X^{\backprime}_1 \int_0^\infty \frac{dk}{k^2 - k^{\backprime 2}} (L_t \cos\epsilon^{\backprime}_1 + M_t \sin\epsilon^{\backprime}_1); \tag*{(16)}$$

in which

$$\left.\begin{aligned} L_t &= P_t k^{\backprime} \cos kx - Q_t k \sin kx, \\ M_t &= P_t k \sin kx + Q_t k^{\backprime} \cos kx, \end{aligned}\right\} \tag*{(17)}$$

$$\begin{aligned} P_t &= s^{\backprime}_1 \Sigma^{n}_{(r)1} (A_{h,r} s_r^{-1} \sin t s_r \,.\, \Sigma^{n}_{(h)1} A_{h,r} A^{\backprime}_{h,1}), \\ Q_t &= \Sigma^{n}_{(r)1} (A_{h,r} \cos t s_r \,.\, \Sigma^{n}_{(h)1} A_{h,r} A^{\backprime}_{h,1}), \end{aligned} \tag*{(18)}$$

$$x = \Sigma^{n}_{(i)1} a^{\backprime}_i x_{g,i}, \tag*{(19)}$$

$$k a^{\backprime}_i = u_i, \quad k^{\backprime} a^{\backprime}_i = u^{\backprime}_i, \quad k^{\backprime 2} = \Sigma^{n}_{(i)1} u^{\backprime 2}_i, \tag*{(20)}$$

and s_r, $A_{h,r}$ are the same functions as before of $u_1, \dots u_n$.

A remarkable conclusion may now be drawn from these expressions, by supposing that all the quantities of the form s_r^2 are not only real but positive, so that the functions $\cos t s_r$ and $\sin t s_r$

---

* [The usual methods of integration lead to an integral of the form $\int_{-\infty}^{\infty} \{f(k+k^{\backprime}) - f(k-k^{\backprime})\}\,dk/k$. This can be written as above $4k^{\backprime}\int_0^\infty f(k)\,dk/(k^2 - k^{\backprime 2})$, where $f(k)$ is an even function of k. This integral, as usual, must be interpreted as Cauchy's Principal Value. Hamilton's method of obtaining these integrals is given on pp. 469 *et seq.*]

are periodic. For in this case the functions $\cos(ts_r \pm kx)$ and $\sin(ts_r \pm kx)$ will vary rapidly, and pass often through all their fluctuations of value, between the limits 1 and -1, while k and the other functions of that variable remain almost unchanged, provided that $t\frac{ds_r}{dk} \pm x$ is large, and that the denominator $k^2 - k^{\backprime 2}$ is not extremely small. We may therefore in general confine ourselves to the consideration of small values of this denominator; and consequently may put it under the form $2k^{\backprime}(k - k^{\backprime})$, making $k = k^{\backprime}$ in the numerator, except under the periodical signs, and integrating relatively to k between any two limits which include $k^{\backprime}$, for example between $-\infty$ and $+\infty$. And because

$$\Sigma^{n}_{(h)1} A^{\backprime}_{h,r} A^{\backprime}_{h,1} = 1, \quad \text{or } = 0,$$

according as $r = 1$ or > 1, we may make

$$P_t = A^{\backprime}_{h,1} \sin ts_1, \quad Q_t = A^{\backprime}_{h,1} \cos ts_1,$$

$$L_t = k^{\backprime} A^{\backprime}_{h,1} \sin(ts_1 - kx), \quad M_t = k^{\backprime} A^{\backprime}_{h,1} \cos(ts_1 - kx)$$

and*

$$\xi_{g,h,t} = \tfrac{1}{2} X^{\backprime}_1 A^{\backprime}_{h,1} \left\{ \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime}x) + \int_{-\infty}^{\infty} dk \frac{\sin(\epsilon^{\backprime}_1 + ts_1 - kx)}{\pi(k - k^{\backprime})} \right\}, \tag{21}$$

that is, nearly, if x be considerably different from $t\frac{ds^{\backprime}_1}{dk}$

$$\xi_{g,h,t} = \tfrac{1}{2} X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime}x) \left\{ 1 + \int_{-\infty} \frac{dk}{\pi k} \sin\left(t\frac{ds^{\backprime}_1}{dk^{\backprime}} - x \right) k \right\} \tag{21$'$}$$

We have therefore the approximate expressions:

$$\xi_{g,h,t} = X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime}x), \quad \text{if} \quad x < t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22}$$

and

$$\xi_{g,h,t} = 0, \quad \text{if} \quad x > t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22$'$}$$

we have also nearly, in general,

$$\xi_{g,h,t} = \tfrac{1}{2} X^{\backprime}_1 A^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime}x), \quad \text{if} \quad x = t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22$''$}$$

but the discussion of the case when x is nearly $= t\frac{ds^{\backprime}_1}{dk^{\backprime}}$ is too long to be cited here.† The formula (22) for $\xi_{g,h,t}$ coincides with the particular integral $(4)^{\backprime}$; and the condition which it involves with respect to x expresses the law according to which this particular integral comes to be (nearly) true for greater and greater positive values of x and t, $\left(\text{if } \frac{ds^{\backprime}_1}{dk^{\backprime}} > 0,\right)$ after having been true only for negative values of x when t was $= 0$.

In the particular case $n = 3$, the foregoing formulae have an immediate dynamical application, and correspond to the propagation of vibratory motion through a system of mutually attracting or repelling particles; and they conduct to this remarkable result, that the velocity with which such vibration spreads into those portions of the vibratory medium which were previously undisturbed is in general different from the velocity of a passage of a given phase

* [(21) is a Principal Value Integral, but when a part is removed by the Principle of Fluctuation (cf. p. 520), this restriction is unnecessary in (21)′.]

† [See pp. 565, 566.]

from one particle to another within that portion of the medium which is already fully agitated; since we have

$$\textit{velocity of transmission of phase} = \frac{s}{k}, \tag{A}$$

but

$$\textit{velocity of propagation of vibratory motion} = \frac{ds}{dk}, \tag{B}$$

if the rectangular components of the vibrations themselves be represented by the formulae

$$XA_1 \cos(\epsilon + st - kx), \quad XA_2 \cos(\epsilon + st - kx), \quad XA_3 \cos(\epsilon + st - kx), \tag{C}$$

t being the time, and x being the perpendicular distance of the vibrating point from some determined plane.

This result, which is believed to be new, includes as a particular case that which was stated in a former communication to the Academy,* on the 11th of February last, (Proceedings, No. 15, p. 269,) respecting the propagation of transversal vibration along a row of equal and equidistant particles, of which each attracts the two that are immediately before and behind it; in which particular question s was $= 2a \sin \frac{k}{2}$, and the velocity of propagation of vibration was $= a \cos \frac{k}{2}$. Applied to the theory of light, it appears to show that if the phase of vibration in an ordinary dispersive medium be represented for some one colour by

$$\epsilon + \frac{2\pi}{\lambda}\left(\frac{t}{\mu} - x\right), \tag{C$'$}$$

so that λ is the length of an undulation for that colour and for that medium, and if it be permitted to represent dispersion by developing the velocity $\frac{1}{\mu}$ of the transmission of phase in a series of the form

$$\frac{1}{\mu} = M_0 - M_1\left(\frac{2\pi}{\lambda}\right)^2 + M_2\left(\frac{2\pi}{\lambda}\right)^4 - \&c., \tag{A$'$}$$

then the *velocity wherewith light of this colour conquers darkness*, in this dispersive medium, by the *spreading of vibration into parts which were not vibrating before, is somewhat less than* $\frac{1}{\mu}$, being represented by this other series

$$M_0 - 3M_1\left(\frac{2\pi}{\lambda}\right)^2 + 5M_2\left(\frac{2\pi}{\lambda}\right)^4 - \&c. \tag{B$'$}$$

For other details of this inquiry it is necessary to refer to the memoir itself, which will be published in the Transactions of the Academy, and will be found to contain many other investigations respecting vibrating systems, with applications to the theory of light.†

---

* [See pp. 576, 577.]

† [This memoir was never published but manuscripts XIX and XX, pp. 451–575, formed undoubtedly material for this work.]

XXIII.

CORRESPONDENCE

[Note Book 39.]

[*To Professor Powell.**]

Observatory, Dublin,
Oct. 24*th*, 1835.

MY DEAR SIR

I received your letter a few days ago, & had even previously been intending to write to you on the subject of dispersion. My opinion of the value of your researches on that subject has not undergone any change, & I continue to consider you successful, and believe you to be the first who has been so, in the arduous & important enterprise of practically establishing the existence of a near agreement between the facts of dispersion & a formula suggested by theory.

But it occurred to me some time ago, and I have been intending to mention it to you, that this formula does not (in my judgment) follow as a *rigorous*, though it does as an *approximate*, consequence from Cauchy's mathematical investigations. In short, I think that you are not strictly warranted in omitting a certain sign of summation, although the final result is only slightly affected by that omission.

To simplify the question, let us conceive a plane wave perpendicular to the axis of x, with vibrations parallel to the axis of y. Then the displacements ξ and ζ will vanish, & we shall have the formula

$$\frac{d^2\eta}{\ } = S\left\{m\,\frac{\mathrm{f}(r)+\cos\beta^2 f(r)}{\ }\,\Delta\eta\right\}, \tag{1}$$

by Cauchy's theory, or by your equations (12) in the Phil. Mag. for Jan. last; η being the value at the time t of the varying displacement of the molecule m which has x, y, z for its rectangular coordinates of equilibrium; & $\eta+\Delta\eta$ being the displacement, at the same moment t, of another molecule m, which has for its rectangular coordinates of equilibrium $x+\Delta x$, $y+\Delta y$, $z+\Delta z$; while r is the distance $\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ between these two molecules in their positions of equilibrium, & β is the angle between this distance & the axis of y; & finally $\mathrm{f}(r)$ and $f(r)$ are functions of r, of which the former (if positive) expresses the law of attraction, or (if negative) the law of repulsion, and the latter is derived from it by the rule

$$f(r) = r\mathrm{f}'(r) - \mathrm{f}(r):$$

and S is a sign of summation, relative to the actions (attractive or repulsive) of all the molecules m.

If now we suppose
$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{2}$$

η_0, η_1, t_0 & λ, μ being constants, we shall have

$$\frac{d^2\eta}{dt^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{3}$$

---

* [Baden Powell, 1796–1860. Savilian Professor of Geometry, Oxford, 1827–60. Published researches on Optics and Radiation.]

and

$$\Delta\eta = -2\eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right)\left(\sin\frac{\pi\mu\Delta x}{\lambda}\right)^2 - \eta_1 \sin\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right)\sin\frac{2\pi\mu\Delta x}{\lambda}; \tag{4}$$

so that, the 2nd part of $\Delta\eta$ disappearing in the summation, the differential equation of vibration (1) is satisfied, if we establish the following relation between the constants λ, μ,

$$\left(\frac{2\pi}{\lambda}\right)^2 = S\left\{2m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{\ }\left(\sin\frac{\pi\mu\Delta x}{\ }\right)^{2}\right. \tag{5}$$

Such is, I think, the law of the velocity $\frac{1}{\mu}$ of propagation of a wave, considered as depending on the periodical time λ of the vibrations of the molecules, & deduced from the principles of Cauchy. If we put for abridgment

$$\frac{\pi\mu\Delta x}{\lambda} = \theta, \tag{6}$$

and

$$\frac{m}{2}\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta x^2 = H^2, \tag{7}$$

(although this quantity H^2 is not necessarily nor always positive), the law becomes

$$\left(\frac{1}{\mu}\right)^2 = S\,.\,H^2\left(\frac{\sin\theta}{\theta}\right)^2, \tag{8}$$

in which it appears necessary, for rigour, to retain the sign of summation. To omit that sign would come to considering the action of only two near molecules; and even to apply the sign to H^2 only, without treating θ as variable, would come to considering only the action of two near layers of molecules, for which Δx^2 is the same: whereas it is likely that the number of such near layers, within the extent of sensible action, is on the contrary very great. Yet, after all, the law of their total action is almost the same in kind as the law of the action of those two. For if we suppose, as it seems to be permitted to do, that θ (which is the *semidifference of phases* of two near molecules) is small, though not insensible, within the extent of sensible action; we may then develope, according to the ascending powers of this small arc, the square of the ratio of its sine to itself,

$$\left(\frac{\sin\theta}{\theta}\right)^2 = 1 - \frac{\theta^2}{3} + \frac{2\theta^4}{45} - \&\text{c.} \tag{9}$$

$$= 1 - \frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2\Delta x^2 + \frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4\Delta x^4 - \&\text{c.},$$

and thus the law (8) will become

$$\left(\frac{1}{\mu}\right)^2 = S(H^2) - \frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2 S(H^2\Delta x^2) + \frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4 S(H^2\Delta x^4) - \&\text{c.} \tag{10}$$

And *if this series converge rapidly enough*, it will give, *nearly*,

$$\left(\frac{1}{\mu}\right)^2 = S(H^2) - \frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{S(H^2)}, \tag{11}$$

$$\frac{1}{\mu} = \sqrt{S(H^2)}\left\{1 - \frac{1}{6}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{\{S(H^2)\}^2}\right\}, \tag{12}$$

and finally

$$\frac{1}{\mu} = H\;\frac{\sin\theta,}{\theta.}, \tag{13}$$

if we put for abridgment

$$H_{,} = {}' \tag{14}$$

and

$$\theta_{,} = \frac{\pi \sqrt{S\,(H^2 \Delta x^2)}}{\lambda S\,(H^2)}. \tag{15}$$

But if we develope $\left(\frac{1}{\mu}\right)^2$ according to the powers of $\left(\frac{\pi}{\lambda}\right)^2$ by the approximate formula (13), we get

$$\left(\frac{1}{\mu}\right)^2 = S\,(H^2) - \frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2 \frac{S\,(H^2\Delta x^2)}{S\,(H^2)} + \frac{2}{45}\left(\frac{\pi}{\lambda}\right)^4 \frac{\{S\,(H^2\Delta x^2)\}^2}{\{S\,(H^2)\}^3} - \&c.; \tag{16}$$

whereas if we develope it by the more accurate formula (10), we find

$$\left(\frac{1}{\mu}\right)^2 = S\,(H^2) - \frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2 \frac{S\,(H^2\Delta x^2)}{S\,(H^2)} + \left(\frac{\pi}{\lambda}\right)^4 \left\{\frac{2}{45}\frac{S\,(H^2\Delta x^4)}{\{S\,(H^2)\}^2} - \frac{1}{9}\frac{\{S\,(H^2\Delta x^2)\}^2}{\{S\,(H^2)\}^3}\right\} - \&c.; \tag{17}$$

and it seems possible enough that the difference of the third terms may be sensible.—Had we taken as our approximate formula

$$\frac{1}{\mu} = H_{,} \quad \sin\theta_{,,} \tag{18}$$

in which

$$H_{,,} = H_{,} = \sqrt{S\,(H^2)}, \tag{19}$$

and

$$= \frac{\pi\mu}{\lambda} \quad \sqrt{\frac{S\,(H^2\Delta x^2)}{S\,(H^2)}} \tag{20}$$

we should have had the approximate development

$$\left(\frac{1}{\mu}\right)^2 = S\,(H^2) - \frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2 S\,(H^2\Delta x^2) + \frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4 \frac{\{S\,(H^2\Delta x^2)\}^2}{S\,(H^2)} - \&c., \tag{21}$$

which would resemble more the original development (10), but still might differ sensibly from it, because the two expressions,

$$S\,(H^2\Delta x^4) \quad \text{and} \quad \frac{\{S\,(H^2\Delta x^2)\}^2}{S\,(H^2)}$$

are not in general coincident.—I conclude therefore that your simplification of Cauchy's results, obtained by omitting the sign of summation, or by attending only to the action of one pair of points, or at most one pair of layers, may be attended with some loss of accuracy, & indeed is likely to be so *if the third term* $\left(\text{or the term proportional to } \frac{1}{\lambda^4}\right)$ *in the development of* $\frac{1}{\mu}$ *be sensible.* And perhaps some of the differences which you have met, between the observed and calculated indices, may have their source in the insufficient expression of this third term. You will judge whether attention to this suggestion might not increase the value of your already valuable researches.—

For my own part, when my attention had been awakened by your success in reconciling so far as you have done the facts with the theory of dispersion, & when I came to perceive that this theory required in rigour the retaining of the sign of summation, (which I did only about a month ago, having been occupied till then by other things, especially by Halley's Comet,) I instituted several numerical calculations, some account of which you may like to have. The immediate result of theory appears, for the reasons that I have given, to be a law which may be expressed by a series of the form

$$\left(\frac{1}{\mu}\right)^2 = A_0 - A_1\left(\frac{\mu}{\lambda}\right)^2 + A_2\left(\frac{\mu}{\lambda}\right)^4 - \&\text{c.} \tag{22}$$

the general term being

$$\frac{\left(-\frac{4\pi^2\mu^2}{\lambda^2}\right)^i}{1\,.\,2\,.\,3\,.\,4\ldots(2i+2)}\, S\left\{m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta x^{2i+2}\right\}, \tag{23}$$

in which we may, if we choose, change Δx to $r\cos\alpha$. And it seems likely that in all useful applications of the law this series will converge; and also that the terms will be alternately positive and negative, or at least that the first term A_0 will be positive and the second term negative. Moreover it may be expected that the resultant development of μ itself, according to the ascending integer powers of $\left(\frac{1}{\lambda}\right)$, will also be convergent; namely the series

$$\mu = a_0 + a_1\lambda^{-2} + a_2\lambda^{-4} + \&\text{c.}, \tag{24}$$

of which the coefficients may be determined by equating to zero the coefficients of the several powers of λ^{-2} in the following identical development, deduced from (22) by substitution of the value (24) for μ, namely

$$\begin{aligned}0 = -1 + A_0(a_0 + a_1\lambda^{-2} + a_2\lambda^{-4} + \&\text{c.})^2 - A_1\lambda^{-2}(a_0 + a_1\lambda^{-2} + \&\text{c.})^4 \\ + A_2\lambda^{-4}(a_0 + \&\text{c.})^6 - \&\text{c.}\end{aligned} \tag{25}$$

For the first three of the new coefficients, we have thus the three equations,

$$\begin{aligned}0 &= -1 + A_0 a_0^2,\\ 0 &= 2A_0 a_0 a_1 - A_1 a_0^4,\\ 0 &= 2A_0 a_0 a_2 + A_0 a_1^2 - 4A_1 a_0^3 a_1 + A_2 a_0^6;\end{aligned} \tag{26}$$

which give

$$\left.\begin{aligned}a_0 &= A_0^{-\frac{1}{2}},\\ a_1 &= \tfrac{1}{2}A_1 A_0^{-\frac{5}{2}},\\ a_2 &= -\tfrac{1}{2}A_2 A_0^{-\frac{7}{2}} + \tfrac{7}{8}A_1^2 A_0^{-\frac{9}{2}}.\end{aligned}\right\} \tag{27}$$

If we may confine ourselves to these three, the law of dispersion becomes simply

$$\mu = a_0 + a_1\lambda^{-2} + a_2\lambda^{-4}, \tag{28}$$

in which a_0 & a_1 at least may be expected to be positive, while all the three coefficients a_0, a_1, a_2 are constant for any single medium, but vary in passing from one such medium to another. Consider now the four rays B, D, F, H. We shall have, for these four rays, and for any single medium, the system of the four equations

$$\left.\begin{aligned}\mu_B &= a_0 + a_1\lambda_B^{-2} + a_2\lambda_B^{-4},\\ \mu_D &= a_0 + a_1\lambda_D^{-2} + a_2\lambda_D^{-4},\\ \mu_F &= a_0 + a_1\lambda_F^{-2} + a_2\lambda_F^{-4},\\ \mu_H &= a_0 + a_1\lambda_H^{-2} + a_2\lambda_H^{-4};\end{aligned}\right\} \tag{29}$$

between which it is possible to eliminate the three medium-constants a_0, a_1, a_2, and so to deduce a general relation, valid for all media, between the 4 indices μ_B, μ_D, μ_F, μ_H and the four periodical times λ_B, λ_D, λ_F, λ_H.

A slight consideration shows that this relation can only involve the two ratios of the three differences of the four indices, & the two ratios of the three differences of the four reciprocals of the squares of the periodical times; so that if we put for abridgment

$$\frac{\mu_D - \mu_B}{\mu_H - \mu_B} = s_D, \qquad \frac{\mu_F - \mu_B}{\mu_H - \mu_B} = s_F, \tag{30}$$

and

$$\frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} = t_D, \qquad \frac{\lambda_F^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} = t_F, \tag{31}$$

the result of the elimination will be a relation between the four quantities s_D, s_F, t_D, t_F only, and will not involve the four other quantities $\mu_B, \mu_H, \lambda_B, \lambda_H$, if we previously substitute for $\mu_D, \mu_F, \lambda_D^{-2}, \lambda_F^{-2}$ their expressions deduced from (30) & (31), namely

$$\mu_D = \mu_B + s_D(\mu_H - \mu_B), \qquad \mu_F = \mu_B + s_F(\mu_H - \mu_B), \tag{32}$$

and

$$\lambda_D^{-2} = \lambda_B^{-2} + t_D(\lambda_H^{-2} - \lambda_B^{-2}), \quad \lambda_F^{-2} = \lambda_B^{-2} + t_F(\lambda_H^{-2} - \lambda_B^{-2}). \tag{33}$$

This result of elimination will $\because$ be the same as if we supposed, in the equations (29),

$$\left.\begin{array}{llll} \lambda_B^{-2} = 0, & \lambda_H^{-2} = 1, & \lambda_D^{-2} = t_D, & \lambda_F^{-2} = t_F, \\ \mu_B = 0, & \mu_H = 1, & \mu_D = s_D, & \mu_F = s_F, \\ a_0 = 0, & a_1 = b, & a_2 = c; & \end{array}\right\} \tag{34}$$

that is, the same as if we eliminated any two new quantities b & c between the three new equations

$$s_D = bt_D + ct_D^2, \quad s_F = bt_F + ct_F^2, \quad 1 = b + c. \tag{35}$$

This last elimination is easy, & gives, as the relation sought, the following:

$$\frac{s_D}{t_D}\,\frac{1 - t_F}{t_D - t_F} + \frac{s_F}{t_F}\,\frac{1 - t_D}{t_F - t_D} = 1. \tag{36}$$

As a verification, we may expand this relation, by substituting the values (30) & (31), so as to put it under the form

$$\begin{aligned} 0 = {} & (\mu_D - \mu_B)(\lambda_H^{-2} - \lambda_B^{-2})(\lambda_F^{-2} - \lambda_B^{-2})(\lambda_H^{-2} - \lambda_F^{-2}) \\ & - (\mu_F - \mu_B)(\lambda_H^{-2} - \lambda_B^{-2})(\lambda_D^{-2} - \lambda_B^{-2})(\lambda_H^{-2} - \lambda_D^{-2}) \\ & + (\mu_H - \mu_B)(\lambda_F^{-2} - \lambda_B^{-2})(\lambda_D^{-2} - \lambda_B^{-2})(\lambda_F^{-2} - \lambda_D^{-2}); \end{aligned} \tag{37}$$

for it will then be found to be satisfied, independently of the three medium-constants a_0, a_1, a_2, by the expressions (29) for the four indices $\mu_B, \mu_D, \mu_F, \mu_H$.

Having found from theory that this relation (36) or (37) was likely to hold good for all media, I proceeded to try it on the ten media observed by Fraunhofer. Adopting first his values of λ, which are for these four rays

$$\lambda_B = 0{,}2541; \quad \lambda_D = 0{,}2175; \quad \lambda_F = 0{,}1794; \quad \lambda_H = 0{,}1464; \tag{38}$$

(the unit of time being here the time which light takes to traverse, in what is called a vacuum, the ten thousandth part of a Paris inch*;) I perceived that within the limits of the errors of these determinations, & indeed to a very great accuracy, the square of the period λ_F is the harmonic mean between the squares of the extreme periods λ_B & λ_H; so that

$$\lambda_F^{-2} = \tfrac{1}{2}(\lambda_H^{-2} + \lambda_B^{-2}), \tag{39}$$

---

* [The metre superseded the French foot as the legal unit of length in France on Nov. 2nd, 1801. The Paris inch equals 0·0254 of a metre.]

or in the notation (31),

$$t_F = \tfrac{1}{2} \tag{40}$$

Availing myself of this circumstance, I put the relation (36) or (37) under the simpler form

$$4 s_F t_D (1 - t_D) - s_D = t_D (1 - 2t_D), \tag{41}$$

or

$$\mu_D - \mu_F = a_D (\mu_F - \mu_B) + b_D (\mu_H - 2\mu_F + \mu_B), \tag{42}$$

a_D and b_D being coefficients independent of the medium, which serve to calculate the index μ_D when the indices μ_B, μ_F, μ_H have been observed, & which have for their expressions, as functions of t_D, or of λ_B, λ_D, λ_H, the following:

$$a_D = -(1 - 2t_D) = -\frac{\lambda_H^{-2} - 2\lambda_D^{-2} + \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}}; \tag{43}$$

$$b_D = -t_D (1 - 2t_D) = -\frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} \cdot \frac{\lambda_H^{-2} - 2\lambda_D^{-2} + \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}}. \tag{44}$$

supposing, as already mentioned, that the observed relation (39) or (40) is accurate. Employing also the observed values (38) of λ_B, λ_D, λ_H, I found

$$\left.\begin{aligned} \log(-a_D) &= \bar{1},80441; \\ \log(-b_D) &= \bar{1},06281. \end{aligned}\right\} \tag{45}$$

For Flint Glass of the kind which he called Nr 13, Fraunhofer had found by observation:

$$\mu_B = 1,6277; \quad \mu_F = 1,6483; \quad \mu_H = 1,6711; \tag{46}$$

and hence, by (42) and (45), I calculated

$$\mu_D = 1,63492. \tag{47}$$

The value observed by Fraunhofer was

$$\mu_D = 1,6350. \tag{48}$$

The difference between calculation and observation is ∵ in this case very small. A similar calculation for each of the ten media, the data being taken to four decimal places from your paper in the Transactions, & the logarithms being taken to five places, (even fewer might here have sufficed,) gave, very easily, results which may thus be collected:

Medium	μ_D *Calcul^d^*	μ_D *Obs^d^*
Flint Glass Nr 13	1,63492	1,6350
Flint Glass Nr 23	1,63350	1,6337
Flint Glass Nr 30	1,63051	1,6306
Flint Glass Nr 3	1,60825	1,6085
Crown Glass *M*	1,55901	1,5591
Crown Glass Nr 13	1,52788	1,5280
Crown Glass Nr 9	1,52945	1,5296
Oil of turpentine	1,47444	1,4744
Solution of potash	1,40270	1,4028
Water	1,33346	1,3336.

(49)

The agreement between the ten observed & the ten calculated values of μ_D is close enough; yet it is worth remarking that in almost every case the calculated value is a little less than the observed. This circumstance led me to think that by altering a little the multipliers a_D, b_D

in the formula (42), the agreement might be improved so far as the *indices* are concerned. But in order to justify an alteration of these multipliers, it is necessary to suppose that the values of λ may be a little altered, or in other words that the *periodical times* of vibration for the four rays have not been determined so very accurately that the errors of their determinations may not have some sensible effect in the calculation of the indices; because the multipliers a_D, b_D may be expressed in general as the following functions of those periodical times,

$$\begin{aligned} a_D &= b_D + (t_D - t_F)\,\frac{1 - t_D}{}, \\ b_D &= (t_D - t_F \quad 1 - t_F \end{aligned} \qquad (50)$$

or thus more fully,

$$a_D = \frac{\lambda_D^{-2} - \lambda_F^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} \left\{ \frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_F^{-}} + \frac{\lambda_H^{-2} - \lambda_D^{-2}}{\lambda_F^{-2} - \lambda_B^{-2}} \right. \qquad (51)$$

and

$$b_D = \frac{\lambda_D^{-2} - \lambda_F^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} \cdot \frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_F^{-2}}; \qquad (52)$$

as appears from the comparison of the formula (42) with the relation (36) or (37). However it seemed to me worth while to try the effect of supposing the values of λ to be quite unknown, & of determining the multipliers a_D, b_D from observations on the indices alone. The formula (42), (which is not dependent on the particular supposition (39) or (40), provided that we conceive a_D, b_D to represent the general & as yet unknown values (51) & (52),) gives for any two different media the two equations:

$$\left. \begin{aligned} \mu_D - \mu_F &= (a_D - b_D)(\mu_F - \mu_B) + b_D(\mu_H - \mu_F), \\ \mu'_D - \mu'_F &= (a_D - b_D)(\mu'_F - \mu'_B) + b_D(\mu'_H - \mu'_F); \end{aligned} \right\} \qquad (53)$$

the accented indices belonging to the second medium: & hence we have, by elimination,

$$\left. \begin{aligned} a_D - b_D &= \frac{(\mu'_H - \mu'_F)(\mu_D - \mu_F) - (\mu_H - \mu_F)(\mu'_D - \mu'_F)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}; \\ b_D &= \frac{(\mu'_D - \mu'_F)(\mu_F - \mu_B) - (\mu_D - \mu_F)(\mu'_F - \mu'_B)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}. \end{aligned} \right\} \qquad (54)$$

To apply these expressions I chose the two media Water & Flint Glass, and among the various specimens of the latter, examined by Fraunhofer, I chose his N^r 23, (specific gravity = 3,724,) because it appears from his Table of indices, extracted by Herschel in article 437 of the Treatise on Light, that he has observed and recorded the phenomena of each of these two media, as they presented themselves in two distinct experiments....

[Numerical details are here omitted and the resulting table simplified.]

...and substituting these values in the expressions (54), I found

$$b_D = -0{,}1547788; \quad a_D - b_D = -0{,}4732852; \quad a_D = -0{,}6280640; \qquad (61)$$

or

$$\log(-b_D) = \bar{1}{,}1897115; \quad \log(b_D - a_D) = \bar{1}{,}6751230; \quad \log(-a_D) = \bar{1}{,}7980039. \qquad (62)$$

Using these two new multipliers $a_D - b_D$ & b_D, and employing equations of the form (53) & the observed values of μ_B, μ_F, μ_H as stated in that Table of Fraunhofer to which I just now referred, I found as follows:

Med^m	$\mu_F-\mu_D$ *Calc*^{d}	$\mu_F-\mu_D$ *Obs*^{d}	*Calc*^{d} − *Obs*^{d}
F.G. 13	0,0132368	0,0132240	+ 0,0000128
F.G. 23	0,0131015	0,0131015	0,0000000
F.G. 30	0,0129154	0,0128810	+ 0,0000344
F.G. 3	0,0116659	0,0115480	+ 0,0001179
C.G. *M*	0,0076340	0,0076660	− 0,0000320
C.G. 13	0,0063462	0,0063550	− 0,0000088
C.G. 9	0,0064643	0,0064650	− 0,0000007
Oil of T.	0,0071984	0,0073020	− 0,0001036
Sol^n of P.	0,0052832	0,0052770	+ 0,0000062
Water	0,0042260	0,0042260	0,0000000.

(63)

Hence

$$\left.\begin{array}{ll}\text{Sum of positive differences} & = +0{,}0001713,\\ \text{Sum of negative differences} & = -0{,}0001451,\\ \text{Sum of all differences} & = +0{,}0000262,\\ \text{Mean of all differences} & = +0{,}0000026.\end{array}\right\} \tag{64}$$

This mean difference may be regarded as insensible; which cannot so well be said of the mean of the differences (49); & therefore the new multipliers a_D-b_D & b_D are probably not far from the truth, & may be thought to be somewhat more exact than the multipliers (45).

To see now what relations they suppose between the periodical times of vibration, we are to resolve the equations (51) & (52), or more simply the two equations (50), so as to deduce the values of t_D and t_F, or of the equivalent expressions

$$\frac{\lambda_D^{-2}-\lambda_B^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}} \quad\text{and}\quad \frac{\lambda_F^{-2}-\lambda_R^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}}.$$

The equations (50) give, when cleared of their divisors,

$$\left.\begin{array}{r}(a_D-b_D)\,t_F=(t_D-t_F)\,(1-t_D),\\ b_D\,(1-t_F)=(t_D-t_F)\,t_D;\end{array}\right| \tag{65}$$

and therefore, by addition,

$$(a_D-b_D)\,t_F+b_D\,(1-t_F)+t_F=t_D. \tag{66}$$

Substituting this expression for t_D in the second equation (65), we find

$$b_D\,(1-t_F)=\{(a_D-b_D)\,t_F+b_D\,(1-t_F)\}\,\{(a_D-b_D)\,t_F+b_D\,(1-t_F)+t_F\}; \tag{67}$$

that is, we have to resolve the following quadratic for t_F,

$$t_F^2\,(a_D-2b_D)\,(a_D-2b_D+1)+2t_F b_D\,(a_D-2b_D+1)+b_D\,(b_D-1)=0. \tag{68}$$

Hence, by the numeric values (61) or (62), we find

$$0=-0{,}2170601\,t_F^2-0{,}2109614\,t_F+0{,}1787352; \tag{69}$$

that is,

$$0=t_F^2+0{,}9719029\,t_F-0{,}8234368; \tag{70}$$

and ∵, attending only to the positive root,

$$t_F=+0{,}5434102; \tag{71}$$

from which we derive, by (66),

$$t_D=+0{,}2155518. \tag{72}$$

The values of these two ratios deduced from the observed values of λ, (38), by the formula (31), are sensibly smaller, namely

$$t_F = +0{,}4999527, \tag{73}$$

and

$$t_D = +0{,}1813010. \tag{74}$$

Yet perhaps when the great difference of the methods is considered, the agreement of results, such as it is, may be thought to give some support to the theory. However, the fairest way to judge of the degree to which the results of these two methods approach to or diverge from each other is to compare the values of the periods λ_D and λ_F, which were deduced by Fraunhofer from a spectrum of interference, formed by a fine grating, (&, if I remember rightly, without any use of the knowledge of the refractive indices,) with the values of the same two mean periods λ_D and λ_F, calculated from Fraunhofer's similarly deduced values of the two extreme periods λ_B and λ_H, by employing the theoretical law of dispersion (28) and the 8 refractive indices $\mu_B, \mu_D, \mu_F, \mu_H, \mu'_B, \mu'_D, \mu'_F, \mu'_H$, for water and for a specimen of flint glass, which were determined by the same optician for the same four definite rays, through a process which did not suppose any knowledge of the periods of vibration. Calculating ∵ the two mean periods λ_D and λ_F, by the method above explained, that is by the formulae (33), (71), (72), from the two extreme periods λ_B, λ_H considered as known by observation of the diffraction-spectrum, namely $\lambda_B = 0{,}2541$, $\lambda_H = 0{,}1464$ in (38), and from the 4 water and 4 flint glass indices, in equations (59) & (60), considered as known by observation of two prismatic spectra, I find

$$\lambda_D = 0{,}21221; \quad \lambda_F = 0{,}17561: \tag{75}$$

values which do not differ very greatly from those which Fraunhofer found from the diffraction-spectrum, namely,

$$\lambda_D = 0{,}2175; \quad \lambda_F = 0{,}1794: \tag{38}$$

since the differences, namely $\quad -0{,}00529 \quad \text{and} \quad -0{,}00379, \qquad (76)$

correspond each, in space, to less than the millionth part of an inch &, as compared with the corresponding period or undulation itself, to less than the fortieth part of such period. I even think that by pursuing, with greater accuracy & to greater extent than I have done, this train of investigation, it will be possible to deduce, from the phenomena of dispersion alone, those relations between the several periods or undulations which answer to my fractions t_D, t_F, &c., namely the ratios of the differences of the inverse squares of those periods, more exactly than from the phenomena of interference. And perhaps I may be tempted hereafter to resume this particular inquiry: since I see that many improvements might be made in the plan of the foregoing calculations.

Meanwhile, as I have been, I am likely to continue to be, diverted for some time from this inquiry, I shall lay before you a few other results of past calculations, which look a little promising. I have hitherto mentioned only combinations of the four rays B, D, F, H. But by combining in exactly the same way the four rays B, F, G, H, I arrived at analogous conclusions. Thus, adopting Fraunhofer's observed value of λ_G,

$$\lambda_G = 0{,}1587, \tag{77}$$

(in which the unit is still the ten thousandth part of a Paris inch, or the time light takes to traverse it in a space void of ponderable matter,) & retaining his values of λ_B, λ_H, I found

$$t_G = \frac{\lambda_G^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} = +0{,}7769594; \tag{78}$$

and hence, approximately, in the formula

$$\mu_G - \mu_F = a_G(\mu_F - \mu_B) + b_G(\mu_H - 2\mu_F + \mu_B), \tag{79}$$

which is analogous to (42), I found, by supposing

$$t_F = \tfrac{1}{2}, \tag{40}$$

these numerical expressions for the multipliers a_G, b_G,

$$\left.\begin{aligned} a_G &= -1 + 2t_G = +0{,}55392,\\ b_G &= t_G(-1 + 2t_G) = +0{,}43037, \end{aligned}\right\} \tag{80}$$

or

$$\left.\begin{aligned} \log a_G &= \bar{1}{,}74345,\\ \log b_G &= \bar{1}{,}63384. \end{aligned}\right\} \tag{81}$$

Adopting, next, the approximate values of μ_B, μ_F, μ_H, as stated, from Fraunhofer's observations, in your paper, & collected in table (49) of this letter, for the ten media mentioned in that paper or in that table, I found for those media, in their order, results which may be thus collected:

Medium	μ_G *Calcul*^{d}	μ_G *Obs*^{d}	*Calcul*^{d} – *Obs*^{d}
F.G. 13	1,66066	1,6603	+0,00036
F.G. 23	1,65910	1,6588	+0,00030
F.G. 30	1,65569	1,6554	+0,00029
F.G. 3	1,63101	1,6308	+0,00021
C.G. *M*	1,57368	1,5735	+0,00018
C.G. 13	1,54001	1,5399	+0,00011
C.G. 9	1,54182	1,5416	+0,00022
Oil of T.	1,48833	1,4882	+0,00013
Sol^n of P.	1,41272	1,4126	+0,00012
Water	1,34140	1,3413	+0,00010.

(82)

Here

$$\left.\begin{aligned} &\text{Sum of differences} = +0{,}00202;\\ &\text{Mean of differences} = +0{,}000202. \end{aligned}\right\} \tag{83}$$

This mean is not insensible; nor does it seem likely that it would be made so, by redressing the little errors of these comparatively rough calculations: for instance I happened to employ the value $\bar{1}{,}74344$ for $\log a_G$ instead of $\bar{1}{,}74345$; but if the latter more accurate logarithm of 0,55392 had been employed it would have tended to increase (though very slightly) the excess of the calculated over the observed μ_G. I proceeded ∵ to calculate the multipliers $a_G - b_G$ & b_G, in the formula

$$\mu_G - \mu_F = (a_G - b_G)(\mu_F - \mu_B) + b_G(\mu_H - \mu_F), \tag{84}$$

by employing the expressions

$$\left.\begin{aligned} a_G - b_G &= \frac{(\mu'_H - \mu'_F)(\mu_G - \mu_F) - (\mu_H - \mu_F)(\mu'_G - \mu'_F)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)},\\ b_G &= \frac{(\mu'_G - \mu'_F)(\mu_F - \mu_B) - (\mu_G - \mu_F)(\mu'_F - \mu'_B)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}, \end{aligned}\right\} \tag{85}$$

which are analogous to (54), & in which the four indices μ_B, μ_F, μ_G, μ_H are those determined for water, and μ'_B, μ'_F, μ'_G, μ'_H are those determined for flint glass (N^r 23), by the means of the two experiments of Fraunhofer on each of those two media. Those mean values of μ_B, μ_F, μ_H

for water, & of μ'_B, μ'_F, μ'_H for that specimen of flint glass, have been already stated in equations (59) & (60) of this letter; with respect to μ_G & μ'_G, they were taken out in the same way, from Fraunhofer's table extracted by Herschel, which gave

$$\left.\begin{aligned}\mu_G&=1{,}3412770,\\ \mu'_G&=1{,}6588485.\end{aligned}\right\} \tag{86}$$

In this manner I found, by (85),

$$a_G-b_G=+0{,}09514275;\quad b_G=+0{,}4433440;\quad a_G=0{,}53848675; \tag{87}$$

$$\log(a_G-b_G)=\bar{2}{,}9783757;\quad \log b_G=\bar{1}{,}6467408. \tag{88}$$

With these new multipliers & the observed values of $\mu_F-\mu_B$, $\mu_H-\mu_F$, as stated in table (63) of this letter for each of the ten media, I found for those media the following values of $\mu_G-\mu_F$, calculated by the formula (84), & the following differences between calculation & observation:

Medium	$\mu_G-\mu_F$ *Calc*^{d}	$\mu_G-\mu_F$ *Obs*^{d}	*Calc*^{d} − *Obs*^{d}
F.G. 13	0,0120606	0,0120250	+ 0,0000356
F.G. 23	0,0120799	0,0120805	− 0,0000006
F.G. 30	0,0119152	0,0119400	− 0,0000248
F.G. 3	0,0107262	0,0107300	− 0,0000038
C.G. *M*	0,0067819	0,0067940	− 0,0000121
C.G. 13	0,0055411	0,0055710	− 0,0000299
C.G. 9	0,0056337	0,0056050	+ 0,0000287
Oil of T.	0,0064507	0,0064620	− 0,0000113
Sol^n of P.	0,0044777	0,0044970	− 0,0000193
Water	0,0034739	0,0034740	− 0,0000001.

(89)

Here

$$\left.\begin{aligned}&\text{Sum of positive differences}=+0{,}0000643;\\ &\text{Sum of negative differences}=-0{,}0001019;\\ &\text{Sum of all the differences}\ \ =-0{,}0000376;\\ &\text{Mean of all the differences}=-0{,}0000038;\end{aligned}\right\} \tag{90}$$

so that the separate differences are small, much smaller than those in table (82), and the mean of all is quite insensible & is less than the fiftieth part of the mean in (83). There is reason ∵ to consider the new values of the multipliers (87) as more exact than the old ones (80), from which however they do not very greatly differ. And substituting the new values of a_G-b_G and b_G in the equations analogous to (50), namely in the following:

$$\left.\begin{aligned}a_G-b_G&=(t_G-t_F)\frac{1-t_G}{t_F},\\ b_G&=(t_G-t_F)\frac{t_G}{1-t_F},\end{aligned}\right\} \tag{91}$$

or in these others, derived from them,

$$t_G=t_F(a_G-2b_G+1)+b_G, \tag{92}$$

and

$$t_F^2(a_G-2b_G)(a_G-2b_G+1)+2t_Fb_G(a_G-2b_G+1)+b_G(b_G-1)=0, \tag{93}$$

I found first this quadratic equation for t_F,

$$0=-0{,}2269571t_F^2+0{,}5779421t_F-0{,}2467901, \tag{94}$$

that is,

$$0=t_F^2-2{,}5464830t_F+1{,}0873863, \tag{95}$$

or, confining ourselves to the root which lies between 0 and 1,

$$t_F = 0{,}5426557; \tag{96}$$

and then,

$$t_G = 0{,}7970463. \tag{97}$$

These values, deduced solely from the indices of four rays observed in two prismatic spectra, differ sensibly from the values

$$t_F = +0{,}4999527 \tag{73}$$

and

$$t_G = +0{,}7769594, \tag{78}$$

which were deduced solely from the spectrum of diffraction; but it may be considered as somewhat confirming the theory that they do not differ still more. And if we employ the prismatic values (96), (97) of t_F, t_G to calculate the two mean periods or undulations λ_F, λ_G from the two extreme periods λ_B, λ_H considered as known by observations on diffraction, we find, by the method already explained,

$$\lambda_F = 0{,}17568; \quad \lambda_G = 0{,}15746, \tag{98}$$

values which differ from those deduced from the observations on diffraction alone, namely from the following

(38)

$$\lambda_F = 0{,}1794 \quad \text{and} \quad \lambda_G = 0{,}1587, \tag{77}$$

only by the little differences

$$-0{,}00372 \quad \text{and} \quad -0{,}00124, \tag{99}$$

of which each is much less that the millionth part of an inch, if expressed in space, & of which the former is about the fiftieth part, & the latter less than the hundredth part of the whole corresponding period or undulation λ. But what appears to me the most remarkable and encouraging result of these calculations, is the near agreement of the two prismatic results, (71) & (96), which give respectively

$$\begin{aligned} \frac{\lambda_F^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} &= +0{,}5434102, \\ \frac{\lambda_F^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} &= +0{,}5426557; \end{aligned} \tag{100}$$

the first having been obtained by combining the rays *B*, *D*, *F*, *H* & the second by combining the rays *B*, *F*, *G*, *H*, as observed in two different spectra produced by prismatic dispersion. So close is this agreement, that, if Fraunhofer's values be taken for λ_B and λ_H, the two resulting values of λ_F, namely

(75)

$$\lambda_F = 0{,}17561 \quad \& \quad \lambda_F = 0{,}17568, \tag{98}$$

when reduced to what they represent in space, do not differ by so much as the hundred-millionth part of an inch in the length of the concluded undulation. If methods more refined had been employed,—for example, the method of least squares in combining more media than two,—& if greater care had been given to the numerical part of the calculation, it seems reasonable to think that the agreement might have been found closer still. But I have reason to think that, for highly dispersive media, even the *fourth* term ($= a_3 \lambda^{-6}$) of the series (24) for the index μ is sensible; and that it must be allowed for, if we would obtain a sufficiently exact expression for the dependence of that index on the period or undulation λ.

If you should not adopt these views; or if, adopting them, you should still choose to continue the use of the formula

$$\frac{1}{\mu}=\frac{H\sin\theta}{\theta}, \tag{101}$$

(in which θ varies inversely as λ, and H is a constant quantity,) as giving a good approximation to the truth; you may possibly find it convenient to employ occasionally this simple rule, which will give quickly a near approach to the value of θ_B:

$$\left.\begin{aligned}\log\theta_B &= 3{,}77348+\log\sin\phi,\\ \log\sec\phi &= \tfrac{1}{2}(\log\mu_H-\log\mu_B).\end{aligned}\right\} \tag{102}$$

The value of θ_B thus found is expressed in minutes. For example, in the case of Flint Glass N^r 13,

$\mu_B=1{,}6277$	$\log=0{,}21157$	Const...	$3{,}77348$
$\mu_H=1{,}6711$	$\log=0{,}22300$	[log] $\sin\phi$...	$\bar{1}{,}20691$
	$2\,\lvert 0{,}01143$		$2{,}98039$
$\phi=9^\circ\,16'$	$0{,}00571/5$	$\theta_B=956'=15^\circ\,56'$.	

This value $\theta_B=15^\circ\,56'$ is very nearly correct, as a solution of the equations

$$\lambda_B\theta_B=\lambda_H\theta_H,\quad \mu_B\lambda_B\sin\theta_B=\mu_H\lambda_H\sin\theta_H, \tag{103}$$

λ_B and λ_H having the values 0,2541 & 0,1464, & μ_B, μ_H having those stated above. A process of correction, with which I shall not trouble you at present, gave me as a next approximation $\theta_B=15^\circ\,57'\,53''{,}4$; on calculating θ_H from which, by the second equation (103), and then calculating θ_B from it by the first of those two equations (103), I was brought back almost exactly to what I set out with, as the second approximation, for I found $\theta_B=15^\circ\,57'\,53''{,}5$. But this exactness appears to be useless here. My formula for the first approximation is

$$\sqrt{\frac{6\left(1-\dfrac{\mu_B}{\mu_H}\right)}{\left(\dfrac{\lambda_B}{\lambda_H}\right)^2-1}}. \tag{104}$$

I forget whether that which I gave you extempore in the section room* was exactly the same as this.

I became greatly interested lately in the Dynamics of Light, & thought that I could in some things improve the theory of Cauchy; especially by bringing the results to agree more closely with those of Fresnel, respecting the laws of double refraction & polarisation; though I was obliged to let some differences remain, such as that striking one concerning the relation of the direction of vibration to the plane of polarisation, & a few others, no one of which seemed capable of being soon decided by experiment. It is likely that I shall resume the whole subject, after the end of my next course of lectures on Astronomy, which always distract me from study. But before that time I hope to see Professor Lloyd, who has not yet (I believe) returned from a continental excursion; and I shall give him your message about the arragonite.—This letter has swelled to an Essay, but I expect to send it in an official frank. Accept my thanks for the new indices in your last letter, & believe me to remain, my dear Sir,

Very truly yours,
WILLIAM R. HAMILTON.

Prof. POWELL.

---

* [British Association Meeting at Dublin, 1835.]

[*To Professor Powell.*]

Observatory, Dublin,
Dec^{r}. 17^{th}, 1835.

MY DEAR SIR

Your letter reached me yesterday or the day before, & I have great pleasure in trying to remove any obscurity into which I may have fallen, in my former long epistle. I ought perhaps to have justified my assumption of the formula

$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right); \tag{2}$$

or at all events I am bound to prevent any mistake of the meaning which I attach to that assumption. For, you will observe that I did not pretend to prove that *no other* formula would satisfy the differential equation

$$\frac{d^2\eta}{dt^2} = S\left\{m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta\eta\right\} \tag{1}$$

I only showed that this *particular* formula (2) was *one solution* of that differential equation, provided that λ & μ are supposed to be connected by the relation

$$\left(\frac{2\pi}{\lambda}\right)^2 = S\left\{2m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{}\left(\sin\frac{\pi\mu\,\Delta x}{\lambda}\right)^2\right\} \tag{5}$$

& thence concluded that periodical *vibrations of the particular kind expressed by the formula* (2) MAY *be propagated* (on Cauchy's principles) *through a system of attracting & repelling points,* PROVIDED *that the slowness μ of propagation is connected with & varies with the periodical time λ of vibration by the law or relation* (5); and even that *this law* or relation is a *necessary condition* for the propagation of *such* vibrations: without pretending to decide whether vibrations of *some other sort* may or may not be propagated according to *other laws* of velocity. I confined myself to the study of vibrations of that *particular* sort which I thought that *induction from phenomena* had led mathematicians to consider; & did not presume to say that I had followed out *deductively* all *possible* mathematical consequences from the differential equation of vibration. Yet, as I have hitherto studied much less the physics than the mathematics of light, I am anxious to have your opinion as to the correctness of my assumption (2), considered as suggested by *induction*; & to know whether it really appears to you to contain anything new or unusual, when I explain fully my meanings for the symbols.

To represent the *phenomena of interference*, it is supposed, in the wave-theory of light, that certain points, or molecules of ether, *vibrate periodically in space and time*; in such a manner that if we consider the propagation of an indefinite series of plane waves parallel to the plane of yz, all the vibrating points, which at any one moment t have any one common x, have also equal and parallel displacements ξ, η, ζ, and velocities $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$, of vibration; the same displacements & velocities recurring periodically, *in time* after all the multiples 1λ, 2λ, 3λ, &c. of some one fixed time-period λ, & *in space* after all the multiples $1\lambda'$, $2\lambda'$, $3\lambda'$, &c. of some one fixed space-period λ': so that we may put

$$\xi = \phi(x, t), \quad \eta = \chi(x, t), \quad \zeta = \psi(x, t),$$

the functions ϕ, χ, ψ being such as to satisfy these conditions of periodicity

$$\begin{aligned}\phi(x,t)&=\phi(x,t+\lambda)=\phi(x+\lambda',t),\\ \chi(x,t)&=\chi(x,t+\lambda)=\chi(x+\lambda',t),\\ \psi(x,t)&=\psi(x,t+\lambda)=\psi(x+\lambda',t).\end{aligned}$$

Besides, by the *conception of the velocity* (v) *of propagation* of a wave, it is equal to the *ratio* $\frac{\lambda'}{\lambda}$ *of the period* λ' *in space to the period* λ *in time*; & accordingly it is thus that the periods in time, at least for the planetary spaces, have been inductively concluded from the periods in space observed in what we call a void (through the phenomenon of interference), and then divided by the common velocity of propagation considered as also known from astronomical observation: and more generally, the *conception of a uniform velocity of propagation* conducts to these formulae of displacement,

$$\xi=\phi\left(\frac{x}{v}-t\right),\quad \eta=\chi\left(\frac{x}{v}-t\right),\quad \zeta=\psi\left(\frac{x}{v}-t\right),$$

in which $v=\frac{\lambda'}{\lambda}$, & $\Delta\xi=0$, $\Delta\eta=0$, $\Delta\zeta=0$, if $\Delta\left(\frac{x}{v}-t\right)=$ any multiple of λ. Again, the *phenomena of refraction*, as explained by the theory of waves, conduct to the conclusion that the *velocity* v *of propagation of light*, of any one colour, in any transparent terrestrial substance, *is less* than in the planetary spaces & less *in the ratio of* 1 *to* μ, μ being the index of refraction: we must ∵ suppose that either the period in space is diminished, or the period in time increased, or at least that the period in space becomes less as compared with the period in time, on entering a transparent terrestrial substance, than it had been in the planetary spaces. And the *phenomena of dispersion* show that *this diminution* of the ratio $\left(\frac{\text{space-period}}{\text{time-period}}\right)$ *is greater for violet than for red light*, because they show that the index of refraction is greater. Besides it was concluded by Newton, though with some diffidence, & it has (*I believe*) been confirmed by subsequent observation of the *phenomena of thin plates*; (but I beg you to set me right if I am mistaken on this point of the induction;) that whatever the colour of the light & whatever the transparent medium may be, the length of a fit,* or (as we would say) the length of an undulation, that is the *period in space*, which I have here called λ', is *inversely proportional to the index of refraction*, so that the product $\mu\lambda'$ is constant for any one colour, & does not vary with the medium, though it varies with the colour, and is less for violet than for red light. And hence I gather, what indeed is very natural & seems to be almost necessary à priori in the theory of waves, and what I thought was generally supposed by undulationists, though I do not remember seeing it expressly stated in any book, and beg of you to set me right if you know of any facts which militate against the conclusion: namely that the *periodical time of vibration*, which I have called λ, in this letter and in the last, *is constant for any one colour and does not vary with the medium*. For if we take as the unit of velocity the velocity of propagation of light in the planetary spaces, we shall have in general $v=\frac{1}{\mu}$; & since we have also $v=\frac{\lambda'}{\lambda}$, we may conclude that $\lambda=\mu\lambda'=$ a quantity depending on colour only. Again to represent the *phenomena of plane*

---

* [This is Newton's phrase. See *Newton's Opticks*, Book Two, Part III, Prop. XII, new edition, Bell (1931), p. 281.]

or rectilinear polarisation, and the *interference of polarised rays*, we suppose the *displacements* of the vibrating molecules to be either exactly or nearly *parallel to or contained upon the plane surface of the wave*, & to have a *line of direction independent of the time*; I therefore, who am not now considering circular polarisation, nor indeed extraordinary media, but who may suppose if I choose that the light is rectilinearly polarised, am at liberty to suppose that while the wave is parallel to the plane of yz the vibration is parallel to the axis of y, & ∵ to take for my formulae of displacement the following:

$$\xi = 0, \quad \eta = \chi(\mu x - t), \quad \zeta = 0,$$

in which the function χ must satisfy the condition $\Delta\eta = \Delta\chi(\mu x - t) = 0$, if $\Delta(\mu x - t) =$ any multiple of λ. The *simplest* way of satisfying this condition, & the one which seemed to me to *agree best with the received way of representing phenomena in the theory of waves*, was to take

$$\eta = A + B\cos\frac{2\pi(\mu x - t)}{\lambda} + C\sin\frac{2\pi(\mu x - t)}{\lambda},$$

or, which comes to the same thing,

$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{2}$$

t_0 being entirely arbitrary & η_0, η_1 being arbitrary but small. The constant η_0 is introduced for greater generality; I have some reasons for thinking it equal to $-\eta_1$, so that the *absolute displacement* would vary as the *versed sine of the phase*: but in the present investigation, it is sufficient if we can rightly represent the laws of *relative displacement*, which are not affected by the introduction of an arbitrary common term; so that if you do not like my η_0 you may for the present expunge it. Having thus shown, perhaps at too great length, my reasons *from induction* for *assuming* the formula (2) as a *probable law of vibration* and one which I thought differed little (if at all) from the assumptions of other undulationists, I may refer to my former letter for the process, taken from Cauchy, or (at the least) suggested by him, by which *I compared that assumed law of vibration with his differential equations of motion* & showed that it was at least a *particular solution* of those equations and ∵ *dynamically possible, if μ be supposed to depend on λ by the relation* (5).

After all, if you choose rather to consider μ as depending on the period in space, which is different for different media, as well as for different colours, & which I shall still call λ', you have only to change λ to $\mu\lambda'$ in that relation (5), & it then becomes

$$\frac{2\pi}{\mu\lambda'}\Big)^2 = S\,\{\,2m\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{}\Big(\sin\frac{\pi\Delta x}{\lambda'}$$

that is, as in formula (8),

$$\left(\frac{1}{\mu}\right)^2 = S\,.\,H^2\left(\frac{\sin\theta}{\theta}\right)^2,$$

in which H^2 is still the (not necessarily positive) quantity

$$\frac{m}{2}\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta x^2,$$

and θ is still the semi-difference of phases of 2 near molecules, represented now by the expression $\theta = \frac{\pi\Delta x}{\lambda'}$, which perfectly resembles Cauchy's. My *essential objection is to the omitting of the* *sign of summation* S and *not* to the considering of periods in space, though I think *periods in time more convenient,* because they do not seem to change in passing from one medium to another. On this last point of physics, I repeat my request that you will set me right if you think me wrong; & as to the mathematics connected with the sign of summation, I shall gladly write again, if you think there is the least occasion for it. My Lectures are just over; but, as I guessed, I feel more anxious at this moment to pursue my general view of dynamics & to expand the principles, connected therewith, of my Calculus of Principal Functions than to resume for some time my suspended optical researches. Believe me to remain very truly yours

W. R. HAMILTON.

[The remainder of the correspondence with Professor Powell is of no scientific interest.]

*To Sir John Herschel.**

Observatory, Dublin,
Feb. 8th, 1839.

MY DEAR SIR

When we parted last Summer on the banks of Ullswater, you were pleased to encourage me to pursue my researches respecting the dynamics of light; & I think, from what you said on the occasion, that you are not likely to consider it as an intrusion, if I sometimes endeavour to take advantage of your greater knowledge of what has been done in mathematics abroad, so as to learn whether my results have been anticipated; & whether, even where this may not have been done, the researches and results of others are sufficiently analogous to my own to deserve and demand my attention.

It is only since the beginning of the present year that I have resumed my speculations upon light; and the problem which I have been recently considering is the *propagation of vibration*, properly so called, as distinguished from the mere *preservation* of a mode of vibration already established. Most of my own former calculations, & nearly all of those which I had seen in the writings of others, at least all those which had appeared to me to be sufficiently exact, were directed to the examination of the conditions under which a mode of vibration, thus once established, might thus be preserved for ever, through an indefinite extent of space and time; they related to the discovery of particular integrals of the general differential equations, giving periodical expressions for the disturbances as functions of x, y, z, t, which were *not discontinuous functions*, at least within the range of any single medium. The whole of each such medium was supposed to be agitated at once; & the question was, what mode of agitation could be *permanent*, the acting forces being taken into account. It was not shown, at least not to my own satisfaction, how the vibration could *spread*, according to the generally admitted laws, from one part of a medium to another part of the same, leaving all beyond and all behind it at rest. Much had been done, perhaps, in the dynamics of *light*; little, I thought, in the dynamics of *darkness*.

To give a very simple example of the difficulty that appeared to me to exist, & at the same time of the questions that I have resolved, let us consider the case of a single row of equal particles, arranged at equal & finite distances on one indefinite straight line,

---

* [Sir John Frederick William Herschel, 1792–1871. For a general account of Hamilton's correspondence with Herschel, see Graves, *Life of Hamilton*.]

$$\begin{matrix} . & . & . & . & . & . & . & . & . & . & . \\ & & -3 & -2 & -1 & 0 & +1 & +2 & +3 & & \end{matrix}$$

so that their abscissae x may be considered as integer numbers, positive, negative or null, & that their ordinates y at first all vanish. Under these conditions, equilibrium subsists and motion will never arise. But if we now remove some or all of these particles,

$$m_{-\circ} \qquad m_{-3},\quad m_{-2},\quad m_{-1},\quad m_0,\quad m_1,\quad m_2,\quad m_3, \qquad m_\alpha$$

indefinitely little on either side from their natural positions on the axis of x, in directions perpendicular to that axis; so as to give to each particle m_x an indefinitely small ordinate y_x, (positive or negative or null,) without changing its abscissa x; the equilibrium will in general be broken, and motion will commence, whether we do or do not give any initial velocities y'_x to the particles. And if for greater simplicity we suppose that the force $m\mathrm{f}(r)$ which any one of these particles exerts on any other is attractive at the distance $r=1$, & insensible at the distance $r=2$ and at all greater distances; we shall have this general differential equation, as regulating the motion of any particle m_x,

$$\frac{d^2}{dt^2}y_{x,t}=m\mathrm{f}(1)(y_{x+1,t}-y_{x,t})+m\mathrm{f}(1)(y_{x-1,t}-y_{x,t}),$$

or more concisely

$$D_t^2 y_{x,t}=\frac{a^2\Delta_x^2}{1+\Delta_x}y_{x,t}, \tag{1}$$

$a^2=m\mathrm{f}(1)$ being the attraction at the unit of distance, that is, the attraction exerted by any particle on either of the two immediately adjacent. A particular integral of this equation is

$$y_{x,t}=\text{const.}+\text{const.}\times\sin(2x\nu-2at\sin\nu+\text{const.}),$$

and a still more particular one is therefore

$$y_{x,t}=\eta\operatorname{vers}(2x\nu-2at\sin\nu), \tag{2}$$

η and ν being any arbitrary constants. This last expression represents therefore a mode of vibration of which the *permanence is possible*; and which would continue for ever, *if it were once established for all values of* x. If even we suppose that, at the origin of t, the values of $y_{x,t}$ & of its differential coefficient $y'_{x,t}=D_t y_{x,t}$ agree with this expression, or in other words that the initial displacements & initial velocities are all such as to agree with this law; so that we have, for all values of x, the conditions

$$y_{x,0}=\eta\operatorname{vers}2x\nu, \quad\text{and}\quad y'_{x,0}=-2a\eta\sin\nu\sin 2x\nu; \tag{3}$$

then the expression (2) for $y_{x,t}$, or the mode of vibration which it represents, will hold good for all values of t and of x, that is, at every moment & for every particle. And this is a fair *specimen*, I think, of the *sort* of results which I had obtained myself, & of the most satisfactory among those which I had seen as obtained by others, respecting the dynamics of light as distinguished from the dynamics of darkness, or the preservation rather than the propagation of vibrations. It gives, for example, in this particular vibratory motion, the expression

$$\sin\nu$$

for the *rapidity with which a given phase travels*; and this is the simplest (though not the most accurate) form of Cauchy's Law of Dispersion. It is in fact the form which Mr Powell at first supposed to follow from Cauchy's investigations on that subject.

But let us now suppose that the initial conditions (3) are satisfied only for a finite range of x, or in other words that only a finite number of the particles have their initial displacements & velocities such as to agree with the law (2). To fix our conceptions, let the length of a wave be an exact multiple n of the molecular interval, so that $\nu = \frac{\pi}{n}$; and let a large but finite number i of such wave-lengths, immediately behind the origin of x, be the extent of initial disturbance; so that the particles $m_0, m_{-1}, m_{-2}, \ldots m_{-in}$, have their initial displacements & initial velocities determined by the conditions (3); the other particles $m_1, m_2, m_3, \ldots m_\infty$, and $m_{-in-1}, m_{-in-2}, \ldots m_{-\infty}$, (and also, by (3), the particles $m_0, m_{-n}, m_{-2n}, \ldots m_{-in+n}, m_{-in}$,) having their initial displacements & initial velocities null. In other words, let the conditions (3) be satisfied for all values of x which are not greater than 0, and not less than $-in$; but let $y_{x,0}$ & $y'_{x,0}$ vanish for all other values of x, and even, by the conditions (3) themselves, for

$$x=0, \quad x=-n, \quad x=-2n, \ldots x=-in+n, \quad x=-in.$$

These things being supposed, it is clear that the law (2) cannot hold good for *all* values of x & t; but the astronomical phenomena of the actual transfer of luminiferous vibration from one part of space to another, & the terrestrial phenomena of Sound, &c., together with the calculations already made respecting these phenomena, appear to make it *probable* that we shall find, with some reasonable degree of approximation, that the law (2) still holds for *those* particular values of x and t which are connected by the relations

$$x < at^{\sin\nu} \qquad x + in > at^{\sin\nu} \tag{2$'$}$$

In fact, if this were rigorously true, it would imply that the *vibration travelled onwards*, with a *constant velocity* $= a\frac{\sin\nu}{\nu}$, and a *constant amplitude* $= 2\eta$, occupying always i *wave-lengths*, or agitating always a row of in particles, but leaving all which are beyond and all which are behind these undisturbed. I thought it therefore important to inquire, *how nearly* these conclusions follow from these assumptions; or, in other words, how nearly the differential equation (1), combined with the initial conditions (3) for all values of x from 0 to $-in$, and with the conditions

$$y_{x,0}=0, \quad y'_{x,0}=0, \tag{3$'$}$$

for all other values of x, conduct to the law (2), combined with the restrictions (2)$'$. And I have found that from these premises results do follow, which have a certain resemblance to the results thus expected or guessed at, but which also differ from them sensibly.

After long circuits of analysis, I obtained this form for the general & rigorous integral of the equation in mixed differences (1):

$$y_{x,t} = \frac{2}{\pi}\,\Sigma_{(l)-\infty}^{\infty}\left(y_{x+l,0} + y'_{x+l,0}\int_0^t dt\right)\int_0^2 d\theta\cos(2at\cos\theta)\cos(2l\theta + l\pi); \tag{1$'$}$$

which I afterwards developed into this other form

$$y_{x,t}=y_{x,0}+\frac{a^2t^2}{1\,.\,2}\Delta_x^2 y_{x-1,0}+\frac{a^4t^4}{1\,.\,2\,.\,3\,.\,4}\Delta_x^4 y_{x-2,0}+\&c.$$

$$+ty'_{x,0}+\frac{a^2t^3}{1\,.\,2\,.\,3}\Delta_x^2 y'_{x-1,0}+\frac{a^4t^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}\Delta_x^4 y'_{x-2,0}+\&c.; \qquad (1)''$$

$y_{x,0}$ & $y'_{x,0}$ being two quite arbitrary functions, representing the arbitrary initial displacements and velocities. Either of these two forms, (1)′ or (1)″, may easily be shown à posteriori to represent generally and rigorously the integral of the equation (1); & the latter of them may receive this elegant symbolic transformation:

$$y_{x,t}=\left\{1-\frac{a^2\Delta_x^2}{1+\Delta_x}\left(\int dt\right)^2\right\}^{-1}(y_{x,0}+ty'_{x,0}); \qquad (1)'''$$

which might have been directly obtained, by a simple process, from the proposed equation (1).

Introducing now the initial conditions, or determining the arbitrary functions so as to have

$$y_{x,0}=0, \quad y'_{x,0}=0, \quad \text{if } x>0 \text{ or } <-in, \qquad (3)'$$

but

$$y_{x,0}=\eta \operatorname{vers}\frac{2x\pi}{n} \qquad y'_{x,0}=-2a\eta\sin\frac{\pi}{n}\sin\frac{2x\pi}{n} \qquad (3)''$$

$$\text{if } x \ngtr 0, \text{ and } \nless -in;$$

I find this other expression, also rigorous, but adapted to this particular question,

$$y_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin in\theta}{\sin\theta}\frac{\cos(2x\theta+in\theta-2at\sin\theta)\,d\theta}{\cos\theta-\cos\frac{\pi}{n}}: \qquad (2)''$$

which may be shown à posteriori to be correct, especially if we observe that

$$\frac{\sin\nu}{\pi}\int_0^\pi\frac{\cos k\theta\,d\theta}{\cos\theta-\cos\nu}=+\sin k\nu,\ 0,\ \text{or}\ -\sin k\nu,$$

according as the integer k is >0, $=0$, or <0.

It remains therefore to discuss the rigorous integral (2)″, & to examine *how nearly* it reproduces the law of vibration (2) with the restrictions (2)′.

The expression (2)″ may be separated into two parts, of which one changes sign with a or t, while the other remains unaltered when the sign of a or t is changed; the latter of these two parts expresses the effect $Y_{x,t}$ of the initial displacements, and the former expresses the effect $Z_{x,t}$ of the initial velocities; we have, therefore,

Effect of initial displacements

$$=Y_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int^\pi\frac{\sin in\theta}{\sin\theta}\frac{\cos(2x\theta+in\theta)\cos(2at\sin\theta)\,d\theta}{\cos\theta-\cos\frac{\pi}{n}}; \qquad (2)'''$$

and

Effect of initial velocities

$$=Z_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin in\theta}{\sin\theta}\frac{\sin(2x\theta+in\theta)\sin(2at\sin\theta)\,d\theta}{\cos\theta-\cos\frac{\pi}{n}}. \qquad (2)^{\text{iv}}$$

But if it be true that the vibration spreads in one determined direction, that direction must depend on the initial velocities, & must change when a changes sign; so that while $Y_{x,t}+Z_{x,t}$ the displacement, whether $=0$ or >0, of the particle m_x at the time t, x & t being supposed for simplicity to be each >0, & the vibration being supposed to spread from m_0 towards m_{∞}, it seems that we ought to have a displacement either exactly or sensibly null, $Y_{x,t}-Z_{x,t}=0$, when, by changing the sign of a, we suppose the vibration to spread the contrary way towards $m_{-\infty}$. We are therefore to inquire whether, for positive values of x and t, the functions $Y_{x,t}$ & $Z_{x,t}$ are exactly, or even nearly equal; that is, whether the effects of the initial displacements & velocities are exactly or nearly the same.

Now, these two functions cannot be *exactly equal* for all positive values of x & t; because one of them, $Y_{x,t}$, is an even function, & the other, $Z_{x,t}$, is an odd function of t. Yet I have found that these two functions approach to a numerical equality, as t or x or either tends to $+\infty$; although one changes sign with t and the other does not.

This result is of the same character as the theorem:

$$\int^{\infty}\frac{\sin ax}{x}\,dx=+\frac{\pi}{2},\ \text{or}\ 0,\ \text{or}\ -\frac{\pi}{2},$$

according as a is >0, or $=0$, or <0; and indeed I make a frequent use of this last cited theorem in the general discussion of the foregoing integrals.

It is instructive to consider first the case of $i=\infty$, that is, the case when all the particles behind m_0 are supposed to be agitated according to the law (2), at the origin of t. This case is simpler in itself than the case of i finite, & the latter can be reduced to the former. The effect of a finite number of initial waves may be considered as the difference of the effects of two systems of such waves, each infinite in number but terminating differently.

Making $i=\infty$, I find that the integrals become

$$Y_{x,t}=\frac{\eta}{2}\left\{1-\cos\frac{2x\pi}{n}\cos\left(2at\sin\frac{\pi}{n}\right)\right\}-\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)\int^{\pi}\frac{\sin 2x\theta\cos(2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}$$

$$Z_{x,t}=-\frac{\eta}{2}\sin\frac{2x\pi}{n}\sin\left(2at\sin\frac{\pi}{n}\right)+\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi}\frac{\cos 2x\theta\sin(2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)},$$

$$y_{x,t}=\frac{\eta}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2at\sin\frac{\pi}{n}\right)-\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi}\frac{\sin(2x\theta-2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}$$

As n, though large, is finite, it is permitted to suppose that at is a good deal greater than n^3; I shall suppose it to be a large multiple r^2 of $\frac{s^2\sin\frac{\pi}{n}}{2\operatorname{vers}\frac{\pi}{n}\ ^{2}}$,

$$at=\frac{}{\left(2\operatorname{vers}\frac{\pi}{n}\right)^2},$$

in which s is a positive number so large that we have, sensibly,

$$\int_{-\infty}^{\infty} \cos\left(\phi \pm \frac{\psi^2}{s^2}\right) \frac{\sin\psi}{\psi}\, d\psi = \pi \cos\phi.$$

Then, for all values of x between the limits

$$\pm\left(at\cos\frac{\pi}{n} - \frac{s}{2}\sqrt{at\sin\frac{\pi}{n}}\right) = \pm\left\{at - \left(1+\frac{1}{r}\right) at \operatorname{vers}\frac{\pi}{n}\right\},$$

I find that we have, nearly, (& more and more exactly as t or r increases without limit,)

$$\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int^{\pi} \frac{\sin 2x\theta \cos(2at\sin\theta)\, d\theta}{\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)} = \tfrac{1}{2}\sin\frac{2x\pi}{n}\sin(2at\sin\theta),$$

$$\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^{\pi} \frac{\cos 2x\theta \sin(2at\sin\theta)\, d\theta}{\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)} = \tfrac{1}{2}\cos\frac{\pi}{n} - \tfrac{1}{2}\cos\frac{2x\pi}{n}\cos(2at\sin\theta);$$

and therefore

$$Y_{x,t} = \frac{\eta}{2}\operatorname{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right),$$

$$Z_{x,t} = \frac{\eta}{2}\operatorname{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \eta\left(\sin\frac{\pi}{2n}\right)^2;$$

so that these two functions are in fact nearly equal, if n be large, that is, if the wave-length contain many molecular intervals, & if at the same time we take t sufficiently large and confine x within the limits just assigned. On the same suppositions, the displacement becomes*

$$y_{x,t} = \eta\operatorname{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \eta\left(\sin\frac{\pi}{2n}\right)^2;$$

it differs therefore from that expressed by the law (2), only by a small and constant quantity. But the limits found for x differ sensibly from those anticipated in (2)′, & conduct to a sensibly different law of the *velocity of propagation*. In fact, on taking $x = at\cos\frac{\pi}{n}$, I find

$$y_{x,t} = \eta\left(\cos\frac{\pi}{2n}\right)^2 - \frac{\eta}{2}\cos\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right),$$

the coefficient of the cosine of the phase being reduced to half its former amount; and this coefficient becomes insensible on taking

$$x = at\cos\frac{\pi}{n} + \frac{s}{2}\sqrt{at\sin\frac{\pi}{n}},$$

which assumption gives

$$y_{x,t} = \eta\left(\cos\frac{\pi}{2n}\right)^2.$$

This value of the displacement remains sensibly constant, till we pass to $x = at - \frac{1}{2}s'(at)^{\frac{1}{3}}$, s' being another positive number so large as to give sensibly

$$\int_{-\infty}^{\infty} \sin\left(\psi \pm \frac{\psi^3}{3s'^3}\right)\frac{d\psi}{\psi} = \pi.$$

[See Appendix, Note 12, pp. 640, 641.]

But while x increases from $at-\frac{1}{2}s'(at)^{\frac{1}{3}}$ to $at+\frac{1}{2}s'(at)^{\frac{1}{3}}$, the displacement $y_{x,t}$ decreases from $\eta\left(\cos\frac{\pi}{2n}\right)^2$ to 0, being $=\frac{1}{3}\eta\left(\cos\frac{\pi}{2n}\right)^2$ when x is $=at$; & when x is greater than $at+\frac{1}{2}s'(at)^{\frac{1}{3}}$, the displacement $y_{x,t}$ is insensible. Now, by the suppositions which we have made, the limit conjectured in (2)′, namely

$$x=at\frac{\sin\nu}{\nu}=\frac{atn}{\pi}\sin\frac{\pi}{n},$$

is greater than $at\cos\frac{\pi}{n}+\frac{s}{2}\sqrt{at\sin\frac{\pi}{n}}$ & less than $at-\frac{1}{2}s'(at)^{\frac{1}{3}}$; we may even consider its excess over the one, & its defect below the other, as increasing nearly proportionally to t. The velocity of propagation therefore cannot be considered as $=a\frac{\sin\nu}{\nu}$; but must rather be represented by

$$a\cos\nu=a\cos\frac{}{n}.$$

In short, if we were to consider $a\frac{\sin\nu}{\nu}$ as the velocity of propagation, we should be obliged to consider as the front of the wave, or rather of the wave-system, a particle m_x which is not properly *vibrating* at all but is only *displaced* by an amount $=\eta\left(\cos\frac{\pi}{2n}\right)^2$, which is common to a great and increasing number of particles beyond and behind it. But when we consider $a\cos\nu$ as the velocity, we get for the front of the wave-system a particle which is not merely displaced but vibrating, with a phase determined by the law (2), though the amplitude of its excursions is only half the amplitude which that law would assign. It is situated in the middle of an interval which does indeed continually increase, but bears continually a less and less ratio to the time t; the particles beyond which interval have no amplitude of excursion, while the particles behind it have the full amplitude required by the law (2). And whatever particle we take, throughout this interval, the ratio of its abscissa x to the time t tends ultimately to the limit

$$a\cos\nu \quad \text{or} \quad a\cos\frac{\pi}{n},$$

and *not* to the limit

$$a\sin\nu \quad \text{or} \quad \frac{an}{\pi}\sin\frac{\pi}{n}.$$

I think then that a distinction is completely established, at least in this particular question, between the *rapidity of progress of a given phase*, in a space *already occupied* by an *indefinite* system of waves, & the *velocity of propagation of vibration*, by which a bounded system *comes to occupy new parts of space*. The two velocities differ little, it is true, if the number of molecular intervals contained in one wave-length be great; but the difference is finite and follows a simple law, namely that of the difference between the cosine of a small arc and the ratio of the sine of the same arc to that arc itself; the latter being the old, & the former being the new result—at least, the result is entirely new to me; I am anxious to learn whether you know of its having been obtained before. Of course I am aware that it would be precarious and premature to speculate on any physical consequences of it, until it shall have been made a nearer approach to nature, by our considering, as no doubt we ought to do, the action on any one particle as being the resultant of the actions not of *two* but of *many* others.

It is remarkable that the same reasonings conduct us to consider still a third different velocity, namely that with which the *disturbance* (as distinguished from the *vibration*) is propagated. This *velocity of propagation of disturbance* is simply $=a$; it is therefore independent of the length of the wave, & is equal to the square root of the attraction exerted by any one particle on that adjacent, the distance between these two adjacent particles being taken as the unit of length.

Disturbance is also propagated *backward* with the same velocity a; in such a manner that (when t is large and positive as before) all negative values of x which are algebraically

$$> -at+\tfrac{1}{2}s'\,(at)^{\frac{1}{3}}$$

give
$$y_{x,t}=\eta\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-\eta\left(\sin\frac{\pi}{2n}\right)^2;$$
the value $x=-at$ gives
$$y_{x,t}=\eta\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-\tfrac{1}{3}\eta\left(\sin\frac{\pi}{2n}\right)^2;$$
and values algebraically $<-at-\tfrac{1}{2}s'\,(at)^{\frac{1}{3}}$ give sensibly
$$y_{x,t}=\eta\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$
that is, they reproduce the law (2) without any sensible error, the effect of the discontinuity of the initial state of the medium not having yet had time to show itself.

It seems unnecessary to trouble you at present with any further details; except to mention generally that when I return to supposing the number i of the wave-lengths primitively agitated to be not infinite but only *very large*, I find results entirely analogous. Disturbance is propagated forward from the front and backward from the rere of the initial multiple wave with a velocity which may still be considered as $=a$; there is also a forward (but not a backward) propagation of vibration, & its velocity is still $=a\cos\frac{\pi}{n}$; the amplitude of the excursion is for a long time undiminished throughout the much larger part of the travelling multiple wave, of which the extent may long be considered as retaining its initial value $=in$; but however great the value of i may be, that is, however many simple waves we suppose to be in vibration at the origin of t, if only this number be finite, the progress of time will at last cause the *terminal diffusion* to encroach more and more upon the advancing i-fold wave, till its vibratory motion will have spent itself to all sense on the particles before and behind it, according to a law which I have been able to assign, and which gives, ultimately for the central particle, $\left(\text{whose abscissa is } -\frac{in}{2}+at\cos\frac{\pi}{n}\right)$, an amplitude of excursion which varies inversely as the square-root of the time.—The formulae allow us also to trace the motion backwards in *time*, so as to say what must *have been* the disturbances at any earlier epoch, in order to produce the assumed state at the moment $t=0$.

Feb. 9th.

It would be unreasonable to ask you to examine at present into the *correctness* of the foregoing results; which may indeed be affected by some mistake of transcription; but perhaps with little trouble you could give me an opinion with regard to their *novelty*, which might prevent me from claiming as my own what had been already discovered by others; or, in the contrary case, might encourage me to pursue the subject farther—I shall perhaps say something on the matter to the R.I.A. on ☽ next; in which event, it will be expected that I should allow some notice of it to be printed in the "Proceedings" or Monthly Notices.

Now, any communication from you which should reach me in about a *week from this date* might be of use in the construction of such a notice; & that without mentioning your name unless you expressly desired or allowed it.—Received Diploma & Letter for Miss H.[*]

W. R. H.

---

[*] [Miss Caroline Herschel. See Graves, *Life of Sir W. R. Hamilton*, II, pp. 290, 291, for Herschel's reply to this letter.]

EDITORS' APPENDIX

EDITORS' APPENDIX

CONTENTS

PAGE

Note 1. On the approximation to t. (I, pp. 37–42) 612

Note 2. On the determination of Hamilton's principal function. (II, pp. 130, 151) . . 613

Note 3. Comparison of Hamilton's and Lagrange's methods. (II, p. 160) 622

Note 4. Explanatory. (III, p. 202) 623

Note 5. The principal function for the undisturbed motion of a planet. (V, p. 217) . . 624

Note 6. On the approximation to the parallactic inequality. (VI, p. 248; VII, pp. 272–275) 627

Note 7. On Hamilton's method of approximating to the motion of the moon. (III, p. 209; VII, pp. 266, 269) 628

Note 8. Hodographic and anthodographic isochronism. (XI, p. 293) 630

Note 9. Hamilton's method of finding the principal function from the partial differential equation. (XIII, pp. 335–339) 631

Note 10. Hamilton's method of solving partial differential equations and its relation to Cauchy's theory. (XV, pp. 391–407) 634

Note 11. Hamilton's luminiferous æther. (XVII, p. 413) 638

Note 12. Example of wave motion. (XX, pp. 565, 566; XXIII, pp. 604–606) 640

NOTE 1. *On the approximation to t*

(I, pp. 37–42.)

All the difficulties here arise, as Hamilton surmised, from the definition of $t_{,}$ in (330) (p. 30). The definition should have been of the form $t = t_{,} + m't_{,,} + mt'_{,,}$. The quantities $t_{,,}$ and $t'_{,,}$ become determinate if we impose the condition that they are symmetrical, and $t_{,}$ can be defined by the equations

$$t_{,} = a_{,}^{\frac{3}{2}}(\theta - \theta_0) = a_{,}'^{\frac{3}{2}}(\theta' - \theta_0');$$

a, a', t can be replaced by $a_{,}$, $a'_{,}$, $t_{,}$ in the terms multiplied by mm'.

Putting $$V = ma^{\frac{1}{2}}(\theta - \theta_0) + m'a'^{\frac{1}{2}}(\theta' - \theta_0') + mm'(H_{,,}t + W),$$

a, a' can be eliminated by the usual process,

$$\frac{\partial V}{\partial a} = t\frac{\partial h}{\partial a}; \quad \frac{\partial V}{\partial a'} = t\frac{\partial h}{\partial a'}; \tag{A}$$

and since

$$\frac{\partial h}{\partial a}\frac{\partial a}{\partial \theta_0'} + \frac{\partial h}{\partial a'}\frac{\partial a'}{\partial \theta_0'} = 0,$$

we can, in finding $\dfrac{\partial V}{\partial \theta_0'}$, treat a and a' as constants.

t occurs in $H_{,,}t + W$; here we can replace t by $t_{,}$, to which we can give the symmetrical value

$$t_{,} = \tfrac{1}{2}a^{\frac{3}{2}}(\theta - \theta_0) + \tfrac{1}{2}a'^{\frac{3}{2}}(\theta' - \theta_0').$$

Hence we get, for initial values,

$$\frac{\partial}{\partial \theta_0'}(H_{,,}t + W) = -\tfrac{1}{2}a'^{\frac{3}{2}}(H_{,,} + C_0) = -aa'n\cos(\epsilon - \epsilon');$$

and finally

$$\frac{\partial V}{\partial \theta_0'} = -m'a'^{\frac{1}{2}} - mm'aa'\cos(\epsilon - \epsilon') = -m'a_{,}'^{\frac{1}{2}} - mm'\left\{aa'n\cos(\epsilon - \epsilon') + \frac{a'_{,,}}{2\sqrt{a'}}\right\},$$

which agrees with (424), (p. 37).

Returning to (A), we get, putting $h = -\dfrac{m}{2(1+m)a} - \dfrac{m'}{2(1+m')a'}$,

$$\tfrac{1}{2}a^{-\frac{1}{2}}(\theta - \theta_0) + m'\{\tfrac{3}{4}a^{\frac{1}{2}}(\theta - \theta_0)(C + H_{,,}) + D\} = \frac{t}{2a^2}(1 - m),$$

$$\tfrac{1}{2}a'^{-\frac{1}{2}}(\theta - \theta_0) + m\{\tfrac{3}{4}a'^{\frac{1}{2}}(\theta - \theta_0)(C + H_{,,}) + D'\} = \frac{t}{2a'^2}(1 - m').$$

From these we can easily obtain a symmetrical form for t in the form $t_{,} + m't_{,,} + mt'_{,,}$ and, by putting $a_{,} + m'a_{,,}$, $a'_{,} + ma'_{,,}$ for a, a' in the first term in each equation, get correct values for $a_{,,}$ and $a'_{,,}$.

NOTE 2. *On the determination of Hamilton's principal function*[*]

(II, pp. 130, 151.)

1. *Introduction*

The use of the characteristic function in optics led Hamilton to the application of this function to dynamics.[†] His first important exposition was in his first essay on dynamics,[‡] at the end of which he introduced the more general function S. In the second essay [§] he studied the chief properties of this function S, which is there called the principal function, proving incidentally that it satisfies two partial differential equations.

Jacobi's first important contribution to the theory was given in the memoir,[||] in which he proves that any complete integral (in Lagrange's sense) of the first of Hamilton's two differential equations suffices to give the paths and time of the dynamical system. He also gives the often referred to, but seldom quoted, criticism which is reproduced here in full.[¶]

"Hamilton scheint mir dadurch seine schöne Entdeckung in ein falsches Licht gesetzt zu haben, ausserdem dass sie dadurch zu gleicher Zeit unnöthig complicirt und beschränkt wird. Auch ist hier der Uebelstand, dass, da man eine Function nicht durch zwei partielle Differentialgleichungen definiren kann, denen sie gleichzeitig genügen soll, ohne zu beweisen, dass eine solche Function auch wirklich möglich ist, sein Theorem, wie er es ausgesprochen hat, nicht an sich, sondern nur mit dem Beweise, den er liefert, verständlich sein kann. Wenn dadurch, dass er gerade diese besondere Function S nimmt, die willkürlichen Constanten die Anfangswerthe der Coordinaten und der nach den Coordinaten-Axen zerlegten Geschwindigkeiten werden, so hat dies kein wesentliches Interesse, da die Einführung dieser Constanten die Form der Integralgleichungen in der Regel complicirter macht, man auch die vollständigen Integralgleichungen aus jeder andern Form in diese bringen kann. Vielleicht ist auch Hamilton dadurch, dass er immer gleichzeitig zwei partielle Differentialgleichungen vor Augen hat, verhindert worden, die allgemeinen Vorschriften, welche Lagrange in den Vorlesungen über die Functionenrechnung für die Integration einer nicht linearen partiellen Differentialgleichung erster Ordnung zwischen drei Variabeln giebt, auf sein Theorem anzuwenden, wodurch ihm, wie ich in einer andern Abhandlung zeigen werde, Resultate von grösster Wichtigkeit für die Mechanik entgangen sind. Ich bemerke noch, dass die Forderung, dass die Function S, nachdem sie der ersten partiellen Differentialgleichung genügt, noch der zweiten genügen solle, auch noch dadurch eine Beschränkung herbeiführt, dass sie den Fall ausschliesst, wo die Kräftefunction U die Zeit t auch explicite enthält, weil für diesen die zweite partielle Differentialgleichung nicht mehr gültig ist."

---

[*] *Proceedings Royal Irish Academy*, Vol. XLI, A (1932), No. 3, pp. 18–25.

[†] See *The Mathematical Works of Sir W. R. Hamilton*, Vol. I, p. 484.

[‡] Pp. 103–161 of this volume.

[§] Pp. 162–211.

[||] "Über die Reduction der Integration der partiellen Differentialgleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen," *Crelle*, Vol. XVII, pp. 97–162, *Jacobi's Gesammelte Werke*, Vol. IV, pp. 57–127.

[¶] *Gesammelte Werke*, Vol. IV, pp. 73, 74.

This criticism contains two points which have not been sufficiently examined. In the first place, no proof is given that a solution of the two simultaneous equations exists. In reply to this, it can be stated that although Hamilton never published a proof, yet he must certainly have possessed one, for in a note-book on the three-body problem, written early in 1833, he gives a rigorous proof for a special case—an orbit under a central force.*

The second criticism,† that his occupation with *two* partial differential equations prevented him from applying Lagrange's method for solving such equations, is more difficult to answer. It *is* a fact that in the note-book referred to this very difficulty made his progress slow and tedious. In the few places where he does form the characteristic function of a special dynamical problem—as, for example, on page 130 in the first essay—we are left in doubt as to what method he employed, for he has given no direct account of it. It seems probable that he first integrated the differential equations of motion and then evaluated by direct integration the right-hand side of the equation

$$V=\int_{t_0}^{t} 2T\,dt.$$

Expressing this in terms of the initial and final configurations he obtained the characteristic function in the appropriate form.

An alternative method‡ is described in the present paper, in which it is shown that the Hamiltonian form of the principal function S, and therefore of the characteristic function V also, can be obtained immediately from the Jacobi complete integral.

2. *The Hamilton integral*

If we are given a dynamical system of N degrees of freedom whose coordinates are $x_1, x_2, \ldots, x_N$ and whose Hamiltonian function is

$$H(x_1, \ldots, x_N, p_1, \ldots, p_N, t),$$

we know that the equations of motion are

$$\frac{dx_r}{dt}=\frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt}=-\frac{\partial H}{\partial x_r}, \tag{1}$$

and that Hamilton's partial differential equation associated with the system is

$$\frac{\partial V}{\partial t}+H\left(x, \frac{\partial V}{\partial x}, t\right)=0. \tag{2}$$

If we know a complete integral of this equation we can easily deduce Hamilton's principal function.

---

* See pp. 79, 80 of this volume.

† Cf. Poisson's criticism: "...c'est à dire l'exprimer en fonction des inconnues ϕ, ψ, θ, etc., de leurs valeurs initiales α, β, γ, etc., et de la constante des forces vives. Cette méthode d'intégration est celle que M. Hamilton a donnée...L'usage serait très bornée et à peu près nul, si l'on voulait, comme l'auteur le propose, le faire servir à trouver toutes les intégrales premières et secondes des équations de mouvement, et qu'il fallât déterminer *à priori* la fonction V, sans connaître aucune de ces intégrales." Poisson, *Liouville, Journal de Mathématiques* (1837), pp. 317–337.

‡ For another method of finding S afterwards proposed by Hamilton himself, see Note 9, p. 631.

Let

$$f(x_1, \ldots, x_N, t, a_1, \ldots, a_N) + a_{N+1}$$

be a complete integral of (2) and let us put

$$S = f(x, t, a) - f(x^0, t^0, a), \tag{3}$$

where x_r^0, t^0 are any given values of x_r, t. We associate with (3) the equations

$$\frac{\partial S}{\partial a_r} = \frac{\partial f}{\partial a_r} - \frac{\partial f_0}{\partial a_r} = 0, \tag{4}$$

in which we have denoted $f(x^0, t^0, a)$ by f_0 for brevity. Equations (4) are just sufficient in number to solve for the a's in terms of x_r, t and x_r^0, t^0 and, on substituting these values in (3), we obtain a function S of the variables x_r, t, x_r^0, t^0. We shall first show that S is an integral of (2), which we shall call *the Hamilton integral.*

We have immediately

$$\frac{\partial S}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial S}{\partial a_m}\frac{\partial a_m}{\partial t} = \frac{\partial f}{\partial t},$$

$$\frac{\partial S}{\partial x_r} = \frac{\partial f}{\partial x_r} + \frac{\partial S}{\partial a_m}\frac{\partial a_m}{\partial x_r} = \frac{\partial f}{\partial x_r},$$

the repeated suffix m implying summation from 1 to N. Since $f(x, t, a)$ satisfies (2), we conclude that

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}, t\right) = 0,$$

showing that S is also an integral. It is a complete integral depending on the $N+1$ arbitrary constants x_r^0, t^0.

3. *The identification of the Hamilton integral with Hamilton's principal function*

We shall next show that the Hamilton Integral

$$S(x, t, x^0, t^0)$$

is Hamilton's principal function.

Let us consider the equations

$$\frac{\partial f}{\partial a_r} = \frac{\partial f_0}{\partial a_r}, \tag{5}$$

where the a's are *arbitrary constants*. Since $f(x, t, a)$ is a complete integral the determinant

$$\frac{\partial^2 f}{\partial a_r \partial x_s}$$

is different from zero, and equations (5) can be solved for x_r, giving equations of the form

$$x_r = x_r(t, t^0, x^0, a). \tag{6}$$

Moreover it is obvious from (5) that $x_r = x_r^0$, $t = t^0$ satisfy these equations, and therefore in (6) x_r reduces to x_r^0 when $t = t^0$. If we also put

$$p_r = \frac{\partial f}{\partial x_r} \tag{7}$$

and associate these last equations with (6), we obtain the set of equations

$$\left.\begin{aligned} x_r &= x_r(t, t^0, x^0, a), \\ p_r &= p_r(t, t^0, x^0, a). \end{aligned}\right\} \tag{8}$$

We shall show that (8) is the general solution of the equations of motion (1).

From (5)

$$\frac{\partial^2 f}{\partial a_r \partial x_s}\frac{dx_s}{dt} + \frac{\partial^2 f}{\partial a_r \partial t} = 0. \tag{9}$$

But we have

$$\frac{\partial f}{\partial t} + H(x, p, t) = 0, \quad p_r = \frac{\partial f}{\partial x_r}, \tag{10}$$

for all values of the a's. Therefore

$$\frac{\partial^2 f}{\partial a_r \partial t} + \frac{\partial H}{\partial p_s}\frac{\partial^2 f}{\partial a_r \partial x_s} = 0, \tag{11}$$

and we obtain from (9) and (11) the result that

$$\frac{dx_r}{dt} = \frac{\partial H}{\partial p_r}.$$

Also from (7)

$$\frac{dp_r}{dt} = \frac{\partial^2 f}{\partial x_r \partial t} + \frac{\partial^2 f}{\partial x_r \partial x_s}\frac{dx_s}{dt} = \frac{\partial^2 f}{\partial x_r \partial t} + \frac{\partial^2 f}{\partial x_r \partial x_s}\frac{\partial H}{\partial p_s}.$$

But when we differentiate (10) partially with respect to x_r we get

$$\frac{\partial^2 f}{\partial x_r \partial t} + \frac{\partial H}{\partial x_r} + \frac{\partial H}{\partial p_s}\frac{\partial^2 f}{\partial x_r \partial x_s} = 0.$$

Hence

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial x_r}.$$

We have, therefore, verified that (8) is the general solution of the equations of motion since it contains the requisite number of arbitrary constants. It gives all the natural motions that pass through the configuration x_r^0 at the time t^0.

Now, Hamilton's principal function is defined as the integral

$$\int_{t^0}^{t}\left(p_r\frac{\partial H}{\partial p_r} - H\right)dt, \tag{12}$$

evaluated along the natural motion which passes through the configurations x_r, x_r^0 at the times t, t^0 respectively, and is expressed in terms of the initial and final variables x_r^0, t^0 and x_r, t. This natural motion is given by (8) for suitable values of the constants a_r, or by the equivalent equations (5) and (7).

We have, consequently,

$$\begin{aligned} \int_{t^0}^{t}\left(p_r\frac{\partial H}{\partial p_r} - H\right)dt &= \int_{t^0}^{t}\left(p_r\frac{dx_r}{dt} - H\right)dt \\ &= \int_{t^0}^{t}\left(\frac{\partial f}{\partial x_r}\frac{dx_r}{dt} + \frac{\partial f}{\partial t}\right)dt = f(x, t, a) - f(x^0, t^0, a), \end{aligned}$$

and we express the integral in terms of x_r, t and x_r^0, t^0 by eliminating the a's by means of (5). This is exactly the definition of $S(x, t, x^0, t^0)$, and we have therefore identified the Hamilton integral with Hamilton's principal function.

If $f(x, t, a)+a_{N+1}$ *is a complete integral of Hamilton's partial differential equation, then Hamilton's principal function is obtained by eliminating* a_r *from the equations*

$$\begin{gathered} S=f(x, t, a)-f(x^0, t^0, a), \\ \frac{\partial f(x, t, a)}{\partial a_r}=\frac{\partial f(x^0, t^0, a)}{\partial a_r}. \end{gathered} \tag{13}$$

4. *Geometrical interpretation*

To find a geometrical interpretation of this construction of Hamilton's principal function we shall take a space of $N+2$ dimensions, whose coordinates are

$$x_r,\ t,\ V.$$

In this manifold the equation

$$V=V(x_r, t)$$

represents a hypersurface, and if V satisfies the partial differential equation (2) we call it *an integral hypersurface.*

If $f(x, t, a)+a_{N+1}$ is a complete integral of (2), then

$$V=f(x, t, a)+a_{N+1}$$

is a family of integral hypersurfaces depending on the $N+1$ parameters $a_1, \ldots, a_{N+1}$. We take from this family those which pass through the point $(x_r^0, t^0, 0)$. The condition for this is

$$f(x^0, t^0, a)+a_{N+1}=0,$$

and the new family is

$$V=f(x, t, a)-f(x^0, t^0, a), \tag{14}$$

depending on N parameters. The envelope of this family is obtained by eliminating the a's from (14) and the N equations

$$\frac{\partial f}{\partial a_r}-\frac{\partial f_0}{\partial a_r}=0.$$

But this elimination leads to the hypersurface

$$V=S(x, t, x^0, t^0). \tag{15}$$

We have, therefore, proved the following theorem:

If $S(x, t, x^0, t^0)$ *is Hamilton's principal function, the hypersurface*

$$V=S(x, t, x^0, t^0)$$

is the envelope of an N-ply infinite family of integral hypersurfaces of Hamilton's partial differential equation, all of which pass through the point $(x_r^0, t^0, 0)$.

5. *An example of the method*

As an example of the method let us consider a dynamical system of three degrees of freedom whose coordinates are x, y, z, and whose Hamiltonian function is

$$\tfrac{1}{2}(p_x^2+p_y^2+p_z^2)+\tfrac{1}{2}(\lambda^2x^2+\mu^2y^2+\nu^2z^2)-\epsilon,$$

where λ, μ, ν, ϵ are constants. The Hamilton partial differential equation is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}=\epsilon-\tfrac{1}{2}(\lambda^2x^2+\mu^2y^2+\nu^2z^2).$$

We can obtain a complete integral of this equation by the method of separation of the variables. The integral is

$$f=-\lambda\int\sqrt{\alpha^2-x^2}\,dx-\mu\int\sqrt{\beta^2-y^2}\,dy-\nu\int\sqrt{\gamma^2-z^2}\,dz+\{\epsilon-\tfrac{1}{2}(\lambda^2\alpha^2+\mu^2\beta^2+\nu^2\gamma^2)\}\,t+\delta,$$

α, β, γ, δ being the arbitrary constants. We have, therefore,

$$S=-\lambda\int_{x_0}^{x}\sqrt{\alpha^2-x^2}\,dx-\mu\int_{y_0}^{y}\sqrt{\beta^2-y^2}\,dy-\nu\int_{z_0}^{z}\sqrt{\gamma^2-z^2}\,dz+\{\epsilon-\tfrac{1}{2}(\lambda^2\alpha^2+\mu^2\beta^2+\nu^2\gamma^2)\}(t-t_0),$$

where, until the integrations are performed, α, β, γ are regarded as constants and afterwards as functions of x, y, z, t, x_0, y_0, z_0, t_0, given by

$$\frac{\partial S}{\partial\alpha}=\frac{\partial S}{\partial\beta}=\frac{\partial S}{\partial\gamma}=0.$$

Let us suppose that

$$\alpha>x_0>x, \text{ etc.}$$

Then from $\dfrac{\partial S}{\partial\alpha}=0$ we get

$$\cos^{-1}\frac{x}{\alpha}-\cos^{-1}\frac{x_0}{\alpha}=\lambda(t-t_0),$$

that is,

$$\alpha=\frac{\sqrt{\{x^2+x_0^2-2xx_0\cos\lambda(t-t_0)\}}}{\sin\lambda(t-t_0)}$$

and

$$\sqrt{(\alpha^2-x^2)}=\frac{x_0-x\cos\lambda(t-t_0)}{\sin\lambda(t-t_0)},\quad\sqrt{(\alpha^2-x_0^2)}=\frac{x_0\cos\lambda(t-t_0)-x}{\sin\lambda(t-t_0)}.$$

We therefore have

$$\begin{aligned}-\int_{x_0}^{x}\sqrt{\alpha^2-x^2}\,dx&=\tfrac{1}{2}\alpha^2\left(\cos^{-1}\frac{x}{\alpha}-\cos^{-1}\frac{x_0}{\alpha}\right)-\tfrac{1}{2}\left(x\sqrt{\alpha^2-x^2}-x_0\sqrt{\alpha^2-x_0^2}\right)\\&=\tfrac{1}{2}\alpha^2\lambda(t-t_0)-\frac{1}{2}\left[\frac{x\{x_0-x\cos\lambda(t-t_0)\}-x_0\{x_0\cos\lambda(t-t_0)-x\}}{\sin\lambda(t-t_0)}\right]\\&=\tfrac{1}{2}\alpha^2\lambda(t-t_0)+\tfrac{1}{2}(x-x_0)^2\cot\lambda(t-t_0)-xx_0\tan\tfrac{1}{2}\lambda(t-t_0).\end{aligned}$$

Hence the final form of S is

$$\begin{aligned}S=\tfrac{1}{2}\lambda(x-x_0)^2\cot\lambda(t-t_0)&+\tfrac{1}{2}\mu(y-y_0)^2\cot\mu(t-t_0)+\tfrac{1}{2}\nu(z-z_0)^2\cot\nu(t-t_0)\\&-\lambda xx_0\tan\tfrac{1}{2}\lambda(t-t_0)-\mu yy_0\tan\tfrac{1}{2}\mu(t-t_0)-\nu zz_0\tan\tfrac{1}{2}\nu(t-t_0)+\epsilon(t-t_0).\end{aligned}$$

6. *The principal function of a binary system*

It is interesting to apply the method also to find an expression for the principal function of a binary system, as it leads directly to a formula for the characteristic function which was used by Hamilton without any indication as to how it arose.* In fact, the expression for the characteristic function just referred to actually suggests the method described above.

We have two masses m_1 and m_2, whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2), and the force-function is $m_1 m_2 f(r)$, where r is the distance between the masses. If (x, y, z) are the coordinates of the centre of gravity of the masses and we put

$$x_1 - x_2 = \xi, \quad y_1 - y_2 = \eta, \quad z_1 - z_2 = \zeta,$$

we have

$$(m_1 + m_2)\, x = m_1 x_1 + m_2 x_2 = (m_1 + m_2)\, x_2 + m_1 \xi = (m_1 + m_2)\, x_1 - m_2 \xi, \text{ etc.},$$

and so

$$x_1 = x + \frac{m_2}{m_1 + m_2}\xi, \text{ etc.}, \quad x_2 = x - \frac{m_1}{m_1 + m_2}\xi, \text{ etc.}$$

We have therefore for the kinetic energy

$$\begin{aligned} T &= \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \\ &= \tfrac{1}{2} (m_1 + m_2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2). \end{aligned}$$

Also

$$U = m_1 m_2 f(r),$$

where

$$r = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

The Hamiltonian function H of the system is easily seen to be

$$H = \frac{1}{2}\frac{1}{m_1 + m_2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}\frac{m_1 + m_2}{m_1 m_2}(p_\xi^2 + p_\eta^2 + p_\zeta^2) - m_1 m_2 f(r),$$

and the partial differential equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\frac{1}{m_1 + m_2}\left\{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right\} + \frac{1}{2}\frac{m_1 + m_2}{m_1 m_2}\left\{\left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2 + \left(\frac{\partial S}{\partial \zeta}\right)^2\right\} - m_1 m_2 f(r) = 0.$$

To solve this equation, let us write

$$S = -Ht + V + V_{,},$$

where H is an arbitrary constant, V is a function of x, y, z and $V_{,}$ a function of ξ, η, ζ. We shall have

$$\frac{1}{m_1 + m_2}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} + \frac{m_1 + m_2}{m_1 m_2}\left\{\left(\frac{\partial V_{,}}{\partial \xi}\right)^2 + \left(\frac{\partial V_{,}}{\partial \eta}\right)^2 + \left(\frac{\partial V_{,}}{\partial \zeta}\right)^2\right\} - 2 m_1 m_2 f(r) = 2H,$$

and it is immediately obvious that we have

$$\frac{1}{m_1 + m_2}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} = 2(H - H_{,}) = 2H_{,,}$$

and

$$\frac{m_1 + m_2}{m_1 m_2}\left\{\left(\frac{\partial V_{,}}{\partial \xi}\right)^2 + \left(\frac{\partial V_{,}}{\partial \eta}\right)^2 + \left(\frac{\partial V_{,}}{\partial \zeta}\right)^2\right\} = 2 m_1 m_2 f(r) + 2H_{,},$$

where $H_{,}$ is a new arbitrary constant.

---

* P. 130.

Taking, first of all, the equation for V, it is immediately seen to be satisfied by

$$V = \lambda x + \mu y + \sqrt{2(m_1 + m_2) H_{,,} - \lambda^2 - \mu^2}\,.\,z,$$

λ, μ being constants, and this is obviously a complete integral.

Secondly, taking the equation for $V_{,}$, we wish to find a complete integral depending on the initial and final radii vectores r, r_0 and the angle θ between them. If α, β, γ are the initial values of ξ, η, ζ,

$$r = \sqrt{\xi^2 + \eta^2 + \zeta^2}, \quad r_0 = \sqrt{\alpha^2 + \beta^2 + \gamma^2},$$

$$r r_0 \cos\theta = \xi\alpha + \eta\beta + \zeta\gamma.$$

Hence

$$\frac{\partial r}{\partial \xi} = \frac{\xi}{r}, \quad \frac{\xi}{r} r_0 \cos\theta - r r_0 \sin\theta \frac{\partial \theta}{\partial \xi} = \alpha,$$

that is,

$$\frac{\partial \theta}{\partial \xi} = \frac{\xi}{r^2} \cot\theta - \frac{\alpha}{r r_0} \operatorname{cosec}\theta.$$

Let us examine if we can find a solution of the partial differential equation of the form

$$V_{,} = R(r) + \Theta(\theta),$$

R being a function of r alone and Θ of θ alone. We have

$$\frac{\partial V_{,}}{\partial \xi} = R' \frac{\partial r}{\partial \xi} + \Theta' \frac{\partial \theta}{\partial \xi} = R' \frac{\xi}{r} + \Theta' \left\{ \frac{\xi}{r^2} \cot\theta - \frac{\alpha}{r r_0} \operatorname{cosec}\theta \right\},$$

and thus the partial differential equation becomes

$$\frac{m_1 + m_2}{m_1 m_2} \left\{ R'^2 + \frac{1}{r^2} \Theta'^2 \right\} = 2 m_1 m_2 f(r) + 2 H_{,},$$

which is satisfied if we put

$$\Theta' = \frac{m_1 m_2}{m_1 + m_2} h,$$

$$R' = \frac{m_1 m_2}{m_1 + m_2} \left\{ 2(m_1 + m_2) \left(f(r) + \frac{H_{,}}{m_1 m_2} \right) - \frac{h^2}{r^2} \right\}^{\frac{1}{2}} = \frac{m_1 m_2}{m_1 + m_2} \rho,$$

h being a constant. These equations give

$$\Theta = \frac{m_1 m_2}{m_1 + m_2} h\theta, \quad R = \frac{m_1 m_2}{m_1 + m_2} \int \rho\, dr.$$

Collecting our results, we have the complete integral of the original differential equation

$$S = -Ht + \lambda x + \mu y + \sqrt{2(m_1 + m_2) H_{,,} - \lambda^2 - \mu^2}\,.\,z + \frac{m_1 m_2}{m_1 + m_2} \left\{ h\theta + \int \rho\, dr \right\}.$$

Applying the above method to find the principal function we put

$$S = -H(t - t_0) + \lambda(x - x_0) + \mu(y - y_0) + \sqrt{2(m_1 + m_2) H_{,,} - \lambda^2 - \mu^2}\,.\,(z - z_0) + \frac{m_1 m_2}{m_1 + m_2} \left\{ h\theta + \int_{r_0}^{r} \rho\, dr \right\},$$

and we have to eliminate λ, μ, H, $H_{,}$, h between this equation and the following,

$$\frac{\partial S}{\partial \lambda} = 0, \quad \frac{\partial S}{\partial \mu} = 0, \quad \frac{\partial S}{\partial H} = 0, \quad \frac{\partial S}{\partial H_{,}} = 0, \quad \frac{\partial S}{\partial h} = 0.$$

It is easy to eliminate λ, μ and we get

$$S = -H(t-t_0) + \sqrt{2(m_1+m_2)H_{\prime\prime}}\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} + \frac{m_1 m_2}{m_1+m_2}\left\{h\theta + \int_{r_0}^{r} \rho\, dr\right\},$$

where we must eliminate H, $H_{\prime}$, h by means of the equations

$$\frac{\partial S}{\partial H} = -(t-t_0) + \sqrt{\frac{m_1+m_2}{2H_{\prime\prime}}}\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = 0,$$

$$\frac{\partial S}{\partial H_{\prime}} = -\sqrt{\frac{m_1+m_2}{2H_{\prime\prime}}}\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} + \frac{m_1 m_2}{m_1+m_2}\int_{r_0}^{r} \frac{\partial\rho}{\partial H_{\prime}}\, dr = 0,$$

$$\frac{\partial S}{\partial h} = \frac{m_1 m_2}{m_1+m_2}\left\{\theta + \int_{r_0}^{r} \frac{\partial\rho}{\partial h}\, dr\right\} = 0.$$

Remembering that $S + H(t-t_0)$ is the characteristic function of the system, we see at once that the above expression for the characteristic function is exactly the form used by Hamilton on p. 130.

NOTE 3. *Comparison of Hamilton's and Lagrange's methods*

(II, p. 160.)

Lagrange's method of variation of elements is as follows (Cf. *Mécanique Analytique*, Tome I, 2nd Part, Section V). If we have n degrees of freedom, each of the coordinates, ξ_r, in the "undisturbed" problem is supposed to be expressed in terms of the time and $2n$ arbitrary constants (C_r). In the disturbed problem these constants are supposed to be functions of the time. Substituting in the n equations of motion we get n relations concerning the (now) variable quantities (C_r). The problem is made determinate and greatly simplified by the following assumption—that the coordinates (ξ_r) and the velocities (ξ'_r) are the same functions of the quantities (C_r) in the disturbed and in the undisturbed motion. This is the meaning of the statement "...calculate at once the perturbations of place and of velocity, by employing the same rules, and altering at once the initial position and initial velocity". If we give a displacement $\Delta\alpha_i$, $\Delta\beta_i$, $\Delta\gamma_i$ to the initial position we get

$$\Delta\alpha'_i = \Sigma_{\prime\prime} \, . \, m_k \left\{ \int_0^t \frac{\delta R^{(i,k)}}{\delta\alpha_i} dt + \qquad {}^{t}\frac{\delta R^{(i,k)}}{\delta\alpha'_i} dt + \qquad {}^{t}\frac{\delta R^{(i,k)}}{\delta\beta'_i} dt + \frac{\delta\gamma'_i}{\delta\alpha_i} \int_0^t \frac{\delta R^{(i,k)}}{\delta\gamma'_i} dt \right.$$
$$+ \frac{\delta\alpha'_i}{\delta\alpha_i}\Delta\alpha_i + \frac{\delta\alpha'_i}{\delta\beta_i}\Delta\beta_i + \frac{\delta\alpha'_i}{\delta\gamma_i}\Delta\gamma_i .$$

The perturbations given in (T⁷.) satisfy these equations, since $\frac{\delta\beta'_i}{\delta\alpha_i} = \frac{\delta\alpha'_i}{\delta\beta_i}$.

The Lagrangian assumption will be satisfied if we can differentiate the equations

$$\xi_i + \Delta\xi_i = \phi_i(\alpha_i + \Delta\alpha_i, \, \ldots, \, \alpha'_i + \Delta\alpha'_i, \, \ldots, \, t)$$

once with respect to the time and get the same result whether we regard $\Delta\alpha_i, \ldots, \Delta\alpha'_i, \ldots$ constant or variable. This gives

$$0 = \frac{\delta\xi_i}{\delta\alpha_i}\frac{d}{dt}(\Delta\alpha_i) + \frac{\delta\xi_i}{\delta\beta_i}\frac{d}{dt}(\Delta\beta_i) + \frac{\delta\xi_i}{\delta\gamma_i}\frac{d}{dt}(\Delta\gamma_i) + \frac{\delta\xi_i}{\delta\alpha'_i}\frac{d}{dt}(\Delta\alpha'_i) + \frac{\delta\xi_i}{\delta\beta'_i}\frac{d}{dt}(\Delta\beta'_i) + \frac{\delta\xi_i}{\delta\gamma'_i}\frac{d}{dt}(\Delta\gamma'_i),$$

and from (T⁷.)

$$0 = \Sigma_{\prime\prime} m_k \left(\frac{\delta\xi_i}{\delta\alpha'_i}\frac{\delta R^{(i,k)}}{\delta\alpha_i} - \frac{\delta\xi_i}{\delta\alpha_i}\frac{\delta R^{(i,k)}}{\delta\alpha'_i} + \ldots \right).$$

Thus the perturbations in (T⁷.) satisfy the Lagrangian requirements, and by noticing the number of conditions we see that the expressions are unique.

NOTE 4. *Explanatory*

(III, p. 202.)

The integrals of angular momentum are

$$\xi y' - \eta x' = C_1, \quad \eta z' - \zeta y' = C_2, \quad \zeta x' - \xi z' = C_3 .$$

These can be written in the form

$$\begin{cases} \xi y' - \eta x' = \kappa - \lambda, \\ \eta z' - \zeta y' = -\sqrt{2\kappa\lambda - \lambda^2}\sin\nu, \\ \zeta x' - \xi z' = \sqrt{2\kappa\lambda - \lambda^2}\cos\nu, \end{cases}$$

which gives equations (1), (2) and (4) of (Q^2.).

The integral of energy gives

$$\frac{M+m}{2M}(x'^2 + y'^2 + z'^2) = \mu + Mf,$$

which is (3) of (Q^2.).

Now

$$r^2 = \xi^2 + \eta^2 + \zeta^2 \text{ and } rr' = \xi\xi' + \eta\eta' + \zeta\zeta' = \frac{M+m}{M}(\xi x' + \eta y' + \zeta z').$$

Hence

$$\kappa^2 = \Sigma(\xi y' - \eta x')^2 = (\xi^2 + \eta^2 + \zeta^2)(x'^2 + y'^2 + z'^2) + (\xi x' + \eta y' + \zeta z')^2.$$

Thus from the equation of energy

$$\kappa^2 = r^2 \cdot \frac{2M}{M+m}(\mu + Mf) - \frac{M^2}{(M+m)^2} r^2 r'^2$$

or

$$\frac{M}{M+m} r'^2 = 2(\mu + Mf) - \frac{M+m}{M}\frac{\kappa^2}{r^2},$$

and equation (5) follows.

Lastly we have

$$(\kappa - \lambda)\zeta = \sqrt{2\kappa\lambda - \lambda^2}(\xi\sin\nu - \eta\cos\nu);$$

$$\therefore \quad (\kappa - \lambda)^2 r^2 = \kappa^2 r^2 - (2\kappa\lambda - \lambda^2)(\xi^2 + \eta^2) - \kappa^2\zeta^2 + (2\kappa\lambda - \lambda^2)(\xi\sin\nu - \eta\cos\nu)^2,$$

and it follows that

$$\xi\cos\nu + \eta\sin\nu = r\left[1 - \frac{\kappa^2\zeta^2}{2\kappa\lambda - \lambda^2}\right]^{\frac{1}{2}}.$$

Also

$$r\zeta' - \zeta r' = \frac{r^2\zeta' - \zeta rr'}{r} = \frac{\xi(\xi\zeta' - \zeta\xi') + \eta(\eta\zeta' - \zeta\eta')}{r} = -\frac{M+m}{M} \cdot \frac{\sqrt{2\kappa\lambda - \lambda^2}}{r}(\xi\cos\nu + \eta\sin\nu).$$

Consequently

$$\frac{}{\sqrt{2\kappa\lambda - \lambda^2}}\frac{d}{dt}\left(\frac{\zeta}{r}\right) = -\frac{M+m}{M}\frac{\kappa}{r^2}\left[1 - \frac{\kappa^2\zeta^2}{2\kappa\lambda - \lambda^2}\right]^{\frac{1}{2}},$$

and thus

$$\frac{d}{dr}\sin^{-1}\left(\frac{\kappa\zeta}{r\sqrt{2\kappa\lambda - \lambda^2}}\right) = -\frac{M+m}{M}\frac{\kappa}{r^2}\frac{dr}{dt}.$$

Equation (6) now follows when we use (5).

NOTE 5. *The principal function for the undisturbed motion of a planet**

(V, p. 217.)

Let (x, y) be the coordinates of the sun and M its mass, and let $(x+\xi, y+\eta)$ be the coordinates of the planet and m its mass, so that (ξ, η) are the heliocentric coordinates of the planet. If $(\overline{x}, \overline{y})$ are the coordinates of the centre of gravity, we have

$$x = x - \frac{m}{M+m}\xi, \qquad y = \overline{y} - \frac{m}{M+m}\eta,$$

$$x+\xi = \overline{x} + \frac{M}{M+m}\xi, \quad y+\eta = \overline{y} + \frac{M}{M+m}\eta.$$

Hence the kinetic energy of the system is

$$T = \tfrac{1}{2}M(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}m\{(\dot{x}+\dot{\xi})^2 + (\dot{y}+\dot{\eta})^2\}$$

$$= \tfrac{1}{2}(M+m)(\dot{\overline{x}}^2+\dot{\overline{y}}^2) + \tfrac{1}{2}\frac{Mm}{M+m}(\dot{\xi}^2+\dot{\eta}^2),$$

and the potential energy is

$$V = -{}^{Mm}$$

Therefore the principal function is

$$S = \int_0^t (T-V)\,dt = \tfrac{1}{2}(M+m)(\dot{\overline{x}}^2+\dot{\overline{y}}^2)\,t + \frac{Mm}{M+m}\int_0^t \left\{\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2) + {}^{M+m} \right.\quad dt$$

$$= \tfrac{1}{2}(M+m)\left\{\frac{(\overline{x}-\overline{x}_0)^2+(\overline{y}-\overline{y}_0)^2}{t}\right\} + \frac{Mm}{M+m}s,$$

where

$$s = \int_0^t \left\{\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2) + \frac{\mu}{r}\right\}dt,$$

μ being equal to $M+m$.

The equations of the orbit in terms of the eccentric anomaly are

$$r = \mathrm{a}(1 - e\cos v),$$

$$\xi = \mathrm{a}(\cos v - e), \quad \eta = \mathrm{a}\sqrt{1-e^2}\sin v,$$

where 2a is the major axis of the orbit and e its eccentricity. From the equations of motion

$$\ddot{\xi} = -\frac{\mu\xi}{r^3}, \quad \ddot{\eta} = -\frac{\mu\eta}{r^3},$$

we easily deduce that

$$\frac{1}{\mathrm{a}^3 \;\; 1 - e\cos v},$$

---

* Cf. pp. 135–137.

that is, on integration

$$\frac{\mu}{a^3}t = v - v_0 - e\sin v + e\sin v_0 = 2\left\{\frac{v-v_0}{2} - e\sin\frac{v-v_0}{2}\cos\frac{v+v_0}{2}\right\}$$

Also

$$\dot{\xi}^2 + \dot{\eta}^2 = a^2(1 - e^2\cos^2 v)\,\dot{v}^2 = \frac{\mu}{a}\frac{1-e^2\cos^2 v}{(1-e\cos v)^2} = \frac{\mu}{a}\frac{1+e\cos v}{1-e\cos v},$$

and so

$$s = \int_0^t \left\{\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2) + \frac{\mu}{r}\right\}dt = \int_0^t \frac{\mu}{2a}\left\{\frac{1+e\cos v}{1-e\cos v} + \frac{2}{1-e\cos v}\right\}dt$$

$$= \frac{\mu}{2a}\int_0^t \frac{3+e\cos v}{1-e\cos v}\,dt = \tfrac{1}{2}\sqrt{\mu a}\int_{v_0}^{v}(3+e\cos v)\,dv$$

$$= \tfrac{1}{2}\sqrt{\mu a}\,\{3(v-v_0) + e(\sin v - \sin v_0)\} = \sqrt{\mu a}\left\{3\,\frac{v-v_0}{2} + e\sin\frac{v-v_0}{2}\cos\frac{v+v_0}{2}\right\}.$$

Now

$$\alpha = \frac{r+r_0}{2} = a\left(1 - e\cos\frac{v-v_0}{2}\cos\frac{v+v_0}{2}\right),$$

and

$$4\alpha^2\sin^2\lambda = (\xi-\xi_0)^2 + (\eta-\eta_0)^2 = a^2(\cos v - \cos v_0)^2 + a^2(1-e^2)(\sin v - \sin v_0)^2$$

$$= 4a^2\sin^2\frac{v-v_0}{2}\left(1 - e^2\cos^2\frac{v+v_0}{2}\right),$$

that is,

$$\alpha\sin\lambda = a\sin\frac{v-v_0}{2}\sqrt{\left(1 - e^2\cos^2\frac{v+v_0}{2}\right)};$$

and consequently

$$\sin\lambda = \frac{\sin\dfrac{v-v_0}{2}\sqrt{\left(1-e^2\cos^2\dfrac{v+v_0}{2}\right)}}{1 - e\cos\dfrac{v-v_0}{2}\cos\dfrac{v+v_0}{2}}.$$

If we put

$$\frac{v-v_0}{2} = v_{,}, \quad e\cos\frac{v+v_0}{2} = e_{,}$$

in the above formulae, we have

$$\alpha = a(1 - e_{,}\cos v_{,}), \quad \sin\lambda = \frac{\sin v_{,}\sqrt{1-e_{,}^2}}{1-e_{,}\cos v_{,}}, \quad \nu = \tfrac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\,t = \frac{v_{,} - e_{,}\sin v_{,}}{(1-e_{,}\cos v_{,})^{\frac{3}{2}}}$$

and

$$s = \sqrt{\mu\alpha}\,\frac{3v_{,} + e_{,}\sin v_{,}}{\sqrt{(1-e_{,}\cos v_{,})}}.$$

We can obviously express $e_{,}$ and $v_{,}$ in terms of ν and $\sin\lambda$, and consequently $s/\sqrt{\mu\alpha}$ is a function of ν and $\sin\lambda$ only.

Moreover, from the fundamental property of the principal function

$$\frac{\partial s}{\partial t} = -h = -\left\{\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2) - \frac{\mu}{r}\right\},$$

and hence, easily,

$$\frac{\partial s}{\partial t}=\frac{\partial s}{\partial \nu}\left(\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\right)=+\frac{\mu}{2\mathrm{a}},$$

that is,

$$\frac{\partial \kappa}{\partial \nu}=\frac{\partial s}{\partial \nu}\Big/\sqrt{\mu\alpha}=\frac{1}{\mathrm{a}}\sqrt{\mu\alpha^3}/\sqrt{\mu\alpha}=\frac{\alpha}{\mathrm{a}}.$$

If the orbit is circular, $e_{,}=e\cos\frac{v+v_0}{2}=0$ and we see that λ and ν both reduce to $v_{,}$, and consequently $\iota=\nu-\lambda=0$ for circular orbits. If the orbit is nearly circular, $e_{,}$ is small and consequently ι is also small.

NOTE 6. *On the approximation to the parallactic inequality*

(VI, p. 248; VII, pp. 272–275.)

In the Theory of the Moon the terms involving $\cos\tau$ and $\sin\tau$ in the expansions of a/r and λ respectively are called the parallactic inequalities. Hamilton was unable to reconcile the corresponding terms in his expansions with those of Plana and Lubbock. By taking into account two further operations, which he overlooked, it is possible to obtain the correct results. Here we confine our attention to r and to terms as far as the order m^2.

The principal function S may be written

$$S = S_0 + s, \quad \text{where} \quad S_0 = \int_0^t \left(\frac{r'^2 + r^2\lambda'^2}{2} + \frac{\mu}{r}\right) dt,$$

$$s = s_1 + s_2 + s_3 + \ldots,$$

$$\text{where} \quad s_1 = n_{,}^2 \int_0^t r^2 P_2\, dt, \quad s_2 = n_{,}^2 \int_0^t r^3 P_3\, dt, \quad s_3 = n_{,}^2 \int_0^t r^4 P_4\, dt, \quad \ldots,$$

$P_2, P_3, P_4, \ldots$ being Legendre coefficients of argument $\cos(\lambda - n_{,}t)$.

As a first trial, taking $S = S_0$, we get the ordinary elliptic values for r_0, λ. Substituting these values in $S_0 + s_1$, we get

$$r = r_0 + \Delta_1 r, \quad \text{where} \quad \Delta_1 r = -\mathrm{a}\left\{-\frac{m^2}{6} - m^2\left(1 + \frac{19m}{6}\right)\cos 2\tau\right\}.$$

Substituting in $S = S_0 + s_1 + s_2$, we get as far as terms in m^2

$$r = r_0 + \Delta_1 r + \Delta_2 r, \quad \text{where} \quad \Delta_2 r = \frac{m\mathrm{a}^2}{a_{,}}\left(\frac{15}{16} + \frac{27m}{32}\right)\cos\tau.$$

The last term according to Plana and Lubbock should be

$$\frac{m\mathrm{a}^2}{a_{,}}\left(\frac{15}{16} + \frac{81\,m}{16}\right)\cos\tau.$$

If however we substituted the *next* approximation $r_0 + \Delta_1 r + \Delta_2 r + \Delta_3 r$, we should find another term $\Delta_2' r$ of the order of $\Delta_2 r$, where $\Delta_2' r = \dfrac{165}{32}\dfrac{m^2\mathrm{a}^2}{a_{,}}$.

The foregoing process will not however give the complete approximation to r. If $S = S_0 + S_{,}$, we have the *accurate* equation

$$\begin{aligned} S_{,} &= s - \frac{1}{2}\int_0^t \left\{\left(\frac{\partial S_{,}}{\partial x}\right)^2 + \left(\frac{\partial S_{,}}{\partial y}\right)^2\right\} dt \\ &= s - \frac{1}{2}\int_0^t \left[\left\{\frac{d(\Delta r)}{dt}\right\}^2 + \left\{\Delta r\frac{\partial\lambda}{\partial t} + r\frac{\partial(\Delta\lambda)}{\partial t} + \Delta r\frac{\partial(\Delta\lambda)}{\partial t}\right\}^2\right] dt. \end{aligned}$$

Substituting the previous approximation in the integral on the right-hand side, we get from the resulting value of $S_{,}$ the term for $\Delta_2'' r = -\dfrac{15}{16}\dfrac{m^2\mathrm{a}^2}{a_{,}}$. We then get finally

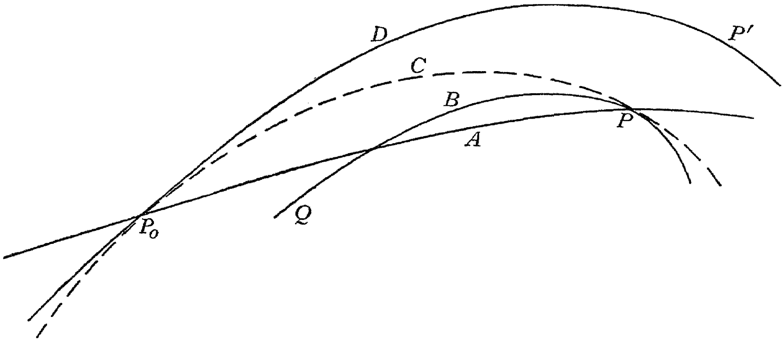
$$\Delta_2 r + \Delta_2' r + \Delta_2'' r = \left(\frac{27}{32} + \frac{165}{32} - \frac{15}{16}\right)\frac{m^2\mathrm{a}^2}{a_{,}} = \frac{81}{16}\frac{m^2\mathrm{a}^2}{a_{,}}.$$

It may be remarked that in getting this term by de Pontécoulant's method (Brown, *Lunar Theory*, pp. 105, 106) it is made up of the same three components.

NOTE 7. *On Hamilton's method of approximating to the motion of the moon*

(III, p. 209; VII, pp. 266, 269.)

Let P, P_0 (see figure) be the positions of the moon's centre at the times t, t_0 and let P_0CP be its actual (geocentric) orbit.



Disregarding the sun's attraction, a body starting at P_0 with the *actual orbital velocity* of the moon will describe an ellipse P_0DP'. The elements of this ellipse, which we may call the fixed ellipse, are constants $a, e, \epsilon, \varpi, \theta, i$. Still disregarding the sun's attraction we can have another ellipse P_0AP which starts at P_0 at the time t_0 and arrives at P at the time t. This is the Hamilton ellipse. Let the elements of this new ellipse be $a+\Delta a$, $e+\Delta e$, ... and the initial velocity at P_0 be $(u_0-\Delta u_0,\ v_0-\Delta v_0,\ w_0-\Delta w_0)$, (u_0, v_0, w_0) being the initial velocity in the fixed ellipse as well as in the orbit at P_0. If, by the ordinary laws of elliptic motion, we have

$$u_0=f(a, e, \epsilon, \ldots),$$

then, *rigorously*,

$$u_0=\Delta u_0+f(a+\Delta a,\ e+\Delta e, \ldots).$$

Here Δu_0, Δv_0, Δw_0 are the components which must be *added* to the initial velocity in the Hamilton ellipse in order to get the correct initial orbital velocity. If Δa, Δe, ... are small, then

$$\Delta u_0+\Delta a\frac{\partial u_0}{\partial a}+\Delta e\frac{\partial u_0}{\partial e}+\ldots=0.$$

Also, since the Hamilton ellipse passes through the same initial point, we have also equations of the type

$$\Delta a\frac{\partial x_0}{\partial a}+\Delta e\frac{\partial x_0}{\partial e}+\ldots=0.$$

These six equations give us Δa, Δe, ... in terms of Δu_0, Δv_0, Δw_0. Finally, if x, y, z be the position of the moving point at the time t in the *fixed* ellipse, then the coordinates of P are $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, where

$$\Delta x = \frac{\partial x}{\partial a}\Delta a + \frac{\partial x}{\partial e}\Delta e + \dots .$$

The actual values of Δu_0, Δv_0, Δw_0 are given by the variational equation

$$\Delta u \,.\, \delta x + \Delta v \,.\, \delta y + \Delta w \,.\, \delta z - \Delta u_0 \,.\, \delta x_0 - \Delta v_0 \,.\, \delta y_0 - \Delta w_0 \,.\, \delta z_0 = \delta \int R\,dt.$$

The Lagrange ellipse, QBP, is an ellipse touching the orbit at P and having there the correct orbital velocity. In fact the fixed ellipse referred to above is the Lagrange ellipse at the period t_0. We now proceed to show that the Hamilton ellipse and the Lagrange ellipse both give the same first approximation. In the variational equations

$$\Sigma\,(p_r\delta q_r - \beta_r\delta\alpha_r) = \delta S, \tag{i}$$

$$\Sigma\,(\Delta p_r\,\delta q_r - \Delta\beta_r\delta\alpha_r) = -\delta\int R\,dt, \tag{ii}$$

we may regard p_r, β_r as functions of q_r, α_r. This type of partial differentiation we shall denote by the symbol δ, whilst if we regard p_r, q_r as functions of α_r, β_r we shall use ∂. We have then equations of the form $\frac{\delta\alpha_s}{\delta\alpha_n} = 0$ $(s \neq n)$ and $\frac{\delta\alpha_s}{\delta\alpha_s} = 1$, and

$$\frac{\partial q_n}{\partial\alpha_r} + \Sigma_s\frac{\partial q_n}{\partial\beta_s}\frac{\delta\beta_s}{\delta\alpha_r} = 0. \tag{iii}$$

From (ii)

$$\Delta\beta_r = \frac{\delta}{\delta\alpha_r}\int R\,dt = \left(\frac{\partial}{\partial\alpha_r} + \Sigma_s\frac{\delta\beta_s}{\delta\alpha_r}\frac{\partial}{\partial\beta_s}\right.$$

$$\Delta q_n = -\Sigma_r\frac{\partial q_n}{\partial\beta_r}\Delta\beta_r = -\Sigma_r\left\{\frac{\partial q_n}{\partial\beta_r}\frac{\partial}{\partial\alpha_r} + \Sigma_s\frac{\partial q_n}{\partial\beta_r}\frac{\delta\beta_s}{\delta\alpha_r}\frac{\partial}{\partial\beta_s}\right\}\int R\,dt = -\Sigma_r\left\{\frac{\partial q_n}{\partial\beta_r}\frac{\partial}{\partial\alpha_r} - \frac{\partial q_n}{\partial\alpha_r}\frac{\partial}{\partial\beta_r}\right\}\int R\,dt,$$

because

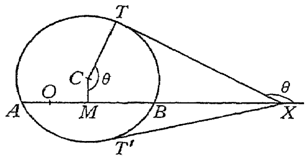
$$\frac{\delta\beta_s}{\delta\alpha_r} = \frac{\delta\beta_r}{\delta\alpha_s},$$

which is the Lagrangian form for this type of variable constants.

NOTE 8. *Hodographic and anthodographic isochronism*

(XI, p. 293.)

These theorems may be proved as follows.



Let ABT be a circular hodograph, centre C and radius a, passing through the given points A and B, and let O be the centre of force of the orbit. If the tangent at any point T cuts AB in X and M is the middle point of AB, we put

$$MX=x, \quad CM=b, \quad MB=c, \quad TX=\tau, \quad \text{and} \quad \angle X=\theta.$$

We have

$$x\sin\theta=a-b\cos\theta, \quad \tau\sin\theta=b-a\cos\theta,$$

and therefore, for a small variation of X,

$$\delta x\sin\theta=\tau\,\delta\theta.$$

Since the velocity of T in the hodograph equals the acceleration of the corresponding point in the orbit,

$$a\frac{d\theta}{dt}=\frac{\mu}{r^2},$$

where r is the radius vector of the particle in the orbit. But $rp=h$, where p is the perpendicular from O on TX and h is the areal velocity. Hence the time of describing the elementary arc of the hodograph at T is

$$dt=\frac{ah^2\,d\theta}{\mu\,OX^2\sin^2\theta}-\frac{ah^2\,dx}{\mu\tau c^2OX^2}(ax-b\tau)=\frac{\mu\,dx}{a\tau c^2OX^2}(ax-b\tau),$$

since $a=\mu/h$. Similarly the time of describing the elementary arc at T' is

$$dt'=\frac{\mu\,dx}{a\tau c^2OX^2}(ax+b\tau).$$

Adding and integrating, we find for the time of description of the hodographic arc TT'

$$\frac{2\mu}{c^2}\int_c^x\frac{x\,dx}{\tau OX^2}=\frac{2\mu}{c^3}\int_0^\phi\frac{d\phi}{(1+k\cos\phi)^2},$$

where $x=c\sec\phi$, $OM=kc$. This expression is the same for all hodographs that pass through A and B and are cut orthogonally by the circle with centre X and radius τ, which proves Hamilton's theorem.

We may also remark that

$$AO\,.\,OB=a^2-OC^2=\frac{\mu^2}{h^2}(1-e^2)=\frac{\mu}{\mathrm{a}},$$

where a is the semi-major axis of the orbit. Hence the orbits of all hodographs passing through A and B have equal major axes.

The *anthode* of a moving particle is obviously the inverse of its hodograph with respect to a unit circle whose centre is O. Consequently the theorem of anthodographic isochronism follows immediately from the above result by inversion.*

---

* The method of inversion was not widely known in 1847. For its early history, see Cajori, *History of Mathematics*, p. 292.

NOTE 9. *Hamilton's method of finding the principal function from the partial differential equation*

(XIII, pp. 335–339.)

In a manuscript dealing with the Calculus of Principal Relations,* Hamilton describes an interesting method of obtaining the principal function from the partial differential equation of the first order which it satisfies, and it is worth describing here in detail how the method is applied in the important dynamical case.

Let us consider a dynamical system of coordinates $x_1, \ldots, x_N$, and having $L(x, \dot{x}, t)$ as its Lagrangian function. Hamilton's principal function S is defined as the integral

$$S=\int_{t_0}^{t} L\,dt, \tag{1}$$

evaluated along a natural motion of the system and expressed as a function of the initial and final values of the x's and of t. It is easy to show that S must satisfy the Hamilton partial differential equation

$$\frac{\partial S}{\partial t}+H\left(x, \frac{\partial S}{\partial x}, t\right)=0, \tag{2}$$

where
$$H(x, p, t)=\dot{x}_r\frac{\partial L}{\partial \dot{x}_r}-L, \quad p_r=\frac{\partial L}{\partial \dot{x}_r}. \tag{3}$$

The question arises how we are to distinguish the solution S from the other solutions of (1). Hamilton identifies it by saying that S must vanish identically when $x_1, \ldots, x_N, t$ take their initial values $a_1, \ldots, a_N$ and t_0, and also that

$$\frac{S}{t-t_0}\to L\left(a_1, \ldots, a_N, \frac{x_1-a_1}{t-t_0}, \frac{x_2-a_2}{t-t_0}, \ldots, \frac{x_N-a_N}{t-t_0}, t_0\right) \tag{4}$$

as $t\to t_0$, a relation which follows immediately from (1). We shall see that these conditions will determine S uniquely.

Now S is a function of $x_1, \ldots, x_N, a_1, \ldots, a_N, t, t_0$. Hence if we put

$$x_r=a_r+\left(\frac{x_r-a_r}{t-t_0}\right)(t-t_0), \quad t=t_0+(t-t_0),$$

we see that S can be expressed as a function of $a_r, t_0, \dfrac{x_r-a_r}{t-t_0}, t-t_0$ and we shall assume that we can expand it in ascending powers of $(t-t_0)$, that is,

$$S=A_1(t-t_0)+A_2(t-t_0)^2+\ldots+A_n(t-t_0)^n+\ldots, \tag{5}$$

where the A's are functions of the remaining variables. This expansion can obviously be written in the form

$$S=S_1+S_2+S_3+\ldots+S_n+\ldots, \tag{6}$$

where S_n is homogeneous of degree n in the variables $x_r-a_r, t-t_0$, and consequently

$$(t-t_0)\frac{\partial S_n}{\partial t}+(x_r-a_r)\frac{\partial S_n}{\partial x_r}=nS_n. \tag{7}$$

---

* XIII, p. 332.

If we substitute this value of S in (2), we obtain

$$\frac{\partial}{\partial t}\left(\sum_1^\infty S_n\right)+H\left\{x,\ \frac{\partial}{\partial x}\left(\sum_1^\infty S_n\right),\ t\right\}=0; \tag{8}$$

or, putting $x_r=a_r+(x_r-a_r)$, $t=t_0+(t-t_0)$, and writing for brevity

$$H_0=H(a,\ p^0,\ t_0),\quad p_r^0=\frac{\partial S_1}{\partial x_r},$$

we can expand (8) in ascending powers of (x_r-a_r) and $(t-t_0)$ as follows

$$\Sigma\frac{\partial S_n}{\partial t}+H_0+\sum_{n=1}^{\infty}\left\{(x_r-a_r)\frac{\partial}{\partial a_r}+(t-t_0)\frac{\partial}{\partial t_0}+\left(\frac{\partial S_2}{\partial x_r}+\frac{\partial S_3}{\partial x_r}+\dots\right)\frac{\partial}{\partial p_r^0}\right\}^n H_0=0. \tag{9}$$

If we write the partial differential equation (2) as $\psi=0$, we may write the expansion (9) in the form

$$\psi_0+\psi_1+\psi_2+\dots+\psi_n+\dots=0,$$

where ψ_n is homogeneous and of degree n in the variables (x_r-a_r), $(t-t_0)$, and, since this partial differential equation must be satisfied identically by $\sum_n^\infty S_n$, we must have

$$\psi_0=0,\ \psi_1=0,\ \psi_2=0,\ \dots,\ \psi_n=0,\ \dots$$

separately satisfied.

Now it is easily seen from (9) that $\psi_0=0$ stands for the equation

$$\frac{\partial S_1}{\partial t}+H\left(a,\ \frac{\partial S_1}{\partial x},\ t_0\right)=0, \tag{10}$$

which gives an equation to be satisfied by S_1. Moreover, it appears from a comparison of (4) and (5) that A_1 or $S_1/(t-t_0)$ should equal the right-hand side of (4) identically, that is, that

$$S_1=(t-t_0)\,L\left(a,\ \frac{x-a}{t-t_0},\ t_0\right). \tag{11}$$

Hence we should expect that (11) satisfies equation (10), a result which is easily verified.

The second equation $\psi_1=0$ becomes

$$\frac{\partial S_2}{\partial t}+(x_r-a_r)\frac{\partial H_0}{\partial a_r}+(t-t_0)\frac{\partial H_0}{\partial t_0}+\frac{\partial S_2}{\partial x_r}\frac{\partial H_0}{\partial p_r^0}=0. \tag{12}$$

But it is easy to prove that

$$\frac{\partial H_0}{\partial p_r^0}=\frac{x_r-a_r}{t-t_0}$$

and therefore (12) becomes

$$(t-t_0)\frac{\partial S_2}{\partial t}+(x_r-a_r)\frac{\partial S_2}{\partial x_r}+(t-t_0)\left\{(x_r-a_r)\frac{\partial H_0}{\partial a_r}+(t-t_0)\frac{\partial H_0}{\partial t_0}\right\}=0,$$

or, using equation (7),

$$2S_2=-(t-t_0)\left\{(x_r-a_r)\frac{\partial H_0}{\partial a_r}+(t-t_0)\frac{\partial H_0}{\partial t_0}\right\}, \tag{13}$$

and so S_2 is given immediately when S_1 is known.

It is easy to see that we can continue in this way; for example, $\psi_2=0$ can be written

$$\frac{\partial S_3}{\partial t}+\frac{\partial S_3}{\partial x_r}\frac{\partial H_0}{\partial p_r^0}+\left\{(x_r-a_r)\frac{\partial}{\partial a_r}+(t-t_0)\frac{\partial}{\partial t_0}+\frac{\partial S_2}{\partial x_r}\frac{\partial}{\partial p_r^0}\right\}^2 H_0=0,$$

and hence, as before,

$$3S_3=-(t-t_0)\left\{(x_r-a_r)\frac{\partial}{\partial a_r}+(t-t_0)\frac{\partial}{\partial t_0}+\frac{\partial S_2}{\partial x_r}\frac{\partial}{\partial p_r^0}\right\}^2 H_0, \tag{14}$$

the right-hand side being known when S_1 and S_2 are known. In general, $\psi_n=0$ can be written

$$\frac{\partial S_n}{\partial t}+\frac{\partial S_n}{\partial x_r}\frac{\partial H_0}{\partial p_r^0}+\left\{(x_r-a_r)\frac{\partial}{\partial a_r}+(t-t_0)\frac{\partial}{\partial t_0}+\frac{\partial S_2}{\partial x_r}\frac{\partial}{\partial p_r^0}\right\} H_0+\dots=0,$$

the remaining terms involving only the S's up to S_{n-1}. Multiplying by $(t-t_0)$ and using the value of $\dfrac{\partial H_0}{\partial p_r^0}$, we have from (7)

$$nS_n=-(t-t_0)\left[\left\{(x_r-a_r)\frac{\partial}{\partial a_r}+(t-t_0)\frac{\partial}{\partial t_0}+\frac{\partial S_2}{\partial x_r}\frac{\partial}{\partial p_r^0}\right\}^n H_0+\dots\right. \tag{15}$$

Consequently S_n can be calculated when S_1, S_2, ..., S_{n-1} are known functions of x_r, a_r, t, t_0.

The process can therefore be continued indefinitely and the function S obtained without any integrations whatever. Moreover the process obviously leads to a unique complete integral of (2) satisfying the condition (4), and we conclude that S must be the principal function.

The method may be a useful one practically if we are dealing with a comparatively short interval of time, since we are furnished by successive approximations to S and in that case only a few terms need be calculated to give a sufficiently approximate value of the principal function.

NOTE 10. *Hamilton's method of solving partial differential equations and its relation to Cauchy's theory*

(XV, pp. 391–407.)

1. *The partial differential equation of the first order*

Let us take a partial differential equation of the first order

$$F(x, y, z, p, q) = 0. \tag{1}$$

We may use geometrical terminology and call (x, y, z, p, q) an element of contact, if the equation (1) is satisfied, and a one-parameter family of elements of contact

$$x = x(u), \quad y = y(u), \quad z = z(u), \quad p = p(u), \quad q = q(u)$$

a multiplicity. Along a multiplicity we must have

$$px' + qy' - z' = 0,$$

where dashes denote derivatives with respect to u.

If we take a neighbouring multiplicity $(x + \delta x, y + \delta y, z + \delta z, p + \delta p, q + \delta q)$, we have

$$0 = \delta(px' + qy' - z') = p\,\delta x' + \delta p\, x' + q\,\delta y' + \delta q\, y' - \delta z', \tag{2}$$

the variations δx, δy, δz, δp, δq not being arbitrary but subject to the condition

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q = 0. \tag{3}$$

Hence, multiplying by a factor λ and integrating,

$$0 = \int \lambda\delta(px' + qy' - z')\,du = \int \lambda(p\,\delta x' + \delta p\, x' + q\,\delta y' + \delta q\, y' - \delta z')\,du$$

$$= [\lambda(p\,\delta x + q\,\delta y - \delta z)] + \int \{-(\lambda p)'\,\delta x - (\lambda q)'\,\delta y + \lambda'\delta z + \lambda x'\delta p + \lambda y'\delta q\}\,du. \tag{4}$$

Let us choose our original multiplicity so that the part under the integral sign vanishes for all neighbouring multiplicities. We must therefore have

$$-(\lambda p)' = \mu'\frac{\partial F}{\partial x}, \quad -(\lambda q)' = \mu'\frac{\partial F}{\partial y}, \quad \lambda' = \mu'\frac{\partial F}{\partial z},$$

$$\lambda x' = \mu'\frac{\partial F}{\partial p}, \qquad \lambda y' = \mu'\frac{\partial F}{\partial q}.$$

Hence, along the multiplicity

$$\frac{dx}{\dfrac{\partial F}{\partial p}} = \frac{dy}{\dfrac{\partial F}{\partial q}} = \frac{dz}{p\dfrac{\partial F}{\partial p} + q\dfrac{\partial F}{\partial q}} = \frac{-dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \frac{-dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}} = \frac{d\lambda}{\lambda\dfrac{\partial F}{\partial z}} = \frac{d\mu}{\lambda}, \tag{5}$$

that is, the multiplicity must be a *characteristic*.

The solutions of these equations are of the form

$$x = x(u, x_0, y_0, z_0, p_0, q_0), \text{ etc.,} \tag{6}$$

where zero suffixes denote initial values of the variables, and if we associate with them the equation

$$F(x_0, y_0, z_0, p_0, q_0) = 0, \tag{7}$$

we see that (6) will satisfy (1) identically for all values of u. Moreover, we have from (4)

$$\lambda (p\,\delta x + q\,\delta y - \delta z) = \lambda_0 (p_0\,\delta x_0 + q_0\,\delta y_0 - \delta z_0),$$

and it follows that p and q will be the partial derivatives of z with respect to x and y respectively, if

$$p_0\,\delta x_0 + q_0\,\delta y_0 - \delta z_0 = 0. \tag{8}$$

In other words, we shall have an integral surface of (1) if we take x_0, y_0, z_0, p_0, q_0 in (6) as any functions of a variable v subject only to the two conditions

$$F(x_0, y_0, z_0, p_0, q_0) = 0, \quad p_0 \frac{dx_0}{dv} + q_0 \frac{dy_0}{dv} - \frac{dz_0}{dv} = 0. \tag{9}$$

In particular, if we are given a curve $x_0 = x_0(v)$, $y_0 = y_0(v)$, $z_0 = z_0(v)$, which is not a characteristic, then (9) determines p_0, q_0 and we have the unique integral of (1) that passes through the given curve.*

2. *The partial differential equation of the second order*

The same method can be applied to the partial differential equation of the second order

$$F(x, y, z, p, q, r, s, t) = 0. \tag{1}$$

Let us call (x, y, z, p, q, r, s, t) a second-order element of contact if it satisfies (1) and let us call a one-parameter family of such elements

$$x = x(u),\ \ y = y(u),\ \ z = z(u),\ \ p = p(u),\ \ q = q(u),\ \ r = r(u),\ \ s = s(u),\ \ t = t(u)$$

a second-order multiplicity, u being the parameter. Along such a multiplicity, we must have

$$\left.\begin{aligned} px' + qy' - z' &= 0, \\ rx' + sy' - p' &= 0, \\ sx' + ty' - q' &= 0, \end{aligned}\right\} \tag{2}$$

where dashes denote derivatives with respect to u. If we take a neighbouring multiplicity, $x + \delta x$, $y + \delta y$, etc., we have

$$\left.\begin{aligned} 0 &= \delta(px' + qy' - z') = p\,\delta x' + q\,\delta y' - \delta z' + \delta p\,x' + \delta q\,y', \\ 0 &= \delta(rx' + sy' - p') = r\,\delta x' + s\,\delta y' - \delta p' + \delta r\,x' + \delta s\,y', \\ 0 &= \delta(sx' + ty' - q') = s\,\delta x' + t\,\delta y' - \delta q' + \delta s\,x' + \delta t\,y', \end{aligned}\right\} \tag{3}$$

where the variations are subject to the condition

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial s}\delta s + \frac{\partial F}{\partial t}\delta t = 0. \tag{4}$$

---

* Cf. Goursat, *Cours d'Analyse*, Tome II (1918), pp. 602–610.

Multiplying equations (3) by three multipliers λ, μ, ν we have, on integrating by parts,

$$\begin{aligned}0 &= \int [\lambda\delta\,(px'+qy'-z')+\mu\delta\,(rx'+sy'-p')+\nu\delta\,(sx'+ty'-q')]\,du\\ &= [\lambda\,(p\,\delta x+q\,\delta y-\delta z)+\mu\,(r\delta x+s\,\delta y-\delta p)+\nu\,(s\,\delta x+t\,\delta y-\delta q)]\\ &\quad+\int[-\{(\lambda p)'+(\mu r)'+(\nu s)'\}\,\delta x-\{(\lambda q)'+(\mu s)'+(\nu t)'\}\,\delta y+\lambda'\delta z\\ &\quad+(\lambda x'+\mu')\,\delta p+(\lambda y'+\nu')\,\delta q+\mu x'\delta r+(\mu y'+\nu x')\,\delta s+\nu y'\delta t]\,du. \qquad (5)\end{aligned}$$

As before, we choose our multiplicity so that the part under the integral sign disappears for all variations subject to (4). We must therefore have, along the multiplicity,

$$\left.\begin{aligned}&(\lambda p)'+(\mu r)'+(\nu s)'+\rho'\,\frac{\partial F}{\partial x}=0,\\ &(\lambda q)'+(\mu s)'+(\nu t)'+\rho'\,\frac{\partial F}{\partial y}=0,\\ &\lambda'=\rho'\,\frac{\partial F}{\partial z},\quad \lambda x'+\mu'=\rho'\,\frac{\partial F}{\partial p},\quad \lambda y'+\nu'=\rho'\,\frac{\partial F}{\partial q},\\ &\mu x'=\rho'\,\frac{\partial F}{\partial r},\quad \mu y'+\nu x'=\rho'\,\frac{\partial F}{\partial s},\quad \nu y'=\rho'\,\frac{\partial F}{\partial t},\end{aligned}\right. \qquad (6)$$

ρ' being another multiplier.* Equations (2) and (6) are eleven homogeneous linear differential equations of the first order between the twelve variables x, y, z, p, q, r, s, t, λ, μ, ν, ρ, and hence these can in general be solved in the form

$$x=x\,(u,\,x_0,\,y_0,\,z_0,\,p_0,\,q_0,\,r_0,\,s_0,\,t_0,\,\lambda_0,\,\mu_0,\,\nu_0,\,\rho_0),\ \text{etc.}, \qquad (7)$$

where zero suffixes denote initial values of the variables.

Moreover it is easy to see that a linear combination of (2) and (6) leads to $F'=0$, and therefore if we have the condition

$$F_0=F\,(x_0,\,y_0,\,z_0,\,p_0,\,q_0,\,r_0,\,s_0,\,t_0)=0, \qquad (8)$$

the solution (7) will satisfy (1) identically.

In addition, we have, from (5),

$$\begin{aligned}&\lambda\,(p\,\delta x+q\,\delta y-\delta z)+\mu\,(r\,\delta x+s\,\delta y-\delta p)+\nu\,(s\,\delta x+t\,\delta y-\delta q)\\ &\quad=\lambda_0\,(p_0\delta x_0+q_0\delta y_0-\delta z_0)+\mu_0\,(r_0\delta x_0+s_0\delta y_0-\delta p_0)+\nu_0\,(s_0\delta x_0+t_0\delta y_0-\delta q_0).\end{aligned}$$

Therefore, making

$$\left.\begin{aligned}p_0\,\delta x_0+q_0\,\delta y_0-\delta z_0=0,\\ r_0\,\delta x_0+s_0\,\delta y_0-\delta p_0=0,\\ s_0\,\delta x_0+t_0\,\delta y_0-\delta q_0=0,\end{aligned}\right\} \qquad (9)$$

it follows that

$$\lambda\,(p\,\delta x+q\,\delta y-\delta z)+\mu\,(r\,\delta x+s\,\delta y-\delta p)+\nu\,(s\,\delta x+t\,\delta y-\delta q)=0, \qquad (10)$$

---

* It is easy to see that the last three equations of (6) become

$$y'^2\,\frac{\partial F}{\partial r}-x'y'\,\frac{\partial F}{\partial s}+x'^2\,\frac{\partial F}{\partial t}=0,$$

when μ, ν, ρ' are eliminated, and this equation is sometimes called the differential equation of the characteristics. Cf. Goursat, *Cours d'Analyse*, Tome III (1915), p. 51.

for all values of the initial variables and their variations. But we have still λ_0, μ_0, ν_0, ρ_0 at our disposal. Hence (10) must be true for arbitrary values of λ, μ, ν and consequently the coefficients of λ, μ, ν in (10) must vanish separately, which means that p, q are the first partial derivatives of z and r, s, t the second partial derivatives. If we make, therefore, $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$ functions of a parameter v satisfying (8) and (9), we shall have an integral surface of the equation (1).

In particular, if we are given a first-order multiplicity

$$x_0 = x_0(v), \quad y_0 = y_0(v), \quad z_0 = z_0(v), \quad p_0 = p_0(v), \quad q_0 = q_0(v),$$

the first of the equations (9) is satisfied and the remaining two together with (8) determine r_0, s_0, t_0. We can therefore find an integral surface of (1) passing through a given first-order multiplicity.*

---

* Cf. Goursat, *Cours d'Analyse*, Tome III (1915), pp. 47–51.

NOTE 11. *Hamilton's luminiferous æther*

(XVII, p. 413.)

Hamilton, following Cauchy's first memoir*, attempted to explain on classical dynamical principles the optical properties of crystals. The investigations contained in Note Book 40 (1835) anticipates, in part, Cauchy's subsequent investigation, and in order to see the place of these unpublished investigations in the history of such attempts by Cauchy, Poisson and Green it seems useful to adopt the idea of a continuous medium and also to make use, like Green, of the potential energy function. If u, v, w denote coordinate displacements and suffixes denote differentiations, the potential energy per unit volume is

$$\begin{aligned} &Gu_x + Hv_y + Iw_z + \tfrac{1}{2}G(u_x^2 + v_x^2 + w_x^2) + \tfrac{1}{2}H(u_y^2 + v_y^2 + w_y^2) + \tfrac{1}{2}I(u_z^2 + v_z^2 + w_z^2) \\ &\quad + \tfrac{1}{2}Lu_x^2 + \tfrac{1}{2}Mv_y^2 + \tfrac{1}{2}Nw_z^2 + \tfrac{1}{2}P(v_z + w_y)^2 + \tfrac{1}{2}Q(w_x + u_z)^2 + \tfrac{1}{2}R(u_y + v_x)^2 \\ &\qquad + Pv_yw_z + Qw_zu_x + Ru_xv_y. \end{aligned}$$

There are thus nine constants (which have here the values given on pp. 424, 425). This limitation arises from the fact that Hamilton assumes a "simple" space-lattice which is uniformly strained from a cubic lattice, and that the force between the single points of attraction in the cells is along the line joining them and a function of their distance apart. The constants G, H, I are equilibrium stresses. The other quantities are connected by the relations

$$M + N = 6P, \quad N + L = 6Q, \quad L + M = 6R$$

(see the expressions, p. 425). If

$$u = \xi \cos k(\alpha x + \beta y + \gamma z - ct), \quad v = \eta \cos k(\alpha x + \beta y + \gamma z - ct), \quad w = \zeta \cos k(\alpha x + \beta y + \gamma z - ct),$$

where ξ, η, ζ are constants and α, β, γ the direction cosines of the wave-normal, we have, the volume density being taken as unity,

$$c^2\xi = \xi\{(L + G)\alpha^2 + (R + H)\beta^2 + (Q + I)\gamma^2\} + 2\alpha\beta R\eta + 2\alpha\gamma Q\zeta, \text{ etc.}$$

Given α, β, γ, we have three values, in general different, for c^2 and three corresponding directions for ξ, η, ζ, which are mutually at right angles to one another. Is it possible to have two of those directions in the wave-front or, what amounts to the same thing, is it possible to have one of these directions parallel to α, β, γ? We thus get, if none of α, β, γ is zero,

$$L + G = M + H = N + I = 3R + G = 3Q + G = \text{etc.},$$

i.e. an isotropic medium. It becomes therefore necessary to make the assumption that two of the vibrations are nearly in the wave-front, it being understood that L, M, N and therefore P, Q, R differ very little from one another. Introducing the quantities ξ', η', ζ' given by

$$\gamma\xi - \alpha\zeta = \xi', \quad \gamma\eta - \beta\zeta = \eta', \quad \alpha\xi + \beta\eta + \gamma\zeta = \zeta'$$

and making use of $M + N = 6P$, etc., we get

$$\begin{aligned} c^2\xi' &= \xi'\{(Q + G)\alpha^2 + (R + H)\beta^2 + (Q + I)\gamma^2\} + (R - P)(\alpha\beta\eta' + 3\alpha\gamma\zeta'), \\ c^2\eta' &= \eta'\{(R + G)\alpha^2 + (P + H)\beta^2 + (P + I)\gamma^2\} + (R - Q)(\alpha\beta\xi' + 3\beta\gamma\zeta'), \end{aligned}$$

---

* *Mém. de l'Acad.* Vol. x, p. 293. General reference may be made to Green, *Math. Papers*, pp. 293–311; Whittaker, *History of Theories of the Æther and Electricity*, Chap. v; Born, *Atomtheorie des festen Zustandes*; Macdonald, *Proc. Lond. Math. Soc.* Vol. xxxii (1900), pp. 311–315.

and

$$c^2\zeta' = \tfrac{1}{2}\alpha\xi'\{L-(L\alpha^2+M\beta^2+N\gamma^2)\} + \tfrac{1}{2}\beta\eta'\{M-(L\alpha^2+M\beta^2+N\gamma^2)\}$$
$$+\gamma\zeta'\{(L+G)\,\alpha^2+(M+H)\,\beta^2+(N+I)\,\gamma^2\}.$$

Hence for the transverse waves we have

$$c^2\xi' = \xi'\{(Q+G)\,\alpha^2+(R+H)\,\beta^2+(Q+I)\,\gamma^2\}+(R-P)\,\alpha\beta\eta',$$
$$c^2\eta' = \eta'\{(R+G)\,\alpha^2+(P+H)\,\beta^2+(P+I)\,\gamma^2\}+(R-Q)\,\alpha\beta\xi'.$$

The elimination of $\xi' : \eta'$ gives us the quadratic for c^2

$$\{c^2-(Q+G)\,\alpha^2-(R+H)\,\beta^2-(Q+I)\,\gamma^2\}\{c^2-(R+G)\,\alpha^2-(P+H)\,\beta^2-(P+I)\,\gamma^2\}$$
$$=(R-P)\,(R-Q)\,\alpha^2\beta^2.$$

In order that this should be in the usual Fresnel form,

$$\frac{}{c^2-c'^2}+\frac{}{c^2-c''^2}+\frac{}{c^2-c'''^2}=0,$$

a number of conditions are necessary. It will be found sufficient, however, to express the fact that

$$\{(Q+G)\,\alpha^2+(R+H)\,\beta^2+(Q+I)\,\gamma^2\}\{(R+G)\,\alpha^2+(P+H)\,\beta^2+(P+I)\,\gamma^2\}-(R-P)\,(R-Q)\,\alpha^2\beta^2$$

contains $\alpha^2+\beta^2+\gamma^2$ as a factor and that neither $R-P$ nor $R-Q$ is zero. This gives

$$(Q+G-P-H)\,(G-H)=0,$$
$$(P+I-R-G)\,(H-I)=0,$$
$$(R+H-Q-I)\,(I-G)=0.$$

We have thus two alternatives: (1st) $Q+G=P+H$, $P+I=R+G$, $R+H=Q+I$. These are satisfied if (see p. 434) $R_0+2R_1+R_2=0$. (2nd) $G=H=I$. These equations can only be, approximately, satisfied (apart from an isotropic medium) if R_0+R_1 is small compared with R_2. When a beam of light is polarised parallel to one of the coordinate planes the two wave-velocities are equal. If, with Fresnel, the vibration is perpendicular to the wave-normal, then, for example, if $\xi:\eta:\zeta=1:0:0$, the two wave-normals have for direction cosines (0, 1, 0) and (0, 0, 1). We get $R+H=Q+I$. Hence this corresponds to the 1st alternative. If we take with MacCullagh and Neumann the vibration in the plane of polarisation, for example, if the wave-fronts have normals (0, 1, 0) and (0, 0, 1) and for the corresponding vibrations $\eta=\zeta=0$ and $\xi=\zeta=0$, we get $Q=H$. Hence these correspond to the 2nd alternative. These correspond, respectively, to Cauchy's Second and First Theories.*

In order to attempt a discrimination between these two theories, Hamilton proceeds to take into account dispersion. This means adding eighteen additional terms like u_{xx}^2, etc., to the potential energy. Several interesting results are obtained, e.g. the difference of the squares of two principal velocities is still proportional to the product of the sines of the inclinations of the wave-normal to two fixed axes. However the position of these axes are independent of the wave-length, and in fact none of the formulae deduced fit in with observed experimental facts.

---

* Whittaker, p. 143. Hamilton knew only the First Theory at this date.

NOTE 12. *Example of wave motion*

(XX, pp. 565, 566; XXIII, pp. 604–606.)

Hamilton considers the motion of an infinite series of particles, which in equilibrium occupy the points $x=0, \pm 1, \pm 2$, etc., when started in a certain way. If we take the curve

$$\eta_{x,t} = 1 - \cos\left(\frac{2\pi x}{n} - 2t \sin\frac{\pi}{n}\right),$$

the initial conditions are that when $t=0$ all the particles on the negative side of the origin will have their positions and velocities given by the above curve, i.e.

$$\eta_{x,0} = 1 - \cos\frac{2\pi x}{n}; \quad \dot{\eta}_{x,0} = -2\sin\frac{\pi}{n}\sin\frac{2\pi x}{n}$$

but that all on the positive side of the origin have $\eta_{x,0}=0$. Hamilton works out the asymptotic values of the related integral for large values of t. In order to illustrate graphically we take $n=6$ so that there are six particles in each wave-length of the curve

$$\eta_{x,t} = 1 - \cos\left(\frac{\pi x}{3} - t\right).$$

The particles are however not marked on the curve. For s and s' we take the values suggested by Hamilton, 1000 and 200 respectively, and we take t to be 10^9. The initial state is shown in Fig. 1.

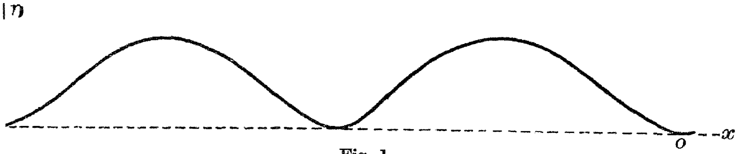


Fig. 1.

When $x > 10{,}001 \times 10^5$, $\eta_{x,t}=0$ but when $x = 10{,}000 \times 10^5$, $\eta_{x,t} = 0{\cdot}039$. On going backwards from the wave-front this disturbance increases to $0{\cdot}117$ (Fig. 2), a value which it retains for a range $9999 \times 10^5 > x > 8772 \times 10^5$. This disturbance has its velocity of propagation $=1$.

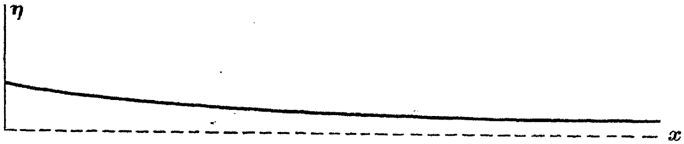


Fig. 2.

In the range $8772 \times 10^5 > x > 8448 \times 10^5$ vibratory motion of gradually increasing magnitude appears. For example, in the neighbourhood of 8660×10^5 we find

$$\eta_{x,t} = \frac{1}{2}\left\{1 - \cos\left(\frac{\pi x}{3} - t\right)\right\} + 0{\cdot}433 \quad \text{(Fig. 3)}.$$

This Hamilton regards as the true wave-front and he finds its velocity to be $\frac{\sqrt{3}}{2}$ ($=0{\cdot}866$) as distinct from the phase velocity $\frac{3}{\pi}$ ($=0{\cdot}955$).

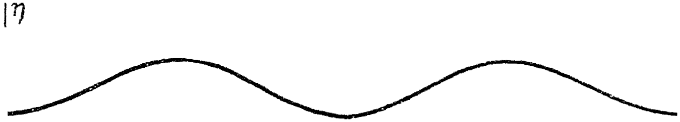


Fig. 3.

The range $8448 \times 10^5 > x > -9999 \times 10^5$ has

$$\eta_{x,t} = \left\{1 - \cos\left(\frac{\pi x}{3} - t\right)\right\} - 0{\cdot}07 \quad \text{(Fig. 4)}.$$

This Hamilton says "corresponds very well to the phenomena of light" (p. 565).

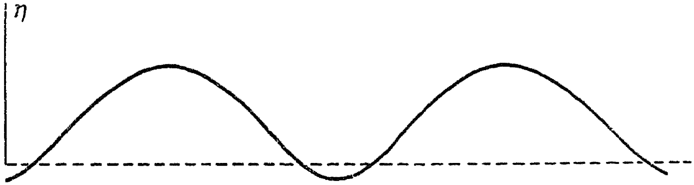


Fig. 4.

For $x = -10{,}000 \times 10^5$,

$$\eta_{x,t} = \left\{1 - \cos\left(\frac{\pi x}{3} - t\right)\right\} - 0{\cdot}023,$$

and for values $x < -10{,}000 \times 10^5$ we have

$$\eta_{x,t} = 1 - \cos\left(\frac{\pi x}{3} - t\right).$$

INDEX

INDEX

References are to pages. In the case of items mentioned in the headings of sections, references are generally to the pages of those headings rather than to the pages of the text where the items occur. For items of frequent occurrence in the text, only the leading references are given.

Aberration of a star, curve of, 290
Academy, Royal Irish, v, 103, 283, 286, 291, 293
Accumulated living force, 107; *see also* Action; Characteristic function
Action, 25, 107, 214
— function, differentiation of, with respect to *vis viva*, x, xii, 49
— law of least, 108, 109
— — of stationary, 96, 98, 109
— — of varying, 105
— — — for binary system, 129
— — — for ternary system, 142
— — — for three-body problem, 53, 97
— — — in general coordinates, 115, 120
— parabolic, 89
— relative, 122, 125, 150
— — approximate value for, 151
— — law of varying, 122, 125
— *see also* Characteristic function
ADARE, Lord, 439
Aether, Faraday-Maxwell, xii
— Hamilton's luminiferous, 638
— law of force between particles of, xiv, 448, 449
— propagation of light in, 446
AIRY, 249, 256, 275, 563
Airy's integral, 563
Alcinoë, 291
Amplitude, 459
— constants of, 454
Angular momentum, integrals of, 623
Anomaly, eccentric, 43, 136, 624
— true, 208
Anthode, xiv, 293, 630
— of a comet, 293
— relation of hodograph to, 630
Anthodographic isochronism, theorem of, xiv, 293, 630
APPELL, 48
Application of calculus of principal relations to lunar theory, xiii, 238, 257
— of characteristic function to dynamics, x, 103
— — — to problem of three bodies, x, xii, 1
— of principal function to dynamics, x, 162, 212
Approximate determination of elements in three-body problem, 14
— expression for *vis viva* in three-body problem, 17
— expressions for perturbations of multiple system, 155
— value of relative action, 151
Approximation, lowering of order of, in lunar theory, 273, 274
— method of, by calculus of principal relations, 250, 257, 278
— to motion of moon, Hamilton's method of, 628
— to parallactic inequality, 248, 272, 277, 627
— to principal integral, 335, 339, 344, 348, 367, 631
— to S, 161, 169, 214, 631
Approximation to S, for lunar theory, 243, 247
— — for motion in resisting medium, 351
— — for nearly circular orbits, 220, 225, 229
— — for particle, 183
— — for projectile, 189, 191
— — in three-body problem, 235
— to time, in three-body problem, 37, 612
— to V, 147, 149, 150
— — for central orbits under various laws of force, 83, 88, 89, 92, 93, 94, 95
— — for multiple system, 150
— — in three-body problem, 25, 27, 32, 44, 53
Approximations to perturbations of dynamical system, 155, 172
Areal velocity, 208
Areas, law of description of, 112, 127
— — — for binary system, 134
— — — for central orbits, 97
Argument of latitude, 208
Astronomical bodies, perturbations of, *see* Perturbations
— refraction, 98
Asymptotic expansions of Bessel functions, xii, 509, 539
— values of integrals, xv, 539, 547, 554
Atmospheric refraction, 96
Attracting systems, motion of, 105, 150, 197
Attraction between particles of aether, law of, xiv, 448, 449
— energy of, 547
Auxiliary function, 160
— — approximation to, 161
— — law of variation of, 160
— — partial differential equations for, 161
— — *see also* Principal function
Axes, optic, 442

BEAUFORT, Admiral Sir F., 103
— D. A., 103
BESSEL, 292, 451
Bessel functions, xii, xv, 451, 458, 509, 511, 526, 547
— — asymptotic expansions of, xii, 509, 539
Biaxal crystal, dispersion in, 437
— — Fresnel's law of polarisation in, 434, 444
— — — — of wave-velocity in, 433, 434, 444, 639
— — optic axes of, 442
— — wave-surface in, 432, 442
— — *see also* Dispersive crystal
Binary system, 128
— — canonical elements for, 202, 206
— — characteristic function for, xii, 128
— — differential equations of varying elements for, 206
— — integrals of motion of, 132, 207
— — law of varying action for, 129
— — partial differential equation for S, 619

Binary system, partial differential equations for V, 130, 135
— — principal function for, 619
— — relative motion of, characteristic function for, 134
— — variational equation for, 128
Biot, 440, 442
Boone, 283
Born, 638
Boscovich, 104, 118
Boscovich's hypothesis, 104
Bremiker, 292
British Association, 256, 257, 267, 278, 410, 595
Brown, 240, 246, 248, 277, 627

Cajori, 202, 630
Calculus of principal functions, *see* Principal relations
— — relations, *see* Principal relations
— of quaternions, *see* Quaternions
— of variations, *see* Variations, calculus of
Canonical elements, xiii, 202, 206
— equations of motion, x, xiii, 166, 215
— forms of equations of motion, Lagrange's, 117
— set of constants, 206
Capital problem of dynamics, 212
Carslaw, 571
Cauchy, xii, 277, 391, 413, 447, 448, 450, 469, 476, 516, 552, 561, 570, 578, 580, 583, 584, 585, 595, 596, 598, 601, 634, 638, 639
Cauchy's law of dispersion, 447, 582, 584, 586, 596, 601
— — of force between particles of aether, 448
— method of characteristics, xii, 391, 634
— principal value, 469, 476, 516, 552, 580
— researches on light-propagation in crystals, comparison with Hamilton, xii, 638
Cayley, 104, 211
Central forces, theorems on, xiii, 286
— orbit, approximation to V for various laws of force, 83
— — characteristic function for, 79
— — curvature of, 100
— — hodograph of, 287, 288
— — partial differential equations for V in, common solution of, 79
— — — — — — transformations of, 95
— — *see also* Binary system; Elliptic motion; Planetary motion
— sun, 291, 292
Centre of gravity, law of motion of, 112, 199
— — motion relative to, 121
Centrobaric coordinates, 2, 124
— — of Jupiter and Saturn, perturbations of, 20
— radius vector, perturbations of, in three-body problem, 57, 67
— velocity, 199
Characteristic function, xiii, 105, 107, 214
— — application of, to dynamics, x, 103
— — applied to three-body problem, xii, 1
— — approximation to, 147, 150
— — — for central orbits, 83, 88, 89, 92, 93, 94, 95
— — — for multiple system, 151, 153
— — — in three-body problem, 25, 27, 32, 44, 53
— — derivative of, with respect to *vis viva*, x, xii, 46, 49, 108
— — disturbing part of, for multiple system, 153, 155
Characteristic function, disturbing part of, for ternary system, 154
— — for binary system, xii, 128
— — for elliptic motion, 44, 46, 135
— — for parabolic motion, 139
— — for planetary motion, 135
— — for relative motion, 121
— — for ternary system, 140
— — for three-body problem, 3
— — integrals of motion deduced from, 107, 108, 115
— — law of variation of, 107, 168
— — — — for binary system, 129, 131
— — — — for relative motion, 121
— — — — for ternary system, 142, 143
— — — — in general coordinates, 115, 120
— — — — in three-body problem, 3, 4, 29
— — — — *see also* Action, law of varying
— — method of improving approximation to, 147
— — partial differential equations for, 109
— — — — — — common solution for, 79
— — — — — — in binary system, 130, 135
— — — — — — in elliptic motion, 45, 79, 138
— — — — — — in general coordinates, 116
— — — — — — in ternary system, 142, 143
— — — — — — in three-body problem, 5
— — relation of principal function to, 160, 168
Characteristics, Cauchy's method of, xii, 391, 634
— of partial differential equation of first order, xii, 392, 634
— — — — of second order, 636
Circular hodograph, law of, 288
— — varying, 290
— orbits, nearly, xiii, 6, 217
— — — characteristic function for, 32, 43
— — — perturbations of, 30, 235
— — — principal function for, 218
— — — — — — approximations to, 220, 228, 235
— — — — — — general expansion for, 225
— — — — — — in polar coordinates, 228
Comet, anthode of, 293
— motion of, 135
— *see also* Parabolic motion; Planetary motion
Common solution of partial differential equations for V, 79
Comparison of Hamilton's and Lagrange's methods of perturbation, 160, 207, 622
— — — — — — in lunar theory, 269, 628
— — — results in perturbations of three-body problem, 73, 77
Composition of forces, statical proof of, xiii, 284
Concave curvature of central orbit, 100
Conics, description of, under central forces, xiii, 286
Constants, canonical set of, 206
— method of variation of, 175, 249; *see also* Parameters, method of variation of
— of amplitude and epoch, 454
— of cubic lattice, 418
— variation of elliptic, in three-body problem, 32
Contact transformation, 179
Convex curvature of central orbit, 100
Coordinates, *see* Centrobaric; General; Geocentric; Heliocentric; Polar; Relative
Coriolis, 106
Correspondence with Herschel, xv, 599
— with Lubbock, xiii, 249
— with Powell, xv, 583

Criticism, Jacobi's, 109, 613
— Poisson's, 614
Crystal, biaxal, dispersion in, 437
— — Fresnel's law of polarisation in, 433, 444
— — — — of wave-velocity in, 433, 434, 444, 639
— — optic axes of, 442
— — wave-surface in, 432, 442
— directions of vibration in, 423, 434
— dispersive, Fresnel's law of propagation in, 438, 444, 445, 639
— — wave-surface in, 444
— — wave-velocity in, 437, 439, 441
— law of *vis viva* for vibrations in, 437
— plane waves in, conditions for, 416
— — — velocity of, 423, 427, 433
— wave-velocities in, 423
Crystalline medium, potential energy of, 638
Crystals, optical properties of, comparison with Cauchy's and Green's results, 638
— propagation of light in, xii, xiv, 413, 450
— reflection and refraction in, 435
Cubic lattice, 418, 450
Curvature of central orbit, 100

Darkness, dynamics of, xii, 541, 555, 599, 600
Degree of periodicity, 485
Depression of order of approximation, in lunar theory, 273, 274
Derivatives, Lagrange's notation for partial and total, 316, 317, 336
Description of areas, law of, *see* Areas
— of conics and sphero-conics under central forces, xiii, 286
— of wave-motion in elastic medium, 563, 604, 640
Determination of principal function, 613, 631
— — — approximate, 335, 339
— — — conditions for, 334, 335, 409
Development of theoretical dynamics, 104
Differential equations, *see* Partial; Total differential equations
— expression of first order, calculus of principal relations for, 332
— — — — principal integral of, 308, 309, 332
— — — — principal supplementary equations of, 333
Diffraction, 569
— spectrum, 591
Direction of light vector, xii
Directions of vibration in crystal, 423, 434
Dispersion, in biaxal crystal, 437
— law of, 447, 582, 584, 586, 596, 601
— of light, 578, 583, 586
— prismatic, 594
Dispersive crystal, law of wave-propagation in, 438, 444, 445, 639
— — wave-surface in, 444
— — wave-velocity in, 437, 439, 441
— — *see also* Biaxal crystal
— medium, law of wave-velocity in, 447, 582, 584, 586, 596, 601
— — propagation of vibration in, 437, 446, 569, 572, 578, 583, 596
— — three-dimensional waves in, 569
Disturbed motion, principal function for, 171
Disturbing function, for particle, 183, 184
Disturbing function, for projectile, 189
— — in general perturbation theory, 172, 173
— — in lunar theory, Laplace's, 239, 257
— — in three-body problem, 32, 34, 44, 49, 53,
— — — — — approximation to, 49, 237
— — — — — Hamilton's and Laplace's co 51
— — — — — heliocentric, 70
— — — — — law of variation of, 37
— part of *S*, 170, 172
— — — for particle, 183
— — — for projectile, 189, 190, 191
— — of *V*, for multiple system, 153, 155
— — — for ternary system, 154
DONKIN, 206
Donkin's theorem, 206
Dublin, University of, 283
Dunraven Castle, 238
Dynamics, application characteristic function to, x, 103
— — of principal function to, x, 163, 212
— capital problem of, 212
— development of theoretical, 104
— fundamental theorem of, 3
— general Method in, xiii, 103, 162, 212
— of darkness, xii, 541, 555, 599, 600
— of light, xii, 411
— of material bodies, 1
— reciprocal theorem in, xv, 451, 458, 490, 493

Earth's annual motion, hodograph of, 290
Eccentric anomaly, 43, 136, 624
Eccentricity, vector of, 290
Elastic medium, description of wave-motion in, 563, 604, 640
— — energy of attraction between particles of, 547
— — equations of vibration in, 527
— — — — general integral of, 547, 548, 576, 601, 602
— — propagation of motion in, xv, 527, 599
— — three-dimensional waves in, 569
— — two-dimensional waves in, 568
Element of contact, 634
Elements, approximate determination of, in three-body problem, 14
— canonical, xiii, 202, 206
— elliptic, 30, 43, 104, 202
— — perturbations of, in lunar theory, 245, 247, 262, 269
— — — in three-body problem, 32
— function of, 163, 177, 179
— — approximations to, 180
— — for multiple system, 209
— — for particle, 185
— — for projectile, 193, 196
— — law of variation of, 179
— — partial differential equations for, 179, 185, 194, 210
— Lagrange's, 163
— method of variation of, 174, 622
— of undisturbed motion, 174, 200
— varying, 163, 174, 177, 209
— — differential equations of, 175, 201, 206
— — for binary system, 206
— — for multiple system, 200

Elements, varying, for particle, 185
— — for projectile, 192
— — in three-body problem, 77
Ellipse, fixed, 57, 628
— Hamilton's, 208, 209, 266, 269, 628
— Lagrange's, 209, 269, 628
— momentary, 77
— varying, 208, 209, 269
Elliptic elements, 30, 43, 104, 202
— — perturbations of, in lunar theory, 245, 247, 262, 269
— — — in three-body problem, 32
— motion, 43, 135, 217, 624
— — characteristic function for, 44, 46, 135
— — principal function for, 217, 624
— orbit, longitude in, 208
— — time in, 139, 140
— variations, in three-body problem, 34
Energy, 106
— of attraction between particles of elastic medium, 547
— of vibrating system, 463
— potential, for crystalline medium, 638
— *see also* Living force; *Vis viva*
Ephemerides, 256
Epoch, 459
— constants of, 454
Equation of limits, 309
— variational, Lagrange's, 105, 164, 212
— — — for binary system, 128
— — — for ternary system, 140
— — — in lunar theory, 239, 257
Equations, canonical, x, xiii, 166, 215
— of motion, Hamilton's, xiii, 166, 215
— — — for particle, 182
— — — for projectile, 186
— — Lagrange's, 117, 164
— — transformation of, 164
— of vibration, in elastic medium, 527
— — — — general integral of, 547, 548, 576, 601, 602
— — of attracting or repelling system, 413, 446, 569, 578
— — of linear system of particles, 452, 576, 600
— *see also* Partial; Total differential equations; Principal supplementary equations
Equivalent vibrations, principle of, 437
Euclid, 289
Euler, 139
Euler's theorem, 139
Extinction, velocity of, 571

Factorials, Vandermonde's notation for, 523
Faraday, xii
Faraday-Maxwell aether, xii
Fit, 597
Five points in space, relation connecting, 146
Fixed ellipse, 57, 628
— sinusoid, 502
Fluctuating functions, xii, xv, 451, 458, 504, 520, 557
Fluctuation, method of, 476, 504, 520, 555, 557
— principle of, 520, 581
Foci, kinetic, 78
Force, approximation to V for various laws of, 83, 88, 89, 92, 93, 94, 95
— between particles of aether, law of, xiv, 448, 449
Force-function, 106, 164, 198
Force-function for binary system, 128
— — for ternary system, 140
— living, *see* Living force; *Vis viva*
— Newtonian law of, 135; *see also* Planetary motion
— — — hodographic representation of, 287, 290
Forces, central, theorems on, xiii, 286
— parallelogram of, statical proof, xiii, 284
Fourier, 529, 579
Fourier's theorem, 529, 579
Fraunhofer, 587, 588, 589, 590, 591, 592, 593, 594
Fresnel, 432, 433, 434, 438, 439, 444, 445, 595, 639
Fresnel's law of polarisation in biaxal crystal, 434, 444
— — of propagation in dispersive crystal, 438, 439, 444, 445, 639
— — of wave-velocity in biaxal crystal, 433, 434, 444, 639
— wave-surface, 432, 442
Function, *see* Action; Bessel; Characteristic; Disturbing; Elements, function of; Fluctuating; Force-; Gamma; Hamiltonian; Principal
Fundamental theorem of calculus of principal relations, 363, 364
— — of dynamics, 3

Galileo, 104
Gamma function, 508
General coordinates, 113, 118, 164, 214
— — characteristic function in, 115, 167, 168
— — equations of motion in, 117, 164
— — Hamiltonian function in, 165, 215
— — integrals of motion in, 115, 167, 215
— — law of varying action in, 115, 120, 125
— — partial differential equations for S in, 168, 631
— — — — — for V in, 116, 169
— — principal function in, 166, 215
Geocentric coordinates of moon, 238
— longitude of moon, 240
— — — perturbations of, 243, 245, 248, 262, 263, 272, 274, 277
— radius vector of moon, 240
— — — — perturbations of, 243, 245, 248, 262, 263, 272, 274, 277
Goursat, 635, 636, 637
Grating, 591
Graves, C., 286
— R. P., 104, 106, 297, 599, 607
Gravity, centre of, *see* Centre of gravity
— motion under, *see* Projectile
Green, 638
Group-velocity, xii; *see also* Velocity of progress of vibration; Velocity of propagation of disturbance; Velocity of propagation of vibration
— — distinguished from phase-velocity, xv, 510, 517, 556, 571, 572, 577, 582, 605, 641

H function, *see* Hamiltonian function; Living force; *Vis viva*
Halley, 585
Hamilton, Life of, 104, 106, 297, 599, 607
Hamilton integral, 614
Hamiltonian function, 165, 215
— — law of variation of, 215

Hamilton's ellipse, 208, 209, 266, 269, 628
— equations of motion, x, xiii, 166, 215
— luminiferous aether, 638
— principle, 167
— theorem on central orbits, xiii, 286
Harmonic oscillator, principal function for, 618
Heat, effect of, on indices of refraction, 440
HEAVISIDE, xv, 542
Heaviside's operator, xv, 542
Heliocentric coordinates, 51, 53, 70, 207, 208
— — perturbations of, in three-body problem, 24, 70, 71, 72
— disturbing function, 70
— longitude, perturbations of Jupiter's, 11
— — — of Saturn's, 14
— plane, perturbation of coordinate perpendicular to, 77
— radius vector, perturbations of, 71
— — — — of Jupiter's, 11
— — — — of Saturn's, 14
— velocity, 208
HERSCHEL, Miss C., 607
— Sir J., xv, 202, 527, 589, 593, 599, 607
Herschel, Correspondence with, xv, 599
Herschel's notation for inverse trigonometrical functions, 202
Hodograph, xiii, 287
— law of circular, 288, 293
— momentary relative, 291
— of central orbit, 287, 288
— of earth's annual motion, 290
— relation of anthode to, 630
— relative, 290
— varying circular, 290
Hodographic isochronism, theorem of, xiv, 293, 630
— representation of moving point, 289
— vector, 291
Hodographs, reciprocal, 288
HOUËL, 211
Hypersurface, 617
Hypothesis, Boscovich's, 104

I, operator, 542
Iceland Spar, double wave-surface in, 441
— — indices of refraction of ordinary ray in, 439, 440
— — velocity of light in, 439, 441, 442
Indices of refraction, effect of heat on, 440
Inequality, parallactic, approximation to, 248, 272, 277, 627
Infinite linear system, vibration of doubly-, 458, 459, 460, 461, 465, 482, 502, 503, 511
— — — — — recapitulation of results, 495
— — — — of semi-, 457, 459, 460, 461, 465, 482, 500, 511
— — — — — recapitulation of results, 492
Integral, Airy's, 563
— complete, determination of principal function from, 614
— — of partial differential equation of first order, 393, 396
— general, of partial differential equation of first order, 329, 330, 393, 396, 401, 402
— — — — — — — connection with principal integral of total differential equation, 329, 330
— Hamilton, 614
Integral of partial differential equation of second order, 401, 402
— principal, *see* Principal integral
— relation, principal, *see* Principal integral
Integrals, asymptotic values of, xv, 539, 547, 554
— of momentum, 113
— of motion, deduced from Q, 167
— — — from S, 167, 213, 215
— — — from V, 108, 115
— — for binary system, 132, 133
— — for particle, 182
— — for projectile, 186
— — verification of, 109
— of undisturbed motion, 171, 174
— — — for binary system, 202
— — — for multiple system, 200
— — — for particle, 183
— — — for projectile, 188, 189
Integration of partial differential equations, by calculus of principal relations, xii, xiv, 391, 634
— — — — by Pfaff's method, 329, 331
Interference of polarised rays, 598
— phenomena of, 591, 596
— spectrum of, 591
Interval of wave, 421
Inverse of hodograph, anthode as, 630
Inverse trigonometrical functions, Herschel's notation for, 202
Inversion, method of, 630
Irish Academy, Royal, v, 103, 283, 286, 291, 293
Isochronism, hodographic and anthodographic, xiv, 293, 630
Isotropic medium, 638, 639

JACOBI, 48, 109, 168, 206, 283, 613, 614
Jacobi's criticism, 109, 613
JEFFREYS, 542
Jupiter, equations of motion of, 2
— perturbations of, 8
— — agreement with Laplace, 11
— — in centrobaric coordinates, 20
— — in heliocentric coordinates, 70
Jupiter's centrobaric longitude, perturbations of, 10, 11
— — radius vector, perturbations of, 9, 10
— heliocentric longitude, perturbations of, 11
— — radius vector, perturbations of, 11

KELVIN, 561
KEPLER, x, 290
Kepler orbit, x; *see also* Planetary motion
Kinetic foci, 78
Known planets in 1834, 104

LAGRANGE, x, 53, 73, 74, 77, 104, 105, 106, 108, 109, 117, 118, 121, 160, 163, 164, 166, 167, 175, 209, 212, 213, 255, 263, 266, 269, 276, 279, 280, 282, 283, 316, 317, 331, 336, 613, 614, 622, 629
Lagrange's elements, 163
— ellipse, 209, 269, 629
— equations of motion, 117, 164

Lagrange's method of variation of parameters, applied to perturbation problem, 255, 279, 280
— — — — — compared with method of principal function, 255, 263, 269, 279, 280, 281, 282, 622
— multipliers, 120, 121
— notation for partial and total derivatives, 316, 317, 336
— results in perturbations of three-body problem, compared with Hamilton's, 77
Lambert, 139
Lambert's theorem, 139
Laplace, x, 8, 11, 12, 49, 51, 105, 109, 519, 563
— agreement with, in perturbations of Jupiter, 11
Laplace's disturbing function, compared with Hamilton's, 51
— — — in lunar theory, 239, 257
Latitude, argument of, 208
Lattice, cubic, 418, 450
Law of circular hodograph, 288, 293
— of description of areas, *see* Areas
— of dispersion, 447, 582, 584, 586, 596, 601
— of force between particles of aether, xiv, 448, 449
— of living force, 107
— of motion of centre of gravity, 112, 199
— of polarisation in biaxal crystal, Fresnel's, 434, 444
— of propagation in dispersive crystal, Fresnel's, 438, 439, 444, 445, 639
— of variation, *see* Action function; Characteristic function; Disturbing function; Elements, function of; Hamiltonian function; Principal function; Principal integral
— of wave-velocity in biaxal crystal, Fresnel's, 433, 439, 444, 445, 639
Laws of least, stationary, varying action, *see* Action
Least action, law of, *see* Action
— squares, method of, 594
Legendre, 508, 627
Legendre's celebrated function, 508
— polynomials, 9, 240, 258, 627
Leibniz, 106
Length of undulation, 446
— wave-, 446
Le Verrier, 292
Le Verrier's planet, 292
Life of Hamilton, 104, 106, 297, 599, 607
Light, dispersion of, 578, 583, 586
— dynamics of, xii, xiv, 411
— — correspondence on, xv, 583
— — researches on, 576
— propagation of, in crystals, xii, xiv, 413, 450
— — *in vacuo*, xiv, 446
— — rectilinearity of, 569
— — velocity of, *see* Velocity
— theory of, researches respecting vibration connected with, xiv, 451, 578
— vector, direction of, xii
— *see also* Crystals; Propagation; Reflection; Refraction; Velocity; Vibration
Limits, equation of, 309
Linear system of particles, energy of vibrating, 463
— — — equations of vibration of, 452, 576, 600
— — — — — general integral of, 452, 490, 492, 576, 601, 602
— — — vibrations of, xiv, 451, 576, 600
— — — *see also* Vibration
Liouville, 277
Living force, 3, 106
Living force, accumulated, 107, 108; *see also* Action; Characteristic function
— — equation of, 107
— — — in three-body problem, 2, 17, 25
— — law of, 107
— — — deduced from hodograph, 289
— — relative, 121, 126, 150
— — *see also Vis viva*
Lloyd, 439, 595
Longitude, in elliptic orbit, 208
— moon's geocentric, 240
— — — perturbations of, 243, 245, 248, 262, 263, 272, 274, 277
— of node, 208
— of perihelion, 208
— perturbation of Jupiter's, 10, 11
— — of Saturn's, 14
Lovett, 160, 175
Lubbock, J. W., xii, xiii, 238, 246, 248, 249, 250, 256, 257, 263, 264, 266, 274, 275, 276, 277, 278, 279, 283
— correspondence with, xiii, 249
— Lord, 249
Lubbock's expression for perturbation of moon's coordinates, 277
— results in lunar theory compared with Hamilton's, 246, 248, 263, 272, 275, 627
Luminiferous aether, Hamilton's, 638
Lunar tables, 249, 256
— theory, xiii, 238, 249
— — application of calculus of principal relations to, 257, 266
— — comparison of Hamilton's, Lubbock's and Plana's results in, 246, 248, 263, 272, 275, 627
— — disturbing function in, Laplace's, 239, 257
— — lowering of order of approximation in, 273, 274
— — method of principal function applied to, xiii, 238, 257, 266
— — perturbational parts of elements in, 242, 245, 247, 262, 268, 269
— — perturbations in, Hamilton's and Lagrange's methods compared, 263, 264, 269, 276, 628
— — principal function in, 238, 239
— — — — approximations to, 243, 247
— — — — variation of, 238, 239, 258
— — variational equation in, 239, 257
— — *see also* Moon, motion of; Moon's coordinates, perturbations of

Maccullagh, xii, 639
Macdonald, 638
Mädler, 291, 292
Malus, 440, 442
Marks of position, 113, 114, 164, 214
— of relative position, 125
Mathematical Papers, Vol. I, 3, 5, 105, 109, 168, 613
Maxwell, xii
Mécanique Analytique, 73, 74, 104, 106, 109, 117, 160, 163, 164, 165, 166, 622
Mécanique Céleste, 9, 10, 11, 24, 51, 71, 264, 285
Medium, crystalline, potential energy of, 638
— — *see also* Crystal
— dispersive, *see* Dispersive
— elastic, *see* Elastic
— resisting, approximation to S for motion in, 351
Method, general, in dynamics, xiii, 103, 162, 212

Method, Hamilton's, of approximating to motion of moon, 628
— of characteristic function, applied to dynamics, x, 103
— — — — to three-body problem, xii, 1
— of fluctuation, 476, 504, 520, 555, 557
— of integration of partial differential equations, 391, 634
— of least squares, 594
— of principal function, *see* Principal function; Principal relations, calculus of
— — relations; *see* Principal relations, calculus of
— of variation of constants, *see* Constants, method of variation of
— — of parameters, *see* Parameters, method of variation of
Methods of approximation, *see* Approximation
— of determining principal function, 334, 335, 613, 631
— of perturbation, *see* Perturbation
— operational, applied to vibration in elastic medium, xv, 527
Mitscherlich, 440
Möbius, 124, 287
Mode of vibration, permanent, 489, 492, 495, 496, 498, 499
— — simple, 464
— — sinusoidal, 488, 489, 492, 495, 497
Molecules of elastic medium, vibration of, 527
Molini, 283
Momenta, generalised, 165, 214
— perturbations of, 172, 173
Momentary ellipse, 77
Momentum, integrals of, 113
Moon, motion of, Hamilton's method of approximating to, 628
— theory of, xiii, 238, 249
— — application of calculus of principal relations to, 257, 266
— — comparison of Hamilton's, Lubbock's and Plana's results in, 246, 248, 263, 272, 275, 627
— — disturbing function in, 238, 239
— — lowering of order of approximation in, 273, 274
— — method of principal function applied to, xiii, 238, 257, 266
— — perturbational parts of elements in, 242, 245, 247, 262, 268, 269
— — perturbations in, Hamilton's and Lagrange's methods compared, 263, 264, 269, 276, 628
— — principal function in, 238, 239
— — — — approximations to, 243, 247
— — — — variation of, 238, 239, 258
— — variational equation in, 239, 257
Moon's coordinates, perturbations of, 243, 245, 248, 262, 263, 272, 274, 277
— — — Hamilton's results compared with Lubbock's and Plana's, 246, 248, 263, 272, 275, 627
Multiple system, 150, 200
— — characteristic function for, approximation to, 150
— — — — disturbing part of, 153, 155
— — function of elements for, 209
— — integrals of undisturbed motion of, 200
— — rigorous transition from binary to, 155
— — with one predominant mass, perturbations of, 150, 155, 200
Multiplicity, 634
Multipliers, Lagrange's, 120, 121

Nearly circular orbits, *see* Circular orbits, nearly
Neptune, discovery of, 104, 291
Neumann, 639
Newton, x, 104, 140, 208, 217, 231, 286, 288, 289, 290, 291, 597
Newtonian law of force, 135, 217
— — — hodographic representation of, 287, 290
— orbit, x; *see also* Elliptic motion; Planetary motion
Node, 454, 456
— longitude of, 208
Notation, Herschel's, for inverse trigonometrical functions, 202
— Lagrange's, for partial and total derivatives, 316, 317, 336
— Vandermonde's, for factorials, 523

Operational methods, applied to vibration in elastic medium, xv, 527
Operator *D*, 544
— ∇, 530
— $(\nabla^{-2}-1)^{-\frac{1}{2}}$, 535
— Heaviside's, xv, 542
— *I*, 542
— tan δ, 543
Optic axes of biaxal crystal, 442
Orbit, plane relative, 135, 136
— varying, 208, 209, 269
— *see also* Central; Elliptic; Kepler; Newtonian; Parabolic; Planetary
Orbits, nearly circular, *see* Circular orbits, nearly
Oscillating sinusoid, 497

Parabolic action, 89
— motion, characteristic function for, 139
— — — — approximation to, 88
— — *see also* Comet; Projectile
— orbit, 75, 76
— — time in, 75, 139, 294
Parallactic inequality, approximation to, 627
Parallelogram of forces, statical proof of, xiii, 284
Parameters, method of variation of, applied to perturbation problem, 255, 279, 280
— — — compared with method of principal function, 255, 263, 269, 279, 280, 281, 282, 622
— — — Poisson's mode of applying, 280
— — — *see also* Constants, method of variation of
Paris inch, 587, 591
Partial differential coefficients, Lagrange's notation for, 316, 317, 336
— — equation of first order, characteristics of, xii, 391, 392, 634
— — — — — complete integral of, 393, 396
— — — — — general integral of, 393, 396
— — — — — integration of, 391, 634
— — — — — — Pfaff's method of, 326, 329, 331
— — — of second order, characteristics of, 636
— — — — — integral of, 401, 402
— — — — — integration of, 397, 635
— — — — — — examples, 403
— — equations for function of elements, 179
— — — — — in motion of particle, 185
— — — — — — of projectile, 194

Partial differential equations for function of elements in multiple system, 210
— — — for principal integral of differential expression, 334
— — — — — of total differential equation, 360, 365, 409
— — — for Q, 169
— — — for S, 161, 168, 214, 631
— — — — for particle, 182
— — — — for projectile, 187
— — — — in elliptic orbit, 217, 219
— — — — — — compatibility of, 219
— — — — in general coordinates, 168, 631
— — — — in three-body problem, 230, 231, 233, 234
— — — for V, 109
— — — — in binary system, 130
— — — — in central orbits, 79, 95
— — — — — — common solution of, 79
— — — — — — transformations of, 95
— — — — in general coordinates, 116, 169
— — — — in relative coordinates, 127
— — — — in ternary system, 142, 143
— — — — in three-body problem, 5
Particle, motion of, 181
— — disturbing function for, 183, 185
— — function of elements for, 185
— — Hamilton's equations for, 182
— — integrals of, 182
— — perturbation theory for, 181
— — principal function for, 182
— — — — disturbing part of, 183
— — varying elements for, 185
Particles, linear system of, *see* Linear system
— of aether, law of force between, xiv, 448, 449
Perihelion distance, 208
— longitude of, 208
— passage, time of, 208
Periodicity, degree of, 485
Permanent mode of vibration of linear system, 489, 492, 495, 496, 498, 499
Perturbation, Lagrange's and Hamilton's methods compared, 159, 160, 255, 279, 280, 281, 282, 622
— — — — — in lunar theory, 263, 264, 269, 276, 628
— problem of, 171, 177
— problems of, calculus of principal relations applied to, 250, 257, 278
— rigorous theory of, by principal function, 170, 174
— theorem of, 73
— theory of, disturbing function in, 172, 173
— — for motion of particle, 181
— — — of projectile, 188
— — — — with variation of gravity, 197
— — for multiple system, 150, 155, 200
Perturbations, independence of, on disturbed mass, to first order, 78
— in three-body problem, 5, 20, 30, 231, 235
— of astronomical bodies, xiii; *see also* Lunar theory; Three-body problem
— of dynamical system, approximations to, 155, 172
— of elliptic elements, in lunar theory, 245, 247, 262, 269
— — — in three-body problem, 32
— of Jupiter, 8
— of Moon, *see* Lunar theory; Moon
— of multiple system, 150, 155, 200
Perturbations of nearly circular orbits, 30, 235
— of Saturn, 12
Pfaff, 326, 329, 331
Pfaff's method, 326, 329, 331
Phase, 459
— change of, 507, 509, 561
— -velocity, xii, 451, 556, 605; *see also* Phase, velocity of passage of; Phase, velocity of transmission of
— — distinguished from group-velocity, xv, 510, 517, 556, 571, 572, 577, 582, 602, 641
— velocity of passage of, 577, 581
— — of transmission of, 459, 487, 491, 494, 496, 510, 517, 582
Plana, xiii, 246, 248, 249, 263, 272, 273, 275, 276, 277, 283, 627
Plana's results in lunar theory, compared with Hamilton's, 246, 248, 263, 272, 275, 627
Plane, momentary, 77
— of ecliptic, 2
— polarisation, 597, 598
— relative orbit, 135, 136
— waves in aether, 446
— — in crystal, 416
— — — velocities of propagation of, 423, 427, 433
Planet, heliocentric coordinates of, 207, 208
— motion of, *see* Central orbit; Elliptic motion; Planetary motion
Planetary motion, 135, 217, 624
— — characteristic function for, 135
— — principal function for, 217, 624
— — *see also* Central orbit; Elliptic motion; Parabolic motion
— orbit, time in, 139, 140
Planets, known in 1834, 104
Plates, phenomena of thin, 597
Pleiades, 291
Poggendorf, 439
Point, motion of single, *see* Particle, motion of
Poisson, xiv, 109, 160, 163, 175, 177, 264, 276, 277, 280, 283, 419, 449, 536, 537, 614, 638
Poisson brackets, 175, 176, 201, 203, 204, 206
— — constancy of, 177
Poisson's criticism, 614
— law of force between particles of aether, 449
— mode of applying Lagrange's method of varying parameters, 280
Polar coordinates, equation of living force in, for three-body problem, 17, 25
— — perturbations of, in three-body problem, 20
— — — of nearly circular orbits in, 228
Polarisation, Fresnel's law of, 434, 444
— plane, 597, 598
Pontécoulant, de, 249, 256, 264, 277, 283, 627
Poole, 542
Position, marks of, 113, 164, 214
— — of relative, 125
— vector of, 287
Potential energy of crystalline medium, 638
Powell, xv, 439, 583, 595, 596, 599, 601
— correspondence with, xv, 583
Principal function, x, xiii, 72, 160, 163, 166, 167, 213, 215
— — application of, to dynamics, x, 162, 212
— — applied to perturbation theory, 170, 174
— — approximations to, 170, 335, 339
— — — for nearly circular orbits, 220, 228, 235

Principal function, approximations to, for particle, 182, 183
— — — for projectile, 189, 190, 191, 351
— — — in lunar theory, 243, 247
— — — in three-body problem, 235
— — determination of, from complete integral, 613
— — — Hamilton's method of, 334, 335, 631
— — disturbing part of, for particle, 183
— — — — for projectile, 189, 190, 191
— — — — in perturbation theory, 170, 172
— — for binary system, xiii, 619
— — for disturbed motion, 171
— — for elliptic orbit, 217, 624
— — for general dynamical system, 160, 166, 167, 213, 215
— — for harmonic oscillator, 618
— — for lunar theory, 238, 239
— — for particle, 182
— — for projectile, 186, 187
— — for relative motion, 200
— — for three-body problem, 230, 231, 232
— — for undisturbed motion, 171
— — general expansion for, for nearly circular orbit, 225
— — integrals of motion deduced from, 167, 213, 215
— — law of variation of, 160, 166, 213, 215
— — — — for elliptic orbit, 217
— — — — for relative motion, 200
— — — — in lunar theory, 238, 239, 258
— — — — in three-body problem, 233
— — method of, applied to lunar theory, xiii, 238, 257, 266
— — — compared with method of varying parameters, 255, 263, 279, 280, 281, 282
— — — of improving approximation to, 169, 214
— — — *see also* Principal relations, calculus of
— — of total differential equations, *see* Principal integral
— — partial differential equations for, 161, 168, 214, 631
— — — — — for particle, 182
— — — — — for projectile, 187
— — — — — in elliptic orbit, 217, 219
— — — — — in three-body problem, 230, 231, 233, 234
— — relation of characteristic function to, 168
— — — of *Q* to, 168
— functions, calculus of, *see* Principal relations, calculus of
— integral of differential expression of first order, 308, 309, 332
— — — — — — approximation to, 335, 339, 631
— — — — — — — for two variables, 344, 348
— — — — — — determination of, 334, 631
— — — — — — partial differential equations for, 334, 409
— — of system of total differential equations of any order, xi, 363, 372
— — — — — — — — law of variation of, 363, 364
— — — — — — of first order, 369
— — — — — — — — law of variation of, 370
— — — — — — of second order, 372
— — of total differential equation of first order, xi, 305, 306, 316, 358, 365, 408
— — — — — — — improvement of approximation to, 367

Principal integral of total differential equation of first order, law of variation of, 359
— — — — — — — partial differential equations for, 360, 365, 409
— — — — — — — relation of general integral to, 329, 330
— relations, calculus of, xi, xiv, 105, 163, 212, 297, 332, 334, 358, 408
— — — applied to lunar theory, xiii, 257, 266
— — — — to perturbation problems, 250, 278
— — — for differential expression of first order, xi, 332
— — — for system of differential equations of any order, xi, 360, 372
— — — — — — of first order, 368
— — — — — — of second order, 372
— — — for total differential equation of first order, xi, 297, 316, 358, 364, 408
— — — fundamental theorem of, 363, 364
— — — particular cases, 375
— supplementary equations of differential expression of first order, 333
— — — of system of differential equations of any order, 360, 362, 372, 373, 374
— — — — — — of first order, 368, 370
— — — — — — of second order, 372
— — — of total differential equation of first order, 302, 303, 316, 358, 364, 408
— value, Cauchy's, 469, 516, 552, 580
Principia, 100, 286
Principle, Hamilton's, 167
— of equivalent vibrations, 437
— of fluctuation, 520, 581
— Rayleigh's, xii, 340
Principles of least, stationary and varying action, *see* Action
Prismatic dispersion, 594
— spectra, 594
Problem, capital, of dynamics, 212
— of perturbation, 171, 177
— of three bodies, *see* Ternary system; Three-body problem
— of two bodies, *see* Binary system; Elliptic motion; Planetary motion
Problems of perturbation, calculus of principal relations applied to, 250, 257, 278
Progress of wave, velocity of, 577; *see also* Group-velocity; Velocity of propagation of vibration
Progressive wave, simple, *see* Travelling sinusoid
Projectile, integrals of motion for, 186, 187, 188
— motion of, 186
— — disturbing function for, 189
— — function of elements for, 193, 196
— — Hamilton's equations for, 186
— — perturbations of, 188
— — principal function for, 186, 187
— — — — approximations to, 189, 190, 191
— — — — disturbing part of, 189, 190, 191
— — varying elements for, 192
— — with variation of gravity, 197
Propagation of disturbance, velocity of, 606
— of light in crystals, xii, xiv, 413, 450
— — *in vacuo*, xiv, 446
— — rectilinearity of, 569
— of motion in elastic medium, 527
— of plane wave in crystal, velocities of, 423, 427, 433
— of vibration, 413, 450, 451, 527, 576, 578, 599

Propagation of vibration, general law of, 571, 572
— — velocity of, 510, 517, 556, 560, 571, 572, 574, 575, 582, 605; *see also* Group-velocity
— slowness of, 417, 596
— wave-, Fresnel's law of, in biaxal crystal, 433, 434, 444, 639
— — — — in dispersive crystal, 438, 439, 444, 445, 639
— — law of velocity of, in dispersive medium, 447, 582, 584, 586, 596

Q function, 167, 170
— — integrals of motion deduced from, 167
— — law of variation of, 167
— — partial differential equations for, 169
— — relation of principal function to, 168
Quantity of vibration, 463
Quaternions, 286, 287, 291, 293

Radius vector, centrobaric, perturbations of, 57, 67
— — heliocentric, perturbations of, 71
— — Jupiter's, perturbations of, 9, 10, 11
— — Moon's geocentric, perturbations of, 243, 245, 248, 262, 263, 272, 274, 277
— — Saturn's, perturbations of, 14
Rayleigh, xii, 340, 490
Rayleigh's principle, xii, 340
Rays, Theory of system of, 105, 214
Reciprocal hodographs, 288
— theorem in dynamics, xv, 451, 458, 490
Rectilinearity of light propagation, 569
Reflection and refraction of light in crystals, 435
— — — — laws of, 437
— of vibrations, 457, 459
Refraction, astronomical, 98
— atmospheric, 96
— indices of, effect of heat on, 440
— — in Iceland Spar, 439, 440
— *see also* Reflection and refraction
Relation between five points in space, 146
Relations, calculus of principal, *see* Principal relations
Relative action, *see* Action
— coordinates, 113, 198
— — for binary system, 207
— hodograph, 290
— living force, 121, 126, 150
— motion, 121, 197
— — characteristic function for, 121
— — — — in binary system, 134
— — — — in elliptic orbit, 136
— — — — in ternary system, 143, 147
— — — — variation of, 121
— — principal function for, 200
— — — — variation of, 200
— orbit, plane, 135, 136
— position, marks of, 125
Resisting medium, approximation to principal function for motion in, 351
Riccati, 563
Rings, phenomena of, 445
Ritz, 340
Routh, 168, 206
Rudberg, 439

S function, *see* Principal function
Salmon, 146
Saturn, equations of motion of, 2
—perturbations of, 12
— — in centrobaric coordinates, 20
— — in heliocentric coordinates, 70
Saturn's longitude, perturbations of, 14
— radius vector, perturbations of, 14
Semi-infinite linear system of particles, vibration of, *see* Infinite linear system
Sinusoid, 488, 489, 492
— fixed, 502
— oscillating, 497
— travelling, 497, 499, 501, 502
Sinusoidal mode of vibration, 488, 489, 492, 495, 497
— wave, 451
Skotodynamics, xii; *see also* Darkness, dynamics of
Slowness of propagation of wave, 417
— vector of, 293
Solar system, approximation to characteristic function for, 83, 94
Solution, common, of partial differential equations for V, 79
Sommerfeld, 571
Spectra, prismatic, 594
Spectrum, diffraction-, 591
— of interference, 591
Sphero-conic, description of, under central force, xiii, 286
Square system of particles, vibration of, 568
Squares, method of least, 594
Star, aberration of, 290
Stationary action, law of, *see* Action
Strained cubic lattice, 418, 450
Sun, central, 291, 292
Supplementary equations, principal, *see* Principal supplementary equations
Surface, wave-, double, in Iceland Spar, 441
— — Fresnel's, 432, 442
— — in dispersive crystal, 444
System of total differential equations, Calculus of principal relations applied to, *see* Principal relations
— — — — Principal integral of, *see* Principal integral
— *see also* Binary; Linear; Multiple; Solar; Square; Ternary
Systems of Rays, Theory of, 105, 214

T function, *see* Living force; *Vis viva*
Tables, lunar, 249, 256
Tait, 106
Taylor, 335, 354
Taylor's theorem, 335, 354
Ternary system, 140
— — characteristic function for, 140
— — — — disturbing part of, 154
— — — — law of variation of, 143
— — force-function for, 140
— — integrals of motion of, 142, 143, 144
— — law of varying action for, 142
— — partial differential equations for V, 142, 143
— — variational equation for, 140
— — with one predominant mass, perturbations of, 150, 200

Ternary system, *see also* Lunar theory; Moon, theory of; Three-body problem
Theorem, fundamental, of calculus of principal relations, 363, 364
— — of dynamics, 3
— *see* Euler's; Fourier's; Hamilton's; Lambert's; Reciprocal; Taylor's
Theoretical dynamics, development of, 104
Theory of Systems of Rays, 105, 214
Theory of light, *see* Light, theory of
— Moon, *see* Lunar theory; Moon, theory of
Thickness of wave, 417, 421
Thin plates, phenomena of, 597
Three bodies, problem of, *see* Three-body problem
Three-body problem, x, xii, 1, 230
— — — application of characteristic function to, xii, 1
— — — characteristic function for, 3
— — — — — approximation to, 25, 27, 32, 44, 53
— — — — — variation of, 3
— — — disturbing function for, 32, 34, 44, 49, 53, 231, 232
— — — — — approximation to, 49, 237
— — — — — Hamilton's and Laplace's compared, 51
— — — — — heliocentric, 70
— — — — — variation of, 37
— — — elements in, approximate determination of, 14
— — — elliptic elements in, perturbations of, 32
— — — partial differential equations for S in, 230, 231, 233, 234
— — — — — — for V in, 5
— — — perturbations in, 5, 20, 30, 231, 235
— — — principal function for, 230
— — — — — approximation to, 235
— — — — — variation of, 233
— — — time in, approximation to, 37, 612
— — — *vis viva* in, approximation to, 17
— — — *see also* Lunar theory; Moon, theory of; Ternary system
Three-dimensional waves in dispersive medium, 569
Time, approximation to, in multiple system, 155, 156
— — in three-body problem, 37, 612
— in parabolic orbit, 75, 139, 294
— in planetary orbit, 139, 140
— obtained from characteristic function, 49, 108
— of perihelion passage, 208
Total derivatives, Lagrange's notation for, 316, 317, 336
— differential equations, *see* Principal integral; Principal relations, calculus of; Principal supplementary equations
Transformation, contact, 179
Transmission of phase, velocity of, 459, 487, 491, 494, 496, 510, 517, 582; *see also* Phase-velocity
Transverse vibrations, *see* Vibration of linear system
Travelling sinusoid, 497, 499, 501, 502
True anomaly, 208
TURNER, 267
Two bodies, problem of, *see* Two-body problem
Two-body problem, *see* Binary system; Elliptic motion; Parabolic motion; Planetary motion
Two-dimensional waves in elastic medium, 568
Two variables, principal integral for differential expression in, 344
— — — — — — examples of, 348

U function, *see* Force-function
Ullswater, 599
Ultimate state of vibration in elastic medium, 555, 565, 566
Undisturbed motion of binary system, 202
— — of general system, integrals of, 171, 174
— — — — principal function for, 171
— — of multiple system, integrals of, 200
— — of particle, 183
— — of planet, principal function for, 217, 624
— — of projectile, 188, 189
University of Dublin, 283
Uranus, 291

V function, *see* Action function; Characteristic function
Vacuo, propagation of light *in*, xiv, 446
Value, asymptotic, xv, 539, 547, 554
VANDERMONDE, 523
Vandermonde's notation for factorials, 523
Variation, law of, *see* Action function; Characteristic function; Disturbing function; Elements, function of; Hamiltonian function; Principal function; Principal integral
— of constants, method of, 175, 249; *see also* Parameters, method of variation of
— of gravity, projectile with, 197
— of parameters, method of, *see* Parameters, method of variation of
— of elements, method of, 174, 622
Variational equation, Lagrange's, 105, 164, 212
— — — for binary system, 128
— — — for ternary system, 140
— — — in lunar theory, 239, 257
Variations, calculus of, xi, 107, 109, 166, 303, 309, 317, 359, 363, 391, 408
— — applied to integration of partial differential equations, xiv, 391
Varying action, *see* Action
— elements, *see* Elements, varying
— ellipse, 208, 209, 269
— orbit, 208, 209, 269
Vector, hodographic, 291
— light, direction of, xii
— of eccentricity, 290
— of position, 287
— of slowness, 293
— of velocity, 287, 293
Velocities, wave-, in crystal, 423
— — in dispersive crystal, 437, 439, 441
Velocity, areal, 208
— centrobaric, 199
— distinction between group- and phase-, xv, 510, 517, 556, 571, 572, 577, 582, 602, 641
— group-, xii; *see also* Velocity of progress of vibration; Velocity of propagation of disturbance; Velocity of propagation of vibration
— heliocentric, 208
— of extinction, 571
— of light in Iceland Spar, 439, 441, 442
— of passage of phase, 577, 581
— of progress of vibration, 577
— of propagation of disturbance, 606

Velocity of propagation of vibration, 510, 517, 556, 560, 571, 572, 582, 605
— of transmission of phase, 459, 487, 491, 494, 496, 510, 517, 582
— phase-, xii, 451, 556, 605; *see also* Velocity of passage of phase; Velocity of transmission of phase
— vector of, 287, 293
— wave-, Fresnel's law of, in biaxal crystal, 433, 434, 444, 639
— — in aether, 447, 448
— — law of, in dispersive medium, 447, 582, 584, 586, 596
Venter, 454, 456
Vibrating system, energy of, 463
— — equations of motion of, 413, 446, 569, 578
Vibration, directions of, in crystal, 423
— general law of propagation of, 571, 572
— in aether, propagation of, 446
— in crystals, propagation of, 413, 450
— in elastic medium, equations of, 527
— — — — general integral of, 547, 548, 576, 601, 602
— — — for large values of t, 555, 565, 566
— — — propagation of, xv, 527, 599
— of cubic system, 418
— of infinite linear system, 458, 459, 460, 461, 465, 482, 502, 503, 511
— — — — recapitulation of results, 495
— of linear system, 451, 576, 600
— — — for large values of t, 503
— — — initially oscillating in simple mode, 463
— — — — — — progressive wave, 487, 497
— — — one particle vibrating in prescribed manner, 511
— — — passage from finite to infinite system, 465, 468, 477, 495, 500
— — — — from infinite to finite system, 478, 482, 485
— — — permanent mode of, 489, 492, 495, 496, 498, 499
— — — recapitulation of results, 488
— — — with arbitrary initial conditions, 451, 576
— of semi-infinite linear system, 457, 459, 460, 461, 465, 482, 500, 511
— — — — — recapitulation of results, 492
— of square system, 568
— quantity of, 463
— researches respecting, connected with theory of light, xiv, 451, 578
— sinusoidal mode of, 488, 489, 492, 495, 497
— velocity of progress of, 577
— — of propagation of, 510, 517, 556, 560, 571, 572, 582, 605
Vibrations in crystals, law of *vis viva* for, 437
— principle of equivalent, 437
— reflection of, 457, 459
Vis viva, 106; *see also* Living force
— — approximations to, in three-body problem, 17
— — derivative of V with respect to, x, xii, 46, 49, 108
— — law of, for vibrations in crystal, 437
Voss, 285

W function, *see* Disturbing function
Watson, 509, 563
Wave-front, xv, 640
— — description of, 563, 604, 605, 606, 640
— interval of, 421
— -length, 446
— motion in elastic medium, description of, 563, 599, 640; *see also* Vibration in elastic medium
— — in linear system, *see* Vibration of linear system
— -normal, 638, 639
— simple progressive, *see* Travelling sinusoid
— sinusoidal, 451
— slowness of, 417
— -surface, double, in Iceland Spar, 441
— — Fresnel's, 432, 442
— — — section by principal plane, 442
— — — optic axes of, 442
— — in dispersive crystal, 444
— thickness of, 417, 421
— -velocities, in crystal, 423
— — in dispersive crystal, 437, 439, 441
— -velocity, in aether, 447, 448
— — in biaxal crystal, Fresnel's law of, 433, 434, 444, 639
— — in dispersive crystal, Fresnel's law of, 438, 439, 444, 445, 639
— — in dispersive medium, law of, 447, 582, 584, 586, 596
— — *see also* Group-velocity; Phase-velocity; Velocity of propagation of vibration; Velocity of transmission of phase
Waves, plane, in aether, 446
— — in crystal, 416
— — — velocities of propagation of, 423, 427, 433
— three-dimensional, in dispersive medium, 569
— two-dimensional, in elastic medium, 568
Whittaker, 139, 166, 286, 638, 639
Work, 106

oung, 106

CAMBRIDGE: PRINTED BY W. LEWIS, M.A., AT THE UNIVERSITY PRESS